Physics Vol. 1, No. 6, pp. 315-328, 1965. Physics Publishing Co. Printed in Great Britain

ELECTRON-PHONON RENORMALIZATION EFFECTS IN HIGH MAGNETIC FIELDS. THE de HAAS-Van ALPHEN EFFECT*

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(Received 17 March 1965)

Abstract

When a metal is placed in a sufficiently strong magnetic field, it is to be expected that the effective mass of the electrons due to electron phonon interactions will be modified. The condition that the field will be strong enough to have an effect is that the cyclotron frequency be comparable to phonon frequencies. It is expected that such effects may be observable with experimentally realizable fields in some special cases. We have calculated the amplitude dependence of the de Haas-van Alphen effect as a function of temperature and magnetic field. This amplitude can be interpreted as giving a field dependent effective mass. The amplitude has been calculated numerically for several cases of interest. It is found that the field effects are rather smooth and depend most directly on the electron self-energy function for pure imaginary values of the frequency.

I. Introduction

ORDINARY metals constitute one of the most interesting classes of interacting systems of elementary "particles". The objects of major interest are the electrons and phonons, and their interactions.

Like the field theory of elementary particles, but perhaps even more dramatically, the field theory of electrons, phonons, and photons must be renormalized. The renormalized electrons and phonons, or elementary excitations, are those which are observed "at zero energy" and "at large distances". In the relativistic case, of course, the two expressions are equivalent to "small four-momentum transfer". At the present time, no one knows at what small distance, if any, quantum electrodynamics begins to break down [1], but one can entertain the idea that at very large momentum transfer, one should observe unrenormalized, or "bare" electrons [2].

In contrast, the electrons in metals are renormalized by a hierarchy of interactions whose scale of energy we understand. There is first, the interaction with the fixed lattice. The relevant energy in this case can usually be taken to be greater than an electron volt. (We

* Supported in part by the U.S. Air Force Office of Scientific Research grant No. AF 735-65, the Advanced Research Projects Agency, and the General Research Board of the University of Maryland.

shall neglect those cases in which spin orbit interactions are important.) Secondly, there is the electron-electron interaction via the Coulomb interaction. The energy scale here is again electron volts. These two interactions will by themselves give rise to a renormalized system of electrons [3]. The scale of energies of the renormalization is electron volts, the scale of wave numbers is inverse angstroms, so that for cases in which the energy and momentum values are much less than these values, one does not expect to find energy and momentum dependent renormalization effects. It turns out that the Coulomb interaction probably contributes very little renormalization or direct scattering at all [4], so we shall say no more about it. The band structure is all important, of course, but is comparatively well understood, so we shall make only a few remarks when necessary.

We see that the most interesting interactions in ordinary metals, as far as low frequency, long wavelength renormalization effects are concerned, have to do with the electron-phonon interactions, It would seem to be of great interest therefore to study the renormalization effects as the transition is made from fully renormalized to unrenormalized. It is the purpose of this note to discuss this transition as it is induced by an increasing magnetic field. In particular, we can predict the result of the most typical experiment which may be observable, namely, the de Haas-van Alphen experiment [5]. Another possible experiment has been discussed by Scher and Holstein [6]. They consider cyclotron resonance at very high fields and high frequencies. We shall later make a few remarks about the relationship between the two experiments.

We first review some of the salient facts concerning the electron-phonon system. The most fundamental approximation is the "weak momentum dependence" approximation, which indicates that the influence of the electron-phonon interaction on the momentum dependence of the self-energy functions is weak. Migdal [7] first recognized clearly how to take advantage of this circumstance, and following his lead, it has been found possible to classify completely the electron-phonon effects, in the normal system, under ordinary conditions. For a time variation small in comparison with the Debye frequency, there are only three experiments in normal metals exhibiting renormalization effects. These are specific heat measurements, cyclotron resonance, and the amplitude dependence of the de Haas-van Alphen oscillations. All three yield the same number, namely an effective mass m^*. Experiments not depending on the mass at all, or depending only on a mobility τ/m are unaffected by renormalization, (i.e. the free time τ is renormalized by the same factor as is the mass). This paucity of effects is a consequence of the fact that the mass renormalization is identical with the wave-function renormalization, which in turn is a consequence of the weak momentum dependence.

In the case of slow time variation, it has been shown that the electron-phonon system obeys a Landau transport equation [8], from which all of these results follow. A more general equation taking into account imposed frequencies of the order of the Debye frequency has been obtained by Holstein [9].

The electronic specific heat might in principle provide a way to measure the variation of the renormalization factor as a function of energy [10], but the phonon specific heat completely swamps it in the temperature range of greatest interest. Thus, we are compelled to study the high magnetic field cases.

It should be remembered that the usual transport coefficients depend markedly on actual electron and phonon scattering events. There is no effect of renormalization on these coefficients, however. On the other hand, the scattering events are inextricably associated with the renormalization, via the Kramers-Kronig relationships, so that the high field effects have to be described as arising from both dispersive and absorptive effects.

Let us estimate the order of magnitude of the fields and frequencies required to observe an effect. Clearly, the crucial relationship will be the comparability of the cyclotron and phonon

frequencies. Now, one million gauss corresponds to a cyclotron frequency, ω_C, for free, electrons, of $1.76 \times 10^{13}\text{sec}^{-1}$, and a Debye temperature of 100°K corresponds to a Debye frequency, ω_D, of $1.31 \times 10^{13}\text{sec}^{-1}$. Since million gauss fields are presently unattainable, experiments are not feasible on the simplest metals. However, band structure effects on small necks or pockets can easily reduce the electronic mass by an order of magnitude, and in the softer materials the transverse Debye frequency could be of the order of, say, 30°K. A mass of one fifth the free mass, and a Debye frequency of 30°, would make a field of 100,000 gauss sufficient to cause the cyclotron frequency to become equal to the Debye frequency. Pulsed field techniques can be used to attain considerably higher de Haas-van Alphen fields, but so far it has not proved feasible to use pulsed fields for measuring the amplitude.

Even smaller masses can of course be found, but then band structural effects are apt to become important in high fields. Such effects, which include magnetic breakdown [11], are very fascinating in their own right, but we do not wish to have to take them into account. In fact, the theory we shall develop is for the Fröhlich [12] model, in the expectation that the band structure effects can be included in an ad hoc fashion to a sufficient degree of accuracy.

Unfortunately, in the case of the de Haas-van Alphen effect, the above estimate is overgenerous. It turns out that there is a 20% effect for fields such that $\omega_C \sim 2\pi\omega_D$. The magnitude of the effect is proportional to $(\omega_C/2\pi\omega_D)^2$. Thus the measurement of the effect is going to be very difficult, and limited to a very few substances until substantially higher fields can be obtained.

As has often been remarked, it is to be expected that the effective mass of the electrons will change as a function of magnetic field from its low field value to the band plus Coulomb interaction mass value in high fields. In addition, there will be a level broadening or lifetime effect from the possibility of phonon emission. This result is often attributed in a vague way to the idea that in sufficiently high fields the electrons go around so fast that the lattice doesn't have time to respond, or to the idea that the electronic orbits become so tight that phonons are radiated away, something like synchrotron radiation. Both of these ideas are quite misleading.

The fallacy in the first idea is that the electrons are not speeded up by the field. In fact, they stay in a given region longer than in the absence of the field. If one based arguments entirely on the electronic velocity, one would arrive at the conclusion that phonon effects are always small. Indeed, this would be the case were it not for the exclusion principle. Because of the exclusion principle, electronic states of very low excitation energy cannot be regarded as wave packets of dimensions comparable with the inverse of their (Fermi) momentum. The exclusion principle is essential in estimating the lifetimes of the states. It is just as important for strong magnetic fields as in the weak field case.

The synchrotron radiation idea can also be discarded. In magnetic fields of the magnitude considered, the cyclotron orbits are very large in comparison with lattice dimensions. The curving of the orbit within a unit cell is quite negligible. We shall see that it is a consequence that the spatial characteristics of the electron-phonon interaction are hardly affected. Since the dependence of the electron mass operator on the momentum variables is weak in weak fields, it remains weak in strong fields. It is this fact that allows a simple solution to the problem.

The field effect is consequently simply one of quantization. The field, by itself, does not cause a decay of the electronic states, nor does it cause the effective mass to change. However, it does cause the kinetic energy values to be discrete. This means that electronic states of energy ω_C away from the Fermi surface come into play in so far as the discreteness is

important, and we know that such states can be strongly influenced by the possibility of phonon emission. The effective mass of such states is also different, in general, from that of the states near the Fermi surface.

In the remainder of this paper, we justify these remarks, and obtain curves illustrating the effect for typical phonon spectra and coupling parameters.

In the next section, we discuss the generalization of Midgal's results to the case of strong magnetic fields. In the third section we obtain the main theorem, namely that the field has no effect on the self energy part of the electrons. Finally, we compute the amplitude of the de Haas-van Alphen oscillations.

II. Analysis of Perturbation Theory

Migdal has shown that, in the absence of a magnetic field, certain graphs dominate in a perturbation analysis of the electron-phonon system,. His analysis is based on the adiabatic approximation, i.e. it rests upon exploiting the smallness of the equivalent parameters $\sqrt{m/M}$ s/v_F , and ω_D/E_F which are characteristically a few tenths of a percent in magnitude. The parameters are electron mass, ion mass, speed of sound, Fermi velocity, Debye frequency and Fermi energy. At the same time, it is important that the spatial parameter of importance in the phonon system is again a, the lattice constant. In the presence of a strong magnetic field, new electronic parameters are acquired, so that the dimensional arguments of Migdal are cast into doubt.

We first examine the phonon polarization graphs (see Fig. 1). As before, it is clear that

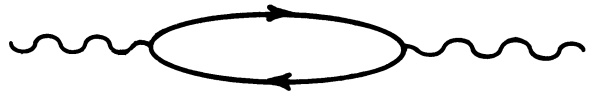

FIGURE 1

Lowest order correction to the phonon propagator (polarization part).

the dominant contribution comes from exciting an electron from well below the Fermi level, to a state well above the Fermi level. This contribution is clearly independent of magnetic field, given that $\omega_C << E_F$. There is a small oscillatory term, coming from those excitations at the Fermi level, but this can be safely neglected. (As usual, the very long wave length phonon have to be treated separately) [13]. Thus, we can conclude that the real phonon frequencies are unaffected by a magnetic field to an adequate approximation.

We next turn to the vertex corrections, Fig. 2. According to Migdal, this graph is small by a factor $\sqrt{m/M}$. The reason is that when an electron and a hole are created by a phonon, they quickly separate since they are moving at an appreciable angle. As the phonon field generated by one of the electrons is strong over a very short distance, the exchange of phonons by the pair can come only during a very short time interval after the creation of the pair. Since there is nothing to enhance the interaction during this period, (no small energy denominator) the effect is much reduced.

It is quite clear that, from this point of view, the magnetic field cannot be expected to

enhance the vertex corrections. Nevertheless, it may be worthwhile to explicitly carry out the estimate, since arguments regarding the electrons as localized are somewhat dangerous.

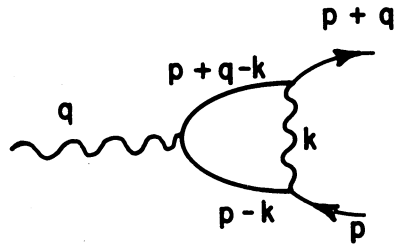

FIGURE 2

Lowest order electron-phonon vertex correction.

In the Landau gauge, $\mathbf{A} = (-Hy, 0, 0)$ it is generally known [14] that the propagator $G(\mathbf{r}_1, \mathbf{r}_2)$ takes the form

$$G(\mathbf{r}_1, \mathbf{r}_2) = \exp\left[i(x_2 - x_1)(y_2 + y_1)/2r_Q{}^2\right] \hat{G}(\mathbf{r}_1 - \mathbf{r}_2) \tag{1}$$

Free electrons in this field are quantized in oscillator states. The spatial dimension of the oscillator's lowest states is given by r_Q where

$$r_Q = \sqrt{\hbar/eH} \tag{2}$$

It is crucial to our approximation scheme that r_Q be not too small. However, a field of a million gauss gives $r_Q = 25\text{A}^\circ$, and since the errors to be incurred are of the order of $(a/r_Q)^2$, where a is the lattice constant, it is seen that no significant error from this source is to be expected. The cyclotron radius is much larger, namely of order $r_Q{}^2 p_F/\hbar$. We shall adopt units such that $m = 1/2$, $r_Q{}^2 = 1/2$, and $\hbar = 1$. Then $\omega_C = eH/m = 4$.

The change of the propagator in the magnetic field can be described as being of two parts. The first is the phase factor, which is simply a reflection of the replacement $p \to p - eA$. The second is the remaining effect on $\hat{G}(\mathbf{r}_1 - \mathbf{r}_2)$. The phase factor alone accounts for the curvature of the classical orbital motion, whereas the main residual effect on $\hat{G}$ is that of quantization.

We are now ready to apply this to an estimation of the vertex correction. Following Migdal, we estimate the graph of Fig. 2 which is given analytically by

$$\Gamma = (2\pi)^{-4} \int d^4k \, |\alpha(k)|^2 D(k) G(p - k) G(p + q - k) \tag{3}$$

Migdal estimates this graph as follows: The phonon propagator with its associated coupling constants are seen to be weakly dependent on momentum, and to be small unless the phonon frequency variable k_0 is comparable to the Debye frequency. On the other hand, the kinetic energies of the electrons are nearly independent for typical q, and range over a scale of energies of the order of electron volts. Thus the vanishing of the energy denominators is unimportant. In other words, the vertex corrections must take place spatially very close to original interaction, but in that case, the frequency integral is small because of the mismatch between the phonon and electron energies. By very close is meant within a distance $1/p_F$.

In the presence of the magnetic field, the vertex function does not have such a simple form,

because of the phase factors appearing in G. However, there is nothing to prevent us from removing a phase factor

$$\exp\left[i(x_3 - x_1)(y_1 + y_3) + i(x_2 - x_3)(y_2 + y_3)\right]$$

from $\Gamma(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3)$ before taking the Fourier transform.

We thus obtain as an estimate of the Fourier transformed vertex function, excluding the phase factors, the expression (3), but with $G(p)$ replaced by $\tilde{G}(p)$, the Fourier transform of $\hat{G}(\mathbf{r})$. Migdal's arguments then show that since $\tilde{G}(p)$ is integrated over a wide range of p values, the most important spatial values in the original $\hat{G}(\mathbf{r})$ are very small. As is well known, and as will be shown in the next section, we may approximate $\tilde{G}(p)$ by $G(p)$, the Green's function in the absence of the field, providing we make the substitution

$$p^2 = p_\perp^2 + p_z^2 \longrightarrow p_n^2 + p_z^2 \tag{4}$$

where

$$p_n^2 = 2m\hbar\omega_C(n + \tfrac{1}{2}) = 4n + 2 \tag{5}$$

and replace $\int dp_\perp^2$ by $4\sum_{n=0}^{\infty}$. The replacement of the integral by a sum does not change the estimate, as can be seen by an application of the Poisson sum formula. The manipulations described here are quite straightforward although tedious, and are very similar to those discussed at greater length in the next section.

We have still to consider those effects which could arise from the omission of the phase factors. However, these cannot be very serious, because it is already clear that $\Gamma(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3)$ is very small unless points 1 and 2 differ by a few lattice constants at most. Thus, the phase factors don't amount to very much, and can't affect the estimate that the vertex correction is small.

III. The Self Energy Part

In this section we compute the self-energy part Σ. The result of the computation is that the self-energy is practically unmodified by the magnetic field, except for the ubiquitous phase factor, and that in particular it does not depend sharply on the momentum variables, but only on the frequency variable.

In real space, the self-energy function will have the form

$$\sum(\mathbf{r}_1, \mathbf{r}_2) = \exp i(x_2 - x_1)(y_1 + y_2)\hat{\sum}(\mathbf{r}_1 - \mathbf{r}_2) \tag{6}$$

Taking the Fourier transform in the $\mathbf{r}_1$ and $\mathbf{r}_2$ variables separately, one obtains, by applying the result of the previous section,

$$\sum(\mathbf{p}_1, \mathbf{p}_2) = i(2\pi)^{-4}\int d^4q\,|\alpha(q)|^2 D(q)G(\mathbf{p}_1 - \mathbf{q}, \mathbf{p}_2 - \mathbf{q}; p_0 - q_0) \tag{7}$$

In order to simplify the calculation, we assume that there is cylindrical symmetry about the magnetic field direction. In this case it is not difficult to show that

$$\hat{G}(\mathbf{r}, E) = \frac{2}{\pi} \sum_{n=0}^{\infty} \int \frac{dp_3}{2\pi} e^{ip_3 z} e^{-\frac{1}{2}\rho^2} L_n(\rho^2) G(n, p_z, E) \tag{8}$$

where L_n is the Laguerre polynomial, and $\rho = (x, y, 0)$. Introduce instead of $\Sigma(\mathbf{p}_1, \mathbf{p}_2)$, $\tilde{\Sigma}(\mathbf{p})$, the Fourier transform of $\hat{\Sigma}(\mathbf{r})$. Using (6) and (7), it is straightforward to obtain the expression

(9)

$$\tilde{\Sigma}(\mathbf{p}, p_0) = \frac{2i}{\pi} \int d^2\rho \sum_{n=0}^{\infty} e^{-\frac{1}{2}\rho^2} L_n(\rho^2) \int \frac{d^4q}{(2\pi)^4} |\alpha(q)|^2 D(q) e^{-i(\mathbf{p}-\mathbf{q})\cdot\rho} G(n, p_z - q_z, p_0 - q_0)$$

The integration over the spatial variables may be carried out and the following expression obtained:

(10)

$$\tilde{\Sigma}(\mathbf{p}, p_0) = i \sum_{n=0}^{\infty} \int \frac{d^4q}{(2\pi)^4} 2(-1)^n e^{-\frac{1}{2}(p_\perp - q_\perp)^2} L_n\left((p_\perp - q_\perp)^2\right) |\alpha(q)|^2 D(q) G(n, p_z - q_z, p_0 - q_0)$$

where $p_\perp = (p_x, p_y, 0)$.

At this stage we discuss the function $\Delta(x) = e^{-\frac{1}{2}x^2} L_n(x^2)$. For given n, there are three regions of x. For x in the region $x^2 < p_n^2 = 4n + 2$, the function is well approximated by

$$\Delta(x) \equiv J_0(x p_n) = (2\pi)^{-1} \int_0^{2\pi} d\varphi \, e^{i\mathbf{p}_n \cdot \mathbf{x}} \tag{11}$$

where we have artificially introduced vector variables in the last equation. For $x \sim p_n$, the oscillations become stationary, and there is no simple expression for the function. The value of the function is rapidly depressed for arguments greater than the critical one $x > p_n$. If $\Delta(x)$ is multiplied by a function whose variation is slow in comparison with the oscillations in the small x region and integrated, it acts like a Dirac δ-function. The correspondence is

$$\Delta(x) \simeq 2(-1)^n \delta(x^2 - p_n^2) \tag{12}$$

If $\Delta(x)$ is multiplied by a function which is itself oscillatory, or peaked, it clearly cannot be treated as a delta function.

According to this discussion, we cannot regard the Δ which appears in (9) as approximately a δ-function, since it is multiplied by the rapidly varying exponential factor. However, it is possible to regard the Fourier transform Δ in equation (10) in this fashion, since there is no exponential factor present. The main caveat is that there must be no rapid variation of the remaining factors.

To show this, we introduce the spectral decomposition of G,

$$G(n, p_z, p_0) = \int \frac{dE}{2\pi} A(n, p_z, E) \left\{ \frac{1 - f(E)}{p_0 - E + i\eta} + \frac{f(E)}{p_0 - E - i\eta} \right\} \tag{13}$$

We may express $A(n, p_z, E)$ as

$$A(n, p_z, E) = \Gamma(n, p_z, E) / \left[(E + E_F - p_n^2 - p_z^2 - \mathrm{Re}\, \Sigma(n, p_z, E))^2 + \Gamma^2/4 \right] \tag{14}$$

In this expression, $\Sigma(n,\ p_z,\ E)$ is given by

$$\sum(n,\ p_z,\ E) = (2\pi)^{-1}\int d^2p_\perp(-1)^n e^{-\frac{1}{2}p_\perp^2}L_n(p_\perp^2)\,\widetilde{\sum}(\mathbf{p},\ E) \tag{15}$$

and by anticipating the final result that $\Sigma(p,\ E)$ is slowly varying in the p variable, we see at once that

$$\sum(n,\ p_z,\ E) = \widetilde{\sum}(p_n,\ p_z,\ E) \tag{16}$$

We have also defined $\Gamma = 2|\mathrm{Im}\ \Sigma|$.

Using this result, we find the expression

$$\widetilde{\sum}(p) = 2\sum_n(-1)^n(2\pi)^{-4}\int d^2p_\perp' dp_z' dE\ e^{-\frac{1}{2}p'^2}L_n(p_\perp'^2)\,|\alpha(\mathbf{p}-\mathbf{p}')|^2\ \times$$

$$\times\ A(n,\ p_z',\ E)\left\{\frac{1-f(E)}{p_0-\omega_{\mathbf{p}-\mathbf{p}'}-E+i\eta}+\frac{f(E)}{p_0+\omega_{\mathbf{p}-\mathbf{p}'}-E-i\eta}\right\} \tag{17}$$

(we record, for simplicity, only the expression at zero temperature, when the Fermi function $f(E)$ becomes the step function). Let us note that both $A(n,\ p_z',\ E)$ and $\Delta(p_\perp'^2)$ are approximately δ-functions in n, so it is not possible to do the n sum directly. On the other hand, in the variables p_z' and p'^2, respectively, these two functions are effectually δ-functions, since the coupling constant and phonon frequencies may be presumed to vary slowly. In particular, the energy denominators are not bothersome, because the phonon frequency is very weakly dependent on the length of the vector $\mathbf{p}'$, depending much more on the angle between $\mathbf{p}$ and $\mathbf{p}'$.

In this way we obtain

$$\sum(p) = \sum_{n=0}^{\infty}\frac{1}{\pi_z(n)}\int\frac{d\varphi'}{2\pi}\int\frac{dE}{2\pi}\,|\alpha(\mathbf{p}-\pi(n))|^2\ \times$$

$$\times\left\{\frac{1-f(E)}{p_0-\omega(\mathbf{p}-\pi(n))-E+i\eta}+\frac{f(E)}{p_0+\omega(\mathbf{p}-\pi(n))-E-i\eta}\right\} \tag{18}$$

where the vector $\Pi(n)$ is $(\mathbf{p}_n,\ \sqrt{E-p_n^2})$ and $\mathbf{p}_n$ is taken parallel to $\mathbf{p}_\perp'$. Upon replacing the sum by an integral, we see that $\Sigma(p)$ is identical to the self energy part in the absence of the field. Since the p_n sum is over a function which varies over changes in p_n of order p_F, it is clear that the errors coming from the replacement of sum by integral are small and may be neglected.

In practice, we will deal with a non-spherical Fermi surface. The values of $\Pi(n)$ will then lie on the Fermi surface. In particular, they won't be confined to a small neck region, so the values of n contributing to the sum is very large, and replacing the sum by an integral is justified.

This completes the discussion of the self-energy part.

IV. The de Haas-van Alphen Amplitude

We can now compute the result of the de Haas-van Alphen experiment, since we know the Green's function $G(n, p_z, p_0)$ to an adequate approximation. The density is given by the standard formula,

$$n = \frac{2}{\pi} \sum_{n=0}^{\infty} \int \frac{dp_z}{2\pi} \int \frac{d\omega}{2\pi} f(\omega) A(n, p_z, \omega) \tag{19}$$

Using the Poisson sum formula in the form

$$\sum_{n=0}^{\infty} f(n + \tfrac{1}{2}) = \int_0^{\infty} dx\, f(x) \left[1 + 2 \sum_{r=0}^{\infty} (-1)^r \cos 2\pi r x \right] \tag{20}$$

we obtain for the oscillating terms in the density,

$$n_r = \frac{2}{\pi} \operatorname{Re} \int_0^{\infty} dx\, e^{2\pi i r x} \int \frac{dp_z}{2\pi} \int \frac{d\omega}{2\pi} f(\omega) A(x - \tfrac{1}{2}, p_z, \omega) \tag{21}$$

By making the usual substitution, $p_\perp{}^2 = 2m\omega_c x$, we cast this into

$$n_r = \operatorname{Re} \int \frac{d^3p}{(2\pi)^3} e^{\pi i r \frac{1}{2} p_\perp{}^2} \int \frac{d\omega}{2\pi} f(\omega) \frac{\Gamma(\omega)}{[\omega - \varepsilon_p - \Sigma(\omega)]^2 + \Gamma^2/4} \tag{22}$$

where $\varepsilon_p = p^2 - E_F$.

Reverting to spherical coordinates, we can evaluate the angular integrals by the method of stationary phase, and find

$$n_r = \operatorname{Re} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} N(\varepsilon)\, e^{-i\pi/4}\, e^{\pi i r(\varepsilon + E_F)/2} \sqrt{\frac{1}{2r(\varepsilon + E_F)}} \int d\omega f(\omega) \frac{\Gamma(\omega)}{(\omega - \varepsilon - \Sigma)^2 + \Gamma^2/4} \tag{23}$$

Here, $N(\varepsilon)$ is the unrenormalized density of states, for one spin. The ε variation in the exponential and A factor is strong, but it is weak elsewhere, and ε can be replaced by ω (or 0) in $N(\varepsilon)$ and under the root symbol. Hence the ε integration can be carried out, to find, in ordinary units,

$$n_r = N(0) \sqrt{\hbar\omega_C / 8rE_F}\, \operatorname{Re}\, e^{-i\pi/4}\, e^{2\pi i r E_F/\hbar\omega_C} \int d\omega f(\omega)\, e^{[2\pi i r(\omega - \Sigma + i\Gamma/2)/\omega_C]} \tag{24}$$

The magnetization can be obtained from this by integrating with respect to the chemical potential E_F, giving the thermodynamic potential, and then differentiating with respect to the magnetic field H. Only the variation of the exponential factor $\exp(2\pi i r E_F/\hbar\omega_C)$ need be taken into account. The result of these steps is,

$$m_r = \left\{ \frac{2\pi i r}{\omega_C} \int_{-\infty}^{\infty} d\omega f(\omega)\, e^{[2\pi i r(\omega - \Sigma + i\Gamma/2)/\omega_C]} \right\} \operatorname{Re} \frac{i}{H} \frac{N(0)}{2\pi} \sqrt{\frac{E_F \hbar\omega_C{}^3}{8r^3}} \exp\left(\frac{2\pi i r E_F}{\hbar\omega_C} - \frac{i\pi}{4} \right) \tag{25}$$

Now, $\Sigma(\omega) - i\Gamma(\omega)/2$ is the boundary value of a function meromorphic in the upper half

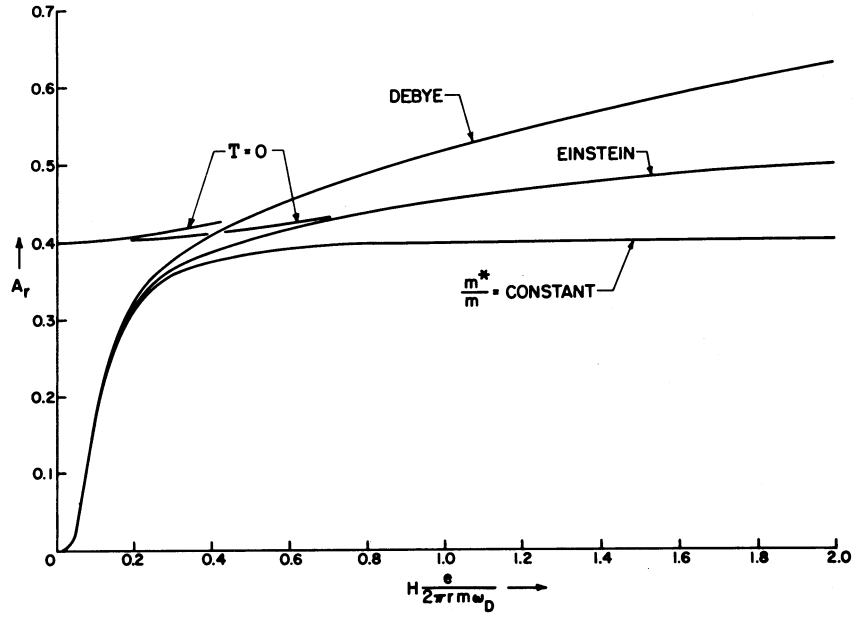

FIGURE 3

The field dependent part Ar of the magnetization amplitude as a function of field for a Debye model and an Einstein model. The behavior for a constant effective mass is given for purposes of comparison.
(Temperature $\pi T = 0.1\ \omega_0$. The $T = 0$ curves are also shown.)

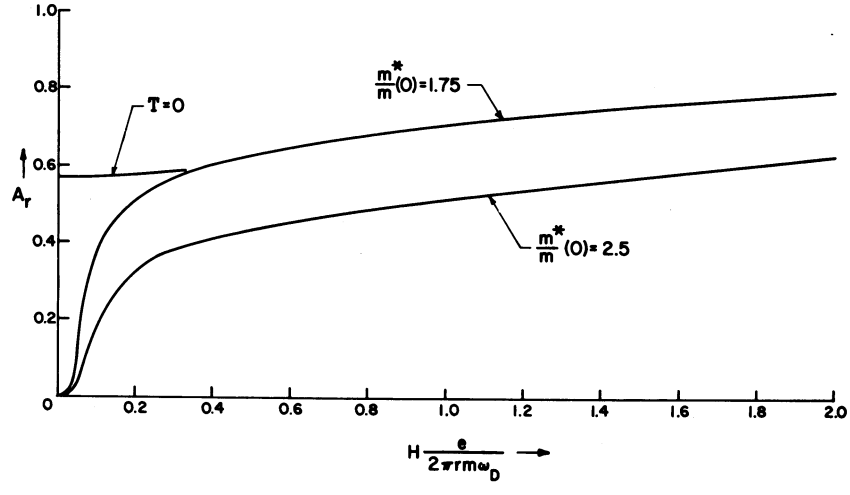

FIGURE 4

The field dependent part A_r of the magnetization amplitude as a function of field for a Debye model for two different zero field effective masses.

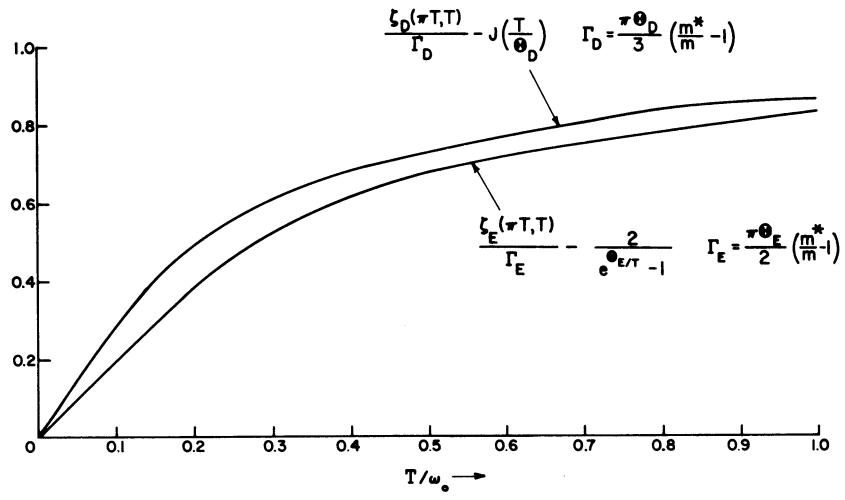

FIGURE 5

Variation of the interesting part of the self energy for the pure imaginary frequency $i\pi T$ in a Debye model and an Einstein model.

frequency plane. Denote $\zeta(y) \equiv i\Sigma(iy)$. We can then define the amplitude factor $A_r(H, T)$ by the bracket in equation (25) which contains all of the interesting field and temperature dependence. The definition is

$$A_r = 4\pi^2 rT/\omega_C \sum_{k=0}^{\infty} \exp\left\{-2\pi r[\omega_k + \zeta(\omega_k, T)]/\omega_C\right\} \tag{26}$$

(Note that ζ depends on temperature explicitly, as well as through $\omega_k = \pi T(2k + 1)$.) Then, the observed r-component of the magnetization is related to the component in the absence of renormalization effects and at zero temperature by

$$m_r = A_r m_r^{(0)} \tag{27}$$

This remains true, even if spin is taken into account, as the electron-phonon effects are diagonal in spin. (The spin merely contributes a factor $2\cos(g\mu_B H 2\pi r/\omega_C)$ to m_r^0. It does make some complication, since the g-factor could well differ significantly from 2 for the small mass regions of greatest interest.)

The quantity A_r is what we hope the experimentalists can measure. At the absolute zero, and for low fields, it clearly becomes m/m^* where

$$\frac{m^*}{m} = 1 + \left.\frac{\partial\zeta(y)}{\partial y}\right|_{y=0} = 1 - \left.\frac{\partial\Sigma}{\partial\omega}\right|_{\omega=0} \tag{28}$$

For very strong fields, such that $\omega_C >> \omega_D$, the factor becomes unity, if the temperature is not too high, since the important ω_k values are of order ω_C, where ζ can be neglected. In the next section, we discuss A_r numerically, for reasonable models of the electron-phonon parameters.

V. Numerical Results

In this section we discuss the expression (26) giving the interesting field and temperature variation of the de Haas-van Alphen amplitude. Although the expression for A_r is not very complex, it cannot be obtained in terms of tabulated functions. We present results for two models, an Einstein model and a Debye model. Neither of these is very realistic, but should serve to provide reasonable estimates and an insight into more general cases. In any case, it should be remembered that the phonon spectra here employed represent the lowest phonon modes. Higher frequency phonons will contribute a field independent renormalization which cannot be modified by the fields presently available.

In the Einstein model, the self-energy part in the absence of field is,

$$\Gamma(E,\ T) = \Gamma_e \left\{2/(e^{\omega_e/T} - 1) + 1 - f(E - \omega_e) + f(E + \omega_e)\right\} \tag{29}$$

where Γ_e, ω_e are constant.

From this, by the dispersion relation,

$$\zeta(y,\ T) = \Gamma_e \left\{1 + \frac{2}{e^{\omega_e/T} - 1} + \frac{2}{\pi}\int_0^\infty dE\ \frac{y}{y^2 + E^2}\left(f(E + \omega_e) - f(E - \omega_e)\right)\right\} \tag{30}$$

At the absolute zero, we have the particular case,

$$\zeta(y,\ 0) = \frac{2}{\pi}\Gamma_e \tan^{-1}(y/\omega_e) \tag{31}$$

In the Debye model, on the other hand, we can take

$$\Gamma(E,\ T) = \Gamma_D \left\{J(T/\omega_D) + 1 + \int_0^{\omega_D} d(\omega/\omega_D)^3 \left[f(E + \omega) - f(E - \omega)\right]\right\} \tag{32}$$

where

$$J(x) = 6x^3\int_0^{1/x} dy\ y^2/(e^y - 1) \tag{33}$$

The particular case of zero temperature yields, this time,

$$\zeta(y,\ 0) = \frac{2}{\pi}\Gamma_D \left\{\tan^{-1}\left(\frac{y}{\omega_D}\right) + \frac{1}{2}\left[\frac{y}{\omega_D} - \left(\frac{y}{\omega_D}\right)^3 \ln\left(1 + \frac{{\omega_D}^2}{y^2}\right)\right]\right\} \tag{34}$$

For the Einstein model, $m^*/m = 1 + 2\Gamma_e/\pi\omega_e$, whereas for the Debye model $m^*/m = 1 + 3\Gamma_D/\pi\omega_D$. For the same effective mass and characteristic temperature, the Debye model will show greater high field effects than the Einstein, (in the case of the de Haas-van Alphen effect).

At the absolute zero of temperature, we can obtain the first high field corrections by expansion. The result is, with $\eta = \omega_C/2\pi r\omega_D$

$$A_r \cong \frac{m}{m^*}\left[1 + 2\left(\frac{m}{m^*}\right)^3\left(\frac{m^*}{m} - 1\right)\eta^2 + \ldots\right] \qquad \text{(Einstein)} \tag{35}$$

and

$$A_r = \frac{m}{m^*}\left[1 + 2\left(\frac{m}{m^*}\right)^3\left(\frac{m^*}{m} - 1\right)\eta^2\ 2\ \ln\ (C'm^*/m\eta) + \ldots\right] \qquad \text{(Debye)} \qquad (36)$$

where C' = Euler's constant/exp (3/2) = 0.3974. Thus we obtain the characteristic logarithmic dependence for the Debye case. These expressions may be interpreted as field-dependent effective masses. As compared with the Eliashberg corrections to the specific heat effective mass, they begin by decreasing the effective mass, rather than increasing it.

Scher and Holstein have also found, in their work, that the cyclotron mass first increases with frequency and then decreases. This can be readily understood in the following way. In the cyclotron resonance experiment, a true time variation is imposed externally, Thus, the self-energy, which is being studied, is evaluated at a real frequency, and no distortion of contour can make it appear that only pure imaginary frequencies are involved. Real excitations are involved in the specific heat mass also, but only virtual ones in this case. Although the expression (25) can be interpreted as giving rise to an amplitude correction because of real effective mass shifts and line broadening separately, it is clear that the two cannot be distinguished by the de Haas-van Alphen effect. Cyclotron resonance on the other hand is more powerful in that it depends on the self energy function for real energies, and thus could show more field dependent structure. (In that case an Einstein spectrum shows more dramatic effects than does a Debye.)

We next consider the case of relatively high temperature, such that only the first term of the sum (26) need be kept. Then we have,

$$-\ln(\omega_C Ar/rT) = 2\pi r(\pi T + \zeta(\pi T,\ T))/\omega_C \qquad (37)$$

At high temperatures the experimental problem may be somewhat eased, since extremal cross sections with large effective masses would not be present to complicate the signal. However, the function $\zeta(\pi T,\ T)$ is rigorously a straight line,

$$\zeta(\pi T,\ T) = \left(\frac{m^*}{m} - 1\right)\pi T \qquad (38)$$

This expression does not depend on the particular model of the phonon spectrum used. The function depends on two parts, one of which simply gives the total scattering rate due to phonons present in their temperature equilibrium and the other arising from the dispersive effects of virtual phonons. The latter is perhaps more interesting. However, at the particular imaginary frequency πT, the two effects cancel. The "interesting" part of $\zeta(\pi T,\ T)$ is plotted in Fig. 5.

Finally, we have numerically performed the sum in (26) for several cases. The results for the amplitude A_r are plotted in Figs. 3 and 4.

Discussions on the experimental aspects of the de Haas-van Alphen measurements with Professor F.J. Koch and J.R. Anderson are gratefully acknowledged. We wish to thank Professors Holstein and Scher for transmitting some of their results prior to publication.

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