Classification of Interacting Topological Floquet Phases in One Dimension

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Periodic driving of a quantum system can enable new topological phases with no analog in static systems. In this paper, we systematically classify one-dimensional topological and symmetry-protected topological (SPT) phases in interacting fermionic and bosonic quantum systems subject to periodic driving, which we dub Floquet SPTs (FSPTs). For physical realizations of interacting FSPTs, many-body localization by disorder is a crucial ingredient, required to obtain a stable phase that does not catastrophically heat to infinite temperature. We demonstrate that 1D bosonic and fermionic FSPT phases are classified by the same criteria as equilibrium phases but with an enlarged symmetry group $\tilde{G}$, which now includes discrete time translation symmetry associated with the Floquet evolution. In particular, 1D bosonic FSPTs are classified by projective representations of the enlarged symmetry group $H^2(\tilde{G}, U(1))$. We construct explicit lattice models for a variety of systems and then formalize the classification to demonstrate the completeness of this construction. We advocate that a prototypical $\mathbb{Z}_2$ bosonic FSPT may be realized by very simple Hamiltonians of the type currently available in existing cold atoms and trapped ion experiments.

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I. INTRODUCTION

Periodic driving of a quantum system enables one to tailor new interactions and achieve interesting quantum phases of matter. Such Floquet engineering has led to various applications in quantum optical contexts, such as the engineering of artificial gauge fields [1], as well as in solid-state contexts, e.g., to produce new Floquet-Bloch band structures [2,3] or understand nonlinear optical phenomena [4]. In addition to providing new tools to engineer phases that could arise as ground states of a different static Hamiltonian, periodic driving also opens up the possibility of engineering entirely new phases with no equilibrium analog [5–14]. In the context of noninteracting particles, various examples of new topological phases that arise from driving are known, including dynamical Floquet analogs of Majorana fermions in 1D [6] and phases with chiral edge modes but vanishing Chern number in 2D [5,7].

Heretofore, such investigations were largely restricted to noninteracting systems, as persistent driving of a generic interacting many-body system typically leads to catastrophic runaway heating towards a featureless infinite-temperature steady state, for which there are no sharp notions of distinct phases. While very rapid driving, with frequency much larger than the natural interaction scales of the Hamiltonian, can postpone this runaway heating for exponentially long times [15], new topological phases that occur exclusively in driven systems can be realized only in moderate-frequency regimes, where clean systems would be susceptible to heating issues [5–9].

However, many-body localized (MBL) systems [16] retain sharp spectral lines for local operators [17] and can therefore avoid energy absorption from off-resonant driving by a local Hamiltonian [18–20]. Interestingly, despite being strongly localized, MBL systems can still exhibit nontrivial topological and SPT order [21–24]. This raises the general conceptual question of which zero-temperature quantum phases can occur in the highly excited states of MBL systems, for which there is a growing systematic understanding [25]. The stability of MBL to Floquet driving enables sharp distinctions between dynamical phases of periodically driven matter [26] and extends this line of inquiry; it raises the prospect of realizing not only familiar ground-state orders but also fundamentally new interacting dynamical topological phases arising from driving. In this paper, we develop a systematic understanding of the structure of topological and symmetry-protected topological (SPT) phases of periodically driven Floquet systems in one spatial dimension.

Following Refs. [22,25,27,28], we begin by formulating a sharp criterion for many-body localizability in terms of the existence of an appropriate set of quasilocal conservation laws. We then study interacting Floquet topological phases of fermions in 1D. After reviewing some ideas and explicit models for noninteracting fermion Floquet SPTs [6,8,9,26], we then address the modification of the fermionic Floquet SPT classification due to interactions in all of the nontrivial classes of the tenfold way [29,30]. In the absence of interactions, periodic driving raises the new possibility of obtaining topologically protected edge modes with quasienergy π, [5,6] in addition to those with zero quasienergy that are familiar from nondriven equilibrium systems [31]. As for the equilibrium SPTs, we find that interactions generally tend to reduce the set of nontrivial phases when the noninteracting classification contains integer topological invariants [32–38]. In the Floquet context, this reduction arises from a nontrivial interplay of the zero- and π-quasienergy modes. In all cases, we find that the fermionic classification can be understood as having projective action of the symmetry group G (graded by fermion parity) combined with an effective integer-valued time-translation symmetry under the Floquet evolution, leading us to hypothesize that such projective representations form a complete classification.

We then turn to the study of Floquet SPTs in bosonic systems (e.g., spin models). Here, we build further evidence towards the hypothesized classification by constructing explicit models whose edge states realize all possible projective realizations of $G \times \mathbb{Z}$, where the extra factor of $\mathbb{Z}$ corresponds to discrete Floquet evolution “symmetry.” Interesting examples include a dynamical analog of the Haldane spin chain [39,40], which exhibits free spin-1/2 edge states that flip under each driving period and only return to their original states after two periods. We also encounter bosonic examples where symmetries protect edge modes with quasienergies that are neither 0 nor π but can be any rational fraction of 2π.

Having built up a repertoire of concrete examples, we then formalize the hypothesized classification of 1D Floquet topological phases by generalizing related classifications of equilibrium SPTs [32,41,42] to periodically driven Floquet systems. We rigorously establish the above-hypothesized equivalence between the Floquet SPT classification with group G and the equilibrium (“weak TI”-like) classification of $G \times \mathbb{Z}$ (or $G \rtimes \mathbb{Z}$ in the case of antiunitary symmetry group G). Recently, von Keyserlingk and Sondhi presented a related but distinct classification with consistent results using a different method [43]. Our results on bosonic Floquet symmetry-protected topological phases (FSPTs) are also consistent with a closely related independent work of Else and Nayak [44].

In addition to solving the formal classification problem, we also identify simple solvable time-dependent Hamiltonians for each phase. A notable example, which can be realized in spin chains with particularly simple Ising interactions of the type commonly realized in trapped ion [45] and ultracold lattice dipolar molecule [46,47] systems, is the bosonic SPT with $\mathbb{Z}_2$ symmetry, which we describe in detail in Sec. VI. This $\mathbb{Z}_2$ FSPT phase has protected edge states that can perfectly store quantum information without cooling, a feature common to all 1D MBL SPT phases [23,24]. However, unlike analogous 1D SPT phases that can be realized in time-independent Hamiltonians and that require complicated symmetries (e.g., at least $\mathbb{Z}_2 \times \mathbb{Z}_2$) and three spin interactions, this bosonic FSPT phase can be obtained with simple two-spin Ising-type interaction terms. As such, the dynamical nature of the FSPT phase dramatically simplifies the experimental implementation of this phase and may well offer the simplest route to realizing a SPT phase in cold-atom systems.

Finally, we discuss the constraints placed by the requirement of many-body localization. These constraints restrict the type of symmetry groups that can protect SPT phases to those with conserved particle number and Abelian symmetry groups and rule out many-body localization of time-reversal (TR)–invariant systems with Kramers doublet particles [48].

II. MANY-BODY LOCALIZED (FLOQUET) HAMILTONIANS

Since the requirement of many-body localization to avoid heating plays a crucial role in the sharp distinction among interacting Floquet phases, we begin by reviewing a widely settled-upon sharp definition for the existence of many-body localizability.

Full many-body localization is best defined through the existence of a complete set of quasilocal conserved quantities $\{n_\alpha\}$ that each take values $\{1\ldots p_\alpha\}$ and together uniquely label an arbitrary eigenstate:

$$|\Psi\rangle = |n_1 n_2 \ldots n_L\rangle, \tag{1}$$

where L is the system size.

By quasilocal, it is meant that each n_α is exponentially well localized near a position r_α, i.e., that the projection operators

$$\Pi_{n_\alpha} = \sum_{n_{\beta\neq\alpha}} |n_1 n_2 \ldots n_\alpha \ldots n_L\rangle\langle n_1 n_2 \ldots n_\alpha \ldots n_L| \tag{2}$$

differ from the identity at position r by an exponentially small amount, i.e., for any local operator $\mathcal{O}(r)$ with bounded support near position r, $(\|[\Pi_{n_\alpha}, \mathcal{O}(\mathbf{r})]\|/\|\mathcal{O}(r)\|) < e^{-|\mathbf{r}-\mathbf{r}_\alpha|/\xi}$, where $\|\ldots\|$ and ξ are an appropriate operator norm and localization length, respectively.

These projectors are exactly conserved quantities, which commute with the Hamiltonian, and hence their values are time independent. More explicitly, the Hamiltonian of a static system can be written as a generic function of these projection operators $H_{\mathrm{MBL}} = \sum_{\{n_\alpha\}} f(\Pi_{n_1}, \Pi_{n_2}, \ldots)$, where f is a (positive-definite) quasilocal function of its arguments (i.e., it is exponentially weakly sensitive to the relative state of two distant projectors). Or similarly, for a Floquet system, governed by a time-dependent Hamiltonian $H(t)$, which is periodic with period T, the time-evolution operator for a fixed period can be expressed as

$$F_{\mathrm{MBL}} = T\{e^{-i\int_0^T H(t)dt}\} = \prod_{\{n_\alpha\}} e^{if(\Pi_{n_1},\Pi_{n_2},\ldots)}, \tag{3}$$

where $T\{\ldots\}$ indicates time ordering.

In what follows, we temporarily put aside the question of localization to focus on the topological aspects of Floquet phases. Our strategy will be to first construct examples of special zero-correlation length models that represent particularly simple realizations of various Floquet topological phases. After building some intuition from these simple models, we give general arguments that the topological features of these zero-correlation length models are stable to generic perturbations and apply over a finite range of parameters; in particular, we examine what constraints are placed by the requirement of localizability.

III. FERMIONIC FLOQUET SPTS

Having sharply defined a notion of many-body localizability, we now turn to the concrete task of systematically understanding 1D Floquet topological and SPT phases. We begin by reviewing some previously known constructions of topological phases in noninteracting fermionic systems with periodic driving and then address how these results are modified upon the inclusion of interactions.

A. Floquet Majorana modes in noninteracting models

In a noninteracting static superconducting wire, Bogolyubov-de Gennes (BdG) excitations with energy E and $-E$ are related by particle-hole conjugation and correspond to complex fermionic excitations unless $E = 0$. However, in a driven system, quasienergy ε is defined modulo 2π (here, and throughout, we normalize the quasienergy with respect to the Floquet period T, such that quasienergies become dimensionless phases between 0 and 2π), and hence $\varepsilon = \pm\pi$ are equivalent, enabling real (i.e., self-conjugate) Majorana modes at energy π. To set the stage for the study, we describe a simple toy model [8,9,26] that exhibits perfectly localized Floquet Majorana Pi modes (MPMs), i.e., real conjugate) fermionic modes with quasienergy exactly quantized to π localized to the edge of a driven superconducting wire. Previous works have given more experimentally achievable proposals for realizing these phases [6]; however, the toy models will be instructive for establishing the proof of existence for more general Floquet SPTs and analyzing the effects of interactions.

Consider a superconducting chain of spinless (complex) fermions $c_j = \frac{1}{2}(a_j + ib_j)$, where j labels sites of the chain, and a and b are real (Majorana) fermion operators satisfying canonical anticommutation relations: $\{a_i, a_j\} = 2\delta_{ij} = \{b_i, b_j\}$, $\{a_i, b_j\} = 0$. A particularly simple construction that realizes a nontrivial Floquet topological phase is obtained by subjecting the chain to a "stroboscopic" periodic time-dependent drive under the Hamiltonian

$$H(t) = \begin{cases} H_1 = \frac{i\lambda_1}{4}\sum_{j=1}^{L} a_j b_j & 0 \le t < T_1 \\ H_2 = \frac{i\lambda_2}{4}\sum_{j=1}^{L-1} b_j a_{j+1} & T_1 \le t < T_1 + T_2. \end{cases} \tag{4}$$

For this alternating drive, the time evolution over the duration of a single period (Floquet operator), $T = T_1 + T_2$, decomposes into the product $F = \mathcal{T} e^{-i\int_0^T H_0(t)dt} = F_2 F_1$ with $F_j = e^{-iH_j T_j}$. Here, H_1 and H_2 are, respectively, zero-correlation length "fixed-point" Hamiltonians for the trivial and topological phases of the superconducting chain (see Fig. 1). In particular, a_0 and b_L do not appear in H_2 and hence would be local Majorana zero-quasienergy modes (MZMs) for $T_1 = 0$.

If we instead choose $(\lambda_1 T_1/4) = (\pi/2)$, which, using the identity $e^{\theta ab} = \cos\theta + \sin\theta ab$, gives $F_1 = \prod_{j=1}^{L} a_j b_j = a_1(\prod_{j=1}^{L-1} b_j a_{j+1}) b_L = a_1 e^{i2\pi H_2/\lambda_1} b_L$, the full Floquet time-evolution operator reads

$$F = a_1 e^{-i\tilde{T}_2 H_2} b_L \equiv e^{-iH_F}, \tag{5}$$

where $\tilde{T}_2 = T_2 - 2\pi/\lambda_1$ and $H_F = \tilde{T}_2 H_2 + (i\pi/2) a_1 b_L$ is the Floquet Hamiltonian for a specific branch cut of $\log F$.

We note that a_1 and b_L are left out of H_2 and hence commute with H_2. Then, $F a_1 F^\dagger = b_L a_1 b_L = e^{i\pi} a_1$ and similarly $F b_L F^\dagger = e^{i\pi} b_L$. Hence, a_1 and b_L are localized Majorana fermion modes with π quasienergy, which we henceforth refer to as Majorana-Pi modes (MPMs).

While we have so far demonstrated the existence of the strictly localized MPMs only for a particular choice of parameters, the MPMs are stable against small perturbations of the driving Hamiltonian and persist over a finite range of parameters centered around the ones chosen above. Just as in nondriven equilibrium quantum systems,

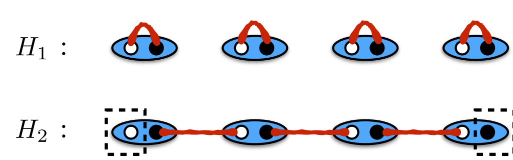

FIG. 1. Schematic picture of stroboscopic Floquet drive. The phase with Floquet Majorana edge states is obtained by alternating time evolution with a trivial Hamiltonian H_1 for time T_1 followed by time evolution under a topological Hamiltonian H_2 for time T_2. Complex fermions c_j (blue ovals) are decomposed into two Majorana fermions a_j and b_j (white and black circles, respectively). Nonzero couplings are represented by wavy red segments.

to assess the stability of the MPMs to generic small local perturbations of the Hamiltonian $H(t) \to H(t) + V(t)$, one can focus on allowed local interactions involving the topological edge modes and ignoring bulk degrees of freedom (DOF).

As written, Eq. (4) describes a clean noninteracting system. In the remainder of the paper, we will instead be interested in MBL systems obtained by strongly disordering λ_2 and including generic weak interactions. In this MBL context, the dynamical spread of the influence of a local perturbation is limited so that the change in the Floquet Hamiltonian, ΔH_F, will also be quasilocal. For small V, i.e., with operator norm $\|V(t)\| \ll (1/T)$, the explicit form of ΔH_F for a given $V(t)$ may be computed through standard time-dependent perturbation theory. Instead of relying on such approximations, we consider the stability to generic ΔH_F without concern for the precise form of the $V(t)$ that gives rise to this perturbation. For $\tilde{T}_2\lambda_2 \neq 2\pi$, the bulk degrees of freedom have quasienergy different from π and are hence separated by an energy gap from mixing with the MPMs at the ends of the wire. Consequently, just as for topological zero modes in static systems, sufficiently small perturbations that mix the MPMs with bulk degrees of freedom simply virtually dress the MPMs with an amplitude decaying exponentially with characteristic distance $\xi \lesssim \big(\log(\tilde{T}_2\lambda_2/|V|T)\big)^{-1}$ away from the edge of the wire. Hence, because of the locality of the perturbation ΔH_F, the change in the coefficient of the nonlocal term $(i\pi/2)a_0 b_L$ will be exponentially small in $e^{-L/\xi}$, such that the quasienergy of these modes is topologically protected at π for asymptotically long wires $(L \to \infty)$.

The above-described model with MPMs serves as a basic building block for constructing general fermionic Floquet SPTs. To this end, we may consider N_0 chains of fermions $c_{n,j} = \frac{1}{2}(a_{n,j} + ib_{n,j})$, with flavor index $n = 1, \ldots, N_0$ and position index $j = 1, \ldots, L$, driven by Eq. (4) with $\lambda_1 = 0$, and N_π chains of fermions $\psi_{m,j} = \frac{1}{2}(\alpha_{m,j} + i\beta_{m,j})$ driven by Eq. (4) with $\lambda_1 = 2\pi/T_1$. These respectively result in N_0 MZMs, $(a_{n,1}, b_{n,L})$ and N_π MPMs, $(\alpha_{m,1}, \beta_{m,1})$ that are strictly localized to the ends of the chain.

B. No symmetry

In the absence of symmetry, there is no topological protection for an even number of MZMs or MPMs. To see this, we may restrict our attention to possible perturbations within the Hilbert space spanned by the topological modes at one end of the chain since coupling the Majorana end states to complex bulk degrees of freedom will simply renormalize the spatial extent of their wave function without perturbing their quasienergy. For concreteness, consider the left end of the chain. The most general noninteracting coupling terms involving the topological modes $\{a_{n,1}, \alpha_{m,1}\}$ that can be generated by a T-periodic perturbation to $H(t)$ are $\Delta H_F = (i/4)\sum_{n\neq n'} a_{n,1} M^{(0)}_{n,n'} a_{n',1} + (i/4)\sum_{m\neq m'} \alpha_{m,1} M^{(\pi)}_{m,m'} \alpha_{m',1}$, where $M^{(0,\pi)}$ are antisymmetric matrices.

1. Dynamical decoupling of 0 and π modes

We show in Appendix A that bilinear couplings between MZMs and MPMs are ineffective and can be ignored. Namely, such couplings may be eliminated by defining new MZM and MPM operators from linear combinations of $a_{n,0}$, $\alpha_{n,0}$.

A more general argument establishing that MZMs and MPMs cannot split each other via noninteracting couplings can be obtained by considering the effective particle-hole "symmetry" of BdG Hamiltonians, which dictates that single-particle levels with quasienergy $(\varepsilon \bmod 2\pi)$ must be related by particle-hole conjugated levels with quasienergy $(-\varepsilon \bmod 2\pi)$. For self-conjugate (real) Majorana modes, this requires that $(\varepsilon \bmod 2\pi) = (-\varepsilon \bmod 2\pi)$, which has only two discrete solutions, $\varepsilon = 0, \pi$–; the latter solution is only possible in a periodically driven system where energy is only conserved modulo 2π, highlighting the special features of the Floquet-driven system. Naively, turning on a weak noninteracting coupling $i\delta\gamma_0\gamma_\pi$ of strength $\delta \ll 1$ between a MZM γ_0 with quasienergy $\varepsilon_1 = 0$ and a MPM γ_π with quasienergy $\varepsilon_2 = \pi$ would split their quasienergies into $\varepsilon_1 = -\mathcal{O}(\delta)$ and $\varepsilon_2 = \pi + \mathcal{O}(\delta)$. However, as illustrated in Fig. 2, one can easily see that this outcome is not compatible with particle-hole symmetry [i.e., in this scenario, there would be no particle-hole conjugate modes at $\tilde{\varepsilon}_1 = +\mathcal{O}(\delta)$ and $\tilde{\varepsilon}_2 = \pi - \mathcal{O}(\delta)$]. Hence, the only possible outcome of turning on the

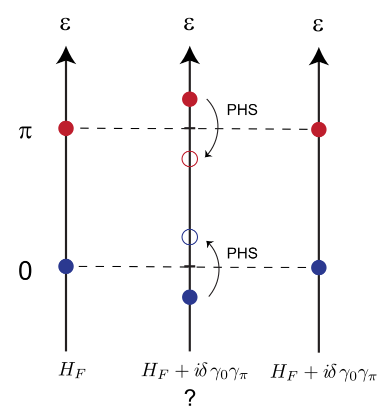

FIG. 2. Schematic picture of quasienergy spectrum of noninteracting fermionic Floquet SPTs. The particle-hole symmetry of BdG equations indicates that a bilinear coupling between a MZM and a MPM is ineffective and cannot move the MZM or MPM away from 0 or π quasienergy, respectively. The leftmost line shows the unperturbed quasienergies. The middle line illustrates that any possible splitting due to the perturbation does not respect particle-hole symmetry (PHS), indicating that the resulting quasienergies (rightmost line) must be identical to the initial unperturbed quasienergies 0 and π (leftmost line)

strength-δ coupling is that the new eigenmodes continue to have quasienergy $\varepsilon_1 = 0$ and $\varepsilon_2 = \pi$, respectively. In a loose sense, such couplings can be thought of as "ineffective" since they do not conserve quasienergy modulo 2π (as defined in terms of the unperturbed Hamiltonian). We caution that the quasienergy modes are an emergent property of the Floquet Hamiltonian and that the way to make the previous statement precise is through the concept of projective symmetry action (as discussed below); however, this "conservation of quasienergy" picture provides a useful intuitive rule of thumb and leads to correct results for all known 1D FSPT cases.

2. *Classification of Floquet phases in the absence of symmetry*

For N_0 (N_π) odd, $H_F + \Delta H_F$ inevitably exhibits a single unpaired Majorana mode with 0 (π) quasienergy. However, for N_0 (N_π) even, ΔH_F can pair the Majorana end states into complex fermions and split them away from 0 (π) quasienergy. Hence, we see that there are four distinct Floquet topological states in the absence of symmetry, characterized by a pair of $\mathbb{Z}_2$ invariants indicating the parity of N_0 and N_π, corresponding to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification of phases.

For any of the topologically nontrivial Floquet phases, there are either one or two edge modes, and hence turning on four-fermion or higher interaction terms does not generate any new possible couplings among the edge states. Hence, we expect the noninteracting classification in the absence of symmetry to coincide with the interacting one. On the other hand, certain symmetries can protect larger numbers of edge modes, in which case interactions offer additional ways to gap out the noninteracting topological edge modes and alter the SPT classification [32].

C. Interacting fermionic Floquet SPTs

The presence of a global symmetry group G constrains the possible form of perturbations (i.e., restricts the entries of $M^{(0,\pi)}$) and can protect multiple MZMs and MPMs. Since noninteracting terms cannot mix the MZMs and MPMs, the analysis of symmetry-allowed mass terms $M^{(0)}$ and $M^{(\pi)}$ each independently follows exactly from the analysis for static noninteracting SPT phases. which are well understood; for a given group G, the group of distinct fermionic SPT phases arising from noninteracting static Hamiltonians, $C_{\mathrm{st}}^{(\mathrm{NI})}[G]$, is known [5–9]. In all cases, nontrivial static 1D SPT phases are characterized as self-conjugate zero-energy edge states. In the noninteracting Floquet context, the most general new possible phases arise from the possibility of also realizing self-conjugate modes at quasienergy π. Hence, from the above considerations, we see that the noninteracting classification of periodically driven Floquet SPT phases then simply yields two independent copies of the noninteracting band invariants—one each for 0 and π quasienergy modes, corresponding to a noninteracting Floquet classification [5–9]:

$$C_F^{(\mathrm{NI})}[G] = C_{\mathrm{st}}^{(\mathrm{NI})}[G] \times C_{\mathrm{st}}^{(\mathrm{NI})}[G]. \tag{6}$$

This noninteracting classification has since been shown to extend to higher dimensions, for the specific case of 2D time-reversal–invariant topological insulators [14], and has since been systematically generalized to other symmetry classes using K theory [49]. The focus of this section of our paper is instead on investigating how interactions modify results in 1D systems and, subsequently, on intrinsically interacting FSPT bosonic phases with no free-particle band-structure description.

For static SPTs, interactions can modify the free-fermion classification. Specifically, in many cases where $C_{\mathrm{st}}^{(\mathrm{NI})} = \mathbb{Z}$, the interacting classification is reduced to $C_{\mathrm{st}}^{(\mathrm{NI})} \to C_{\mathrm{st}} = Z_N$, where N is some even integer. A simple guess based on the above considerations would be that the corresponding Floquet classification would again follow simply from the static classification as $C_F \overset{?}{=} C_{\mathrm{st}} \times C_{\mathrm{st}}$. However, we will see that the situation is more subtle and that interactions can effectively enable Floquet analogs of Umklapp-type terms that conserve quasienergy only modulo 2π and can mix the MPM and MZM sectors in nontrivial ways.

To understand whether interactions reduce the noninteracting classification, we again consider perturbations ΔH_F to the Floquet Hamiltonian that couple the topological zero- and π-quasienergy modes but allow for interaction terms involving two- or higher-body interaction terms with products of four or more edge modes.

For concreteness, we start with the specific illustrative example of spinless, TR symmetric superconducting chains, corresponding to Altland-Zirnbauer (AZ) class BDI, and then give general results for all of the symmetry classes corresponding to the tenfold way.

D. Spinless TR-invariant superconductors (class BDI)

Before diving into the analysis of the interacting Floquet phases, a comment on the notion of TR symmetry in time-dependent quantum systems is in order. Whereas for static Hamiltonians the dynamics may be invariant in the reversal of time about any reference time t_0, periodic time-dependent Hamiltonians can, at most, exhibit a discrete set of time inversion centers t_0 such that $H(t_0 + t) = \mathcal{T} H(t_0 - t)\mathcal{T}^{-1}$. Since the Hamiltonians considered are time periodic, we are free to shift the origin of our time interval by any amount to fix the inversion center t_0 to a convenient value. Floquet evolution operators with different time "origins" are related by a finite-depth unitary transformation and hence have the same universal long-time properties. Hence, we are free to shift the time-inversion center to the center of the period such that

$t_0 = T/2$. In this case, acting on the Floquet Hamiltonian, discrete time reversal acts on the Floquet Hamiltonian $H_F = i \log F$ as an ordinary antiunitary operator $\mathcal{T}$, just as for a static Hamiltonian. In particular, the Floquet evolution for the full period transforms under time reversal as $\mathcal{T}F\mathcal{T}^{-1} = \mathcal{T}e^{-iH_F}\mathcal{T}^{-1} = e^{+i\mathcal{T}H_F\mathcal{T}^{-1}}$. For time-reversal–invariant Floquet Hamiltonians $\mathcal{T}H_F\mathcal{T}^{-1} = H_F$, this implies that

$$\mathcal{T}F\mathcal{T}^{-1} \overset{(\mathrm{TRS})}{=} F^{-1}. \tag{7}$$

For general time-reversal symmetry about an inversion center at time $t_0 \neq T/2$ within the period, the condition for time-reversal symmetry reads $\mathcal{T}F\mathcal{T}^{-1} = U(t_0)^\dagger F^{-1} U(t_0)$, where $U(t) = T\{e^{-\int_0^t ds H(s)}\}$ is the time evolution operator up to time t. The additional unitary transformation by $U(t_0)$ does not alter the group structure of time reversal and Floquet time translation shown in Eq. (7).

For example, the Hamiltonian, Eq. (4), exhibits a time-reversal symmetry that is defined by $\mathcal{T}c_n\mathcal{T}^{-1} = c_n$, $\mathcal{T}\psi_m\mathcal{T}^{-1} = \psi_m$, i.e.,

$$\mathcal{T}\begin{pmatrix} a_n \\ b_n \\ \alpha_m \\ \beta_m \end{pmatrix}\mathcal{T}^{-1} = \begin{pmatrix} a_n \\ -b_n \\ \alpha_m \\ -\beta_m \end{pmatrix}. \tag{8}$$

The relative minus signs in Eq. (8) indicate that single-particle couplings between MZM or MPM edge states of the form $ia_n a_{n'}$ or $i\alpha_m\alpha_{m'}$ are odd under TR and hence forbidden by symmetry. Moreover, as previously remarked, time-periodic single-particle couplings cannot mix zero- and π-quasienergy modes. Hence, the noninteracting phases are characterized by two integer invariants (N_0, N_π), respectively, indicating the number of MZMs and MPMs localized to a boundary of the wire, corresponding to a $\mathbb{Z}\times\mathbb{Z}$ classification of Floquet SPT phases. As is conventional for equilibrium SPTs in this symmetry class, we define $N_{0/\pi}$ to be positive (negative) for $+$ $(-)$ chirality Majorana modes, i.e., those residing on the a/α (b/β) sublattices.

We note that fermionic Hamiltonians may only include terms with even numbers of fermion operators and are therefore inevitably invariant under the fermion parity operator $P_f = (-1)^{N_F}$, where N_F is the total number of fermions in the system. For this reason, fermion parity is sometimes taken to be part of the symmetry group of a fermionic system. Since $P_f^2 = 1$, this corresponds to an extra factor of $\mathbb{Z}_2$ in G, typically denoted $\mathbb{Z}_2^F$. However, unlike a conventional $\mathbb{Z}_2$ symmetry, $\mathbb{Z}_2^F$ cannot be broken even spontaneously by interactions.

While no single-particle perturbations can disturb these topological edge modes, it is known from the study of static systems that four-body interactions can fully remove the degeneracy associated with eight MZMs (or similarly, with any integer multiple of eight MZMs) [32,50]. We briefly recall the key ideas behind this result. First, note that any arbitrary interaction involving an odd number of MZMs will leave behind at least one exact MZM, such that any phase with odd N_0 is nontrivial.

For $N_0 = 2$, the most general edge-state perturbation is the noninteracting term $P_f^{(\mathrm{loc})} = ia_1a_2$, which is odd under time-reversal symmetry and hence cannot be generated by any symmetry-preserving perturbation. The operator $P_f^{(\mathrm{loc})} = 2f_{12}^\dagger f_{12} - 1$ squares to 1 and consequently has eigenvalues ± 1, corresponding to the fermion parity of the complex fermion zero mode $f_{12} = \frac{1}{2}(a_1 + ia_2)$. Therefore, for $N_0 = 2$, $P_f^{(\mathrm{loc})}$ represents the local action of the fermion parity operator acting within the low-energy subspace spanned by MZM edge states. While $\mathcal{T}$ commutes with the total fermion parity P_f, it anticommutes (i.e., commutes only up to an overall phase of -1) with the local action of fermion parity on the edge-state zero modes, $\mathcal{T}P_f^{(\mathrm{loc})}\mathcal{T}^{-1} = -P_f^{(\mathrm{loc})}$; thus, the $N_0 = 2$ MZM edge forms a projective representation of $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$. It is generally true that static 1D SPT phases are systematically classified by a projective representation of G (for bosonic systems) or $G\times\mathbb{Z}_2^F$ (for fermionic systems). Namely, any nontrivial projective action of symmetry action on the edge modes requires an edge-mode Hilbert space of dimension larger than one (all 1D representations of G are Abelian and hence nonprojective)—i.e., it requires an edge-state degeneracy that cannot be lifted without sacrificing symmetry. Moreover, since local bulk degrees of freedom necessarily transform under an ordinary representation of the symmetry group, there is no way for them to form a nondegenerate symmetry singlet by interacting with the edge modes. The projective representations form an Abelian group, with each group element corresponding to a distinct static topological phase of matter.

For $N_0 = 4$ MZMs $a_{1,\ldots,4}$, the interaction term $V = \lambda(ia_1a_2)(ia_3a_4)$ is allowed by symmetry. This divides the fourfold degenerate space spanned by the four MZMs into two doublets: $\{|00\rangle, |11\rangle \equiv f_{34}^\dagger f_{12}^\dagger|00\rangle\}$ and $\{|10\rangle \equiv f_{12}^\dagger|00\rangle, |01\rangle \equiv f_{34}^\dagger|00\rangle\}$ labeled by the occupation numbers $f_{12} = \frac{1}{2}(a_1 + ia_2)$ and $f_{34} = \frac{1}{2}(a_3 + ia_4)$. However, the smaller twofold degeneracy of these doublets is protected by symmetry and cannot be removed by any symmetry-preserving perturbation. To see this, consider the subspace spanned by one such doublet, say, $\{|00\rangle, |11\rangle\}$, and define Pauli-like spin operators $\sigma^z = |00\rangle\langle 00| - |11\rangle\langle 11|$, $\sigma^x = |00\rangle\langle 11| + |11\rangle\langle 00|$, and $\sigma^y = -i(|00\rangle\langle 11| - |11\rangle\langle 00|)$. Since both $|00\rangle$ and $|11\rangle$ have even fermion parity, P_f acts like the identity operator in the subspace of this even doublet. However, the action of time reversal on the doublet is unconventional. Namely, note that

$\mathcal{T} f_{ij} \mathcal{T} = f_{ij}^\dagger$ and hence time reversal flips the occupation number of the two zero modes. Hence, we may choose the relative phase of $|00\rangle$ and $|11\rangle$ such that $\mathcal{T}|00\rangle = |11\rangle$. On the other hand, $\mathcal{T}^2|00\rangle = \mathcal{T}|11\rangle = \mathcal{T} f_{12}^\dagger \mathcal{T}^{-1} \mathcal{T} f_{34}^\dagger \mathcal{T}^{-1} \mathcal{T}|00\rangle = f_{12} f_{34} f_{12}^\dagger f_{34}^\dagger |00\rangle = -|00\rangle$. This is another example of the projective action of symmetry since $\mathcal{T}^2 = +1$ on any bulk degree of freedom, whereas $\mathcal{T}^2 = -1$ on the edge-mode doublets—indicating that the edge modes form a Kramers doublet whose twofold degeneracy is protected by $\mathcal{T}$. We note that the fact that fermion parity acts trivially on the local edge states indicates that the presence of local fermion degrees of freedom is unimportant for realizing this particular SPT order. Indeed, the same projective realization of symmetry can be realized by the edge modes of a purely bosonic SPT, indicating that the fermionic system with $N_0 = 4$ reduces to a bosonic one in the presence of interactions.

For $N_0 = 6$, the edge modes transform as a combination of the projective properties of $N_0 = 2$ and $N_0 = 4$; namely, the local action of symmetry on the edge states satisfies $\mathcal{T} P_f \mathcal{T}^{-1} = -P_f$ and $\mathcal{T}^2 = -1$, indicating a protected fourfold symmetry.

For $(N_0, N_\pi) = (8, 0)$, we can readily see that doubling the (-1) phase factors for the above-described $(N_0, N_\pi) = (4, 0)$ results in an ordinary (nonprojective) action of symmetry on the MZM edge states, indicating that there is no special topological protection of these modes. Indeed, we can concretely confirm this suspicion by combining the eight MZMs into two bosonic doublets, one consisting of the even fermion parity configurations of $a_{1,\ldots,4}$ and another from those of $a_{5,\ldots,8}$. In analogy to the $N_0 = 4$ case described above, we can introduce the Pauli operators σ_1 and σ_2 acting on each of these doublets, which both transform like Kramers doublets under $\mathcal{T}$. However, e.g., the Heisenberg interaction $V = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ clearly preserves $\mathcal{T}$, despite the Kramers nature of $\sigma_{1,2}$, and it removes the degeneracy of the zero modes by selecting a pseudospin singlet combination of $\sigma_{1,2}$.

We now perform a similar analysis of the perturbative stability for edge modes of the periodically driven system. To conserve quasienergy modulo 2π, interacting edge perturbations must involve Floquet-Umklapp-type terms that couple even numbers of MZMs and MPMs. Hence, phases in which $N_0 - N_\pi$ is odd remain nontrivial, in particular, even when the total number of topological edge modes $N_0 + N_\pi$ is an integer multiple of eight. Again, by repeating the above considerations from static systems, one can easily verify that four MPMs can be symmetrically coupled to produce a degenerate bosonic doublet spanned by the spin-1/2 operators $\boldsymbol{\sigma}_{\boldsymbol{\pi}}$, which transforms as $\mathcal{T}^2 = -1$ under time reversal and is static under the Floquet evolution, just as for the nondriven phase with four MZMs.

Hence, for a phase with $(N_0, N_\pi) = (4, -4) \simeq (4, 4)$, we may add the symmetry-preserving interaction $-V \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_\pi$ to completely lift the edge degeneracy. This shows that the noninteracting $(4, -4)$ phase reduces to a trivial phase in the presence of interactions, due to the nontrivial Floquet-Umklapp interaction between MZMs and MPMs, i.e., that having four MZMs is topologically equivalent to having four MPMs.

On the other hand, the $(2, -2)$ state remains topologically nontrivial even in the presence of interactions, due to a dynamical winding property. To see this, let us start with the noninteracting $(2, -2)$ state and, as before, add an edge perturbation $\sim a_1 a_2 \beta_1 \beta_2$ to break the edge-state sector into bosonic degrees of freedom. For example, in the even-fermion parity sector with $|00\rangle$, $|11\rangle = f^\dagger \psi^\dagger |00\rangle$, where $f = \frac{1}{2}(a_1 + i a_2)$ is a complex zero mode, and $\psi = \frac{1}{2}(\beta_1 + i\beta_2)$ is a complex π mode, we may define the bosonic pseudospin: $\sigma^z = |00\rangle\langle 00| - |11\rangle\langle 11|$. Since f and ψ are conjugated by $\mathcal{T}$, $\mathcal{T}$ must flip the state of σ^z. However, since these complex fermions acquire a relative (-1) phase under $\mathcal{T}$, $\mathcal{T} f \mathcal{T}^{-1} = f^\dagger$, $\mathcal{T}\psi\mathcal{T}^{-1} = -\psi$, σ behaves like a non-Kramers singlet ($\mathcal{T}^2 = 1$) under $\mathcal{T}$. Hence, we may represent the local action of $\mathcal{T}$ on the $(2, -2)$ edge as $\mathcal{T} = \sigma^x K$. On the other hand, $|00\rangle$ and $|11\rangle$ have quasienergies that differ by π and hence acquire a relative (-1) phase under the Floquet evolution, such that the local action of Floquet time translation on the edge states is represented as $F = \sigma^z$. Combining these two properties, we see that $\mathcal{T}$ and F act projectively on the topological edge states of the $(2, -2)$ phase:

$$\mathcal{T} F \mathcal{T}^{-1} = (-1) F^{-1}, \tag{9}$$

in contrast to the nonprojective action ($\mathcal{T} F \mathcal{T}^{-1} = F^{-1}$) for bulk degrees of freedom. This nontrivial projective edge action holds, even though the static symmetry group generated by $\mathcal{T}$, P_f acts trivially on the edge. We can picture this phase as having a free pseudospin-1/2 edge degree of freedom σ that rotates by π around the z axis over the course of each period. While we could add a symmetry-preserving field $h\sigma^x$ to try to pin this edge spin, the effect of this field would average to zero over a sequence of two driving periods because of the nontrivial Floquet dynamics of the edge spin.

The set of interacting Floquet SPT phases that arises from these considerations can be generated by combinations of two "root" phases: $(N_0, N_\pi) = (1, 0)$ and $(1, -1)$. N-fold combinations of the former phase for $N = 0, 1, \ldots, 7$ realize all of the static, nondriven topological phases that realize projective edge representations of time reversal (and fermion parity). The latter sequence of phases generated by combinations of $(1, -1)$ transforms ordinarily under the static symmetry group but has nontrivial interplay of symmetry and topological Floquet dynamics that produces a projective edge action of symmetry and time translation. The $(1, -1)$ phase has $\{F, P_f\} = 0$ at the edge, the $(2, -2)$ phase has $\{F, \mathcal{T}\} = 0$, the $(3, -3)$ phase has

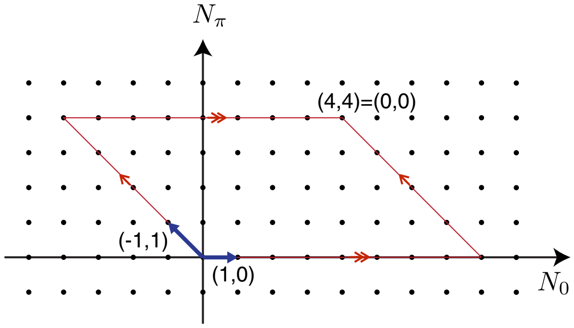

FIG. 3. Group structure of interacting Floquet SPT phases of spinless TR-invariant superconductors (class BDI). In the absence of interactions, each point corresponds to a topological phase with N_0 MZMs and N_π MPMs at the edge. With interactions, only points on a discrete torus (bounded red region, with arrows indicating periodic boundary conditions) correspond to distinct topological phases of the $\mathbb{Z}_8 \times \mathbb{Z}_4$ classification. The blue vectors (1,0) and $(-1, 1)$ are generators of the subgroups $\mathbb{Z}_8$ and $\mathbb{Z}_4$, respectively.

both $\{F, P_f\} = 0$ and $\mathcal{T}F\mathcal{T}^{-1}F = -1$, and the $(4, -4)$ phase is trivial and should be identified with (0,0).

Thus, we see that the noninteracting Floquet classification has been reduced from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_8 \times \mathbb{Z}_4$. The associated group structure of the SPT phases is shown in Fig. 3. These phases exhaust all possible projective representations of $\mathcal{T}$, P_f, and F (see Table 1), which leads us to hypothesize that the full classification of interacting Floquet SPT phases with symmetry group G is given by the group of projective representations of $G \times \mathbb{Z}$ graded by Z_2^F fermion parity "symmetry," where the extra factor of $\mathbb{Z}$ corresponds to time-translation symmetry.

E. Other symmetry groups

So far, we have analyzed the case of no symmetry (class D) and spinless time-reversal symmetry (class AIII). We can repeat the above perturbative stability analysis of the noninteracting Floquet classifications for the other nontrivial 1D SPT symmetry classes in the tenfold way. In the first pass, we will ignore the requirement of many-body localizability, which is necessary to avoid runaway heating by the drive frequency, and just study the topological outcomes. It turns out that only the cases with no symmetry (class D) and spinless time reversal (BDI) permit many-body localized Floquet SPT phases that are stable against heating. The other nontrivial 1D symmetry classes all have group structures with irreducible representations of dimension greater than one, which, as we will show below, protect local degeneracies that spoil the possibility of having a symmetry-preserving many-body localized phase. However, it is instructive to consider other examples to build our intuition. Moreover, if interactions are weak, it is conceivable that the Floquet SPT phases described in these other classes may survive without catastrophic heating for adequately long times to be of interest for experiments.

TABLE I. *Time-reversal symmetric 1D fermion FSPTs*.—The classification of 1D interacting fermionic Floquet SPTs with time reversal (class BDI) has a $\mathbb{Z}_8 \times \mathbb{Z}_4$ group structure. Topological phases are labeled by the number of Majorana zero- and π-quasienergy modes, (N_0, N_π), present in the absence of interactions. The projective edge algebra that defines these phases is shown in the right column. The phases labeled (B) are topologically equivalent to bosonic FSPT phases.

Phase (N_0, N_π)	Defining edge characteristic
(1, 0)	Unpaired Majorana
(2, 0)	$\mathcal{T}P_f = -P_f\mathcal{T}$
(4, 0) (B)	$\mathcal{T}^2 = -1$
(1,-1)	$FP_f = -P_fF$
(2,-2) (B)	$\mathcal{T}F\mathcal{T}^{-1} = -F$

The results are summarized in Table II. In each case, we find precise agreement between the stability analysis and projective representations of $G \times \mathbb{Z}$ graded by $\mathbb{Z}_2^F$ fermion parity symmetry, further supporting the hypothesis that this represents a complete classification.

For Kramers-doublet fermions with time-reversal symmetry (class DIII), the noninteracting classification is unchanged by interactions. The nontrivial phases are characterized by two $\mathbb{Z}_2$ invariants that represent the presence or absence of a Kramers pair of MZMs or MPMs, respectively.

Systems with fermions with a conserved $U(1)$ charge that is time-reversal odd, $G = U(1) \times \mathbb{Z}_2^T$ (class AIII), are derived directly from the study with only time reversal (BDI), but only the subgroup of phases with an even number of N_0, N_π are compatible with the $U(1)$ charge conservation. Equilibrium examples of this class include spinless fermions with random hopping amplitudes, which

TABLE II. *Classification of 1D fermionic Floquet SPTs*.—Group structure of nontrivial topological classes for 1D fermionic systems with discrete on-site symmetries, listed by physical symmetry group and equivalent AZ class. $C \to C'$ indicates that the noninteracting classification C is changed by interactions to C'. AZ classes marked with * indicate symmetry groups that are incompatible with many-body localization and are therefore unstable to runaway heating in the presence of generic bulk interactions.

Symmetry	AZ Class	Static (free → int)	Floquet (free → int)
None	D	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^T$, $(\mathcal{T}^2 = P_f)$	DIII*	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^T$ $(\mathcal{T}^2 = 1)$	BDI	$\mathbb{Z} \to \mathbb{Z}_8$	$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_8 \times \mathbb{Z}_4$
$U(1) \times \mathbb{Z}_2^T$	AIII*	$\mathbb{Z} \to \mathbb{Z}_4$	$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_4 \times \mathbb{Z}_2$
$U(1) \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$, $(\mathcal{T}^2 = P_f)$	CII*	$\mathbb{Z} \to \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2$

satisfy an antiunitary particle-hole symmetry that may be regarded as time reversal.

Finally, in the absence of interactions, systems with a conserved $U(1)$ charge and both time reversal and charge-conjugation symmetries (class CII, $G = U(1) \rtimes (Z_2^T \times \mathbb{Z}_2^C)$), one obtains only the subset of BDI phases with multiples of four MZMs or MPMs. Since four MPMs are equivalent to four MZMs, in the presence of interactions, the periodic driving does not enable any new nonequilibrium phases.

IV. BOSONIC FLOQUET SPTS

In the above fermionic classification of Floquet SPTs with symmetry groups in the tenfold way with some combination of charge-conservation, time-reversal, and particle-hole symmetries, we found that only topological superconducting classes [i.e., those without a conserved $U(1)$ charge] permit stable, localizable Floquet SPTs. Unfortunately, in quantum-optics-based setups, such as cold atoms, superfluid phases are unsuitable for many-body localization because of the presence of a nonlocalizable Goldstone mode.

To uncover potentially experimentally relevant Floquet SPT phases in fermionic systems, we need to look into other symmetry groups. Alternatively, we may examine the prospect of finding Floquet SPTs in bosonic (e.g., spin) systems. Here, we use many examples of localizable Floquet SPTs that are stable against heating.

A. Time-reversal symmetry $(\mathcal{T}^2 = 1)$

To begin, let us consider bosonic models, such as integer-spin chains, with time-reversal symmetry that squares to unity. The ground-state classification of such systems includes a single nontrivial phase first explained by Haldane [39,40], which exhibits free spin-$1/2$ edge states that transform as $\mathcal{T}^2_{\text{edge}} = -1$ under time-reversal symmetry. This Haldane phase is not localizable with full spin-rotation symmetry; however, one may introduce time-reversal symmetric exchange anisotropies into the originally rotation-invariant Haldane model in order to resolve all local degeneracies and obtain a localizable phase. The static phases are hence characterized by a single $\mathbb{Z}_2$ invariant corresponding to $\mathcal{T}^2_{\text{edge}} = \pm 1$.

Given our experience with fermionic systems, it is natural to expect that the classification of driven spin chains includes an extra topological phase that exhibits edge spins that transform projectively under the combination of time reversal and Floquet time evolution: $\mathcal{T} F \mathcal{T}^{-1} = -F^{-1}$ (in addition to the static SPT phases). This phase can be viewed as a dynamical analog of the Haldane phase, in which the spin-$1/2$ edge states flip under each cycle of the Floquet drive and hence require two periods to return to their original state. This edge-state flipping can be viewed as a spin-echo procedure that dynamically decouples the edge spins from bulk excitations with perfect (topologically protected) fidelity.

This suggests that the Floquet phases permit an additional dynamical $\mathbb{Z}_2$ topological invariant labeling whether time reversal commutes or anticommutes with the Floquet operator when acting on edge spins, corresponding to a total classification of $C_F[\mathbb{Z}_2^T] = \mathbb{Z}_2 \times \mathbb{Z}_2$.

1. Construction I: From fermions to spins

In fact, we have already encountered phases with these precise realizations of all the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Floquet SPT invariants in the fermionic systems with spinless time-reversal symmetry (class BDI) described above: Namely, the interacting version of the phase with $(N_0, N_\pi) = (2, -2)$ realizes the nontrivial Floquet invariant, and the (4,0) phase realizes the nontrivial static invariant. However, the fermionic nature of this problem is unimportant since the fermion parity symmetry plays no role in the projective action of symmetry on the edge states. Hence, by adding strong interactions among the fermions, we may reduce this fermionic system to a strongly localized Mott insulator with trivially localized fermionic excitations both in the bulk and at the edges, without changing the underlying SPT order. An explicit example of an interaction term that accomplishes this task, for the $(2, -2)$ phase, in the notation of Sec. III A is $\sum_j U_j a_{1,j} a_{2,j} \alpha_{1,j} \alpha_{2,j} + b_{1,j} b_{2,j} \beta_{1,j} \beta_{2,j}$.

2. Construction II: Spin chain

We can also explicitly construct the phase with nontrivial static and Floquet SPT invariants directly in a purely bosonic model with spin-1 degrees of freedom. As with the AKLT construction for the ground-state SPT phase [40], it is useful to construct the Floquet drive in a two-stage procedure where we first view each spin-1 degree of freedom, $\mathbf{S}_i$, as being formed from two notional spin-$1/2$ degrees of freedom, $\sigma_{A,i}$ and $\sigma_{B,i}$, which each transform projectively under time reversal: $\mathcal{T} = \prod_j \sigma^y_{A,j} \sigma^y_{B,j} K$. The construction of the Floquet SPT phase is simple in the spin-$1/2$ description (e.g., for the Haldane phase, it consists of just nearest-neighbor projectors onto singlets), and it then yields a local Hamiltonian for the original nonprojective degrees of freedom upon applying a local projection onto the original spin-1 degrees of freedom.

In the notional spin-$1/2$ language, the desired Floquet SPT phase can again be achieved by a two-step stroboscopic Floquet evolution $F = F_2 F_1$, where

$$F_1 = e^{-iH_{\text{AKLT}}},$$

$$F_2 = e^{i\pi/2 \sum_j \sigma^x_{A,j} \sigma^x_{B,j}} = i^L \prod_{j=1}^{L} \sigma^x_{A,j} \sigma^x_{B,j}, \tag{10}$$

where $H_{\text{AKLT}} = \lambda \sum_{j=1}^{L-1} \boldsymbol{\sigma}_{B,j} \cdot \boldsymbol{\sigma}_{A,j+1}$ is the AKLT Hamiltonian whose eigenstates exhibit the SPT order of the Haldane phase.

In precisely the same manner as for the fermionic models described above, we may rewrite (dropping an irrelevant overall phase) $F_2 = \sigma^x_{A,1}\sigma^x_{B,L} \times e^{i\pi/2\sum_{j=1}^{L-1}\sigma^x_{B,j}\sigma^x_{A,j+1}}$, from which one may readily verify that the full Floquet evolution operator reads

$$F = F_2F_1 = \sigma^x_{A,1}\sigma^x_{B,L}e^{-i\tilde{H}_{\mathrm{AKLT}}}, \tag{11}$$

where $\tilde{H}_{\mathrm{AKLT}}=\sum_{j=1}^{L-1}\lambda(\sigma^x_{B,j}\sigma^x_{A,j+1}+\sigma^y_{B,j}\sigma^y_{A,j+1})+[\lambda-(\pi/2)]\sigma^x_{B,j}\sigma^x_{A,j+1}$ is an anisotropic analog of the AKLT Hamiltonian.

As for the Haldane phase, the edge spins $\sigma_{A,1}$ and $\sigma_{B,L}$ are left out of $\tilde{H}_{\mathrm{AKLT}}$, and hence their only dynamics is set by the preceding $\sigma^x_1\tau^x_L$ factor. In the σ^z basis, the edge spins flip from up to down over the course of each Floquet period, producing the desired projective edge realization of $F_{\mathrm{edge}} = \sigma^x$, $\mathcal{T}_{\mathrm{edge}} = i\sigma^yK$, such that $(\mathcal{T}F\mathcal{T}^{-1}F)_{\mathrm{edge}} = -1$.

From this model defined in terms of notional spin-1/2 degrees of freedom, we may obtain a corresponding Floquet evolution in terms of the original spin-1 degrees of freedom, $\mathbf{S}_j$, by projecting onto the triplet sector of each site. The terms of the AKLT Hamiltonian that appear in F_1 project to $H_{\mathrm{AKLT}} \to \sum_j\mathbf{S}_j\cdot\mathbf{S}_{j+1} + \frac{1}{3}(\mathbf{S}_j\cdot\mathbf{S}_{j+1})^2$. The projection of the terms, $\sigma^x_i\tau^x_i$ in F_2, onto the spin-1 Hilbert space, $\frac{1}{2}(\sigma^x_{A,i}+\sigma^x_{B,i}) \to S^x_i$, can be obtained by rewriting $\sigma^x_{A,i}\sigma^x_{B,i} = \frac{1}{2}[(\sigma^x_{A,i}+\sigma^x_{B,i})^2 - 2]$, such that the Floquet drive in terms of the spin-1 variables reads $F = F_2F_1$, where

$$F_1 = e^{-i\sum_j\mathbf{S}_j\cdot\mathbf{S}_{j+1}+\frac{1}{3}(\mathbf{S}_j\cdot\mathbf{S}_{j+1})^2},$$
$$F_2 = e^{-i\pi\sum_j(S^x_j)^2}. \tag{12}$$

To obtain a stable many-body localized phase, we may make the exchange couplings random and further introduce spin-exchange anisotropies, $\lambda_j\mathbf{S}_j\cdot\mathbf{S}_{j+1} \to \sum_{\alpha=x,y,z}\lambda_{j,\alpha}S^\alpha_jS^\alpha_{j+1}$, to remove any unwanted local degeneracies due to continuous spin-rotation symmetry.

We note further that the last step of projection onto a spin-1 degree of freedom is not strictly necessary to demonstrate a proof-of-principle construction of the SPT phase. Rather, we may instead view the model defined in terms of $\sigma_{A/B,i}$ as a complete lattice Floquet Hamiltonian for four-state quantum degrees of freedom. In the following sections, we hence drop the superfluous projection step.

B. $\mathbb{Z}_n$ symmetry

In the previous section, we considered Floquet analogs of time-reversal protected bosonic (spin) SPTs. We may also consider bosonic Floquet SPTs protected by unitary on-site symmetries. In this case, to achieve a localized phase that is stable against heating, we may only consider Abelian symmetry groups. If we further restrict to finite-Abelian groups, then the most general symmetry group may be represented by factors of $\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\ldots\mathbb{Z}_{n_p}$ for integers $n_{1,\ldots,p}\in\mathbb{Z}$. A prototype for this general case is to just consider a single unitary $G = \mathbb{Z}_n$ symmetry.

We explicitly construct models that realize all of the projective realization of $\mathbb{Z}_n\times\mathbb{Z}$, further supporting the hypothesized classification of general interacting Floquet SPTs. In this spirit, we first consider an AKLT-like model in which each site is an N^2-state quantum with states $|m_j\rangle$, with $m = 1\ldots N^2$, that can be viewed as a tensor product of two N-state $\mathbb{Z}_N$ "rotors," $|m_j\rangle = |m_{A,j}m_{B,j}\rangle$, with sublattice labels A, B, and defined to be eigenstates of the generator g of $\mathbb{Z}_N$:

$$\begin{aligned} g|m_{A,j}m_{B,j}\rangle &\equiv g_{A,j}g_{B,j}|m_{A,j}m_{B,j}\rangle \\ &= \varphi^{m_{A,j}+m_{B,j}}|m_{A,j}m_{B,j}\rangle, \end{aligned} \tag{13}$$

where $\varphi = e^{2\pi i/N}$ and $m_{A/B}\in\{0\ldots N-1\}$. We can also write the (unitary) cyclical raised and lowered operators:

$$\sigma^\pm = \sum_{m=0}^{N-1}|m\pm 1 \mod N\rangle\langle m|. \tag{14}$$

The $\mathbb{Z}_n$-symmetry generator is $g = \prod_{j=1}^L g_{A,j}g_{B,j}$, where $g_{A/B,j}|m_{A/B,j}\rangle = \varphi^{m_{A/B,j}}$ implements the symmetry action on a fraction of a site labeled by A or B.

Then, we may realize a nontrivial Floquet SPT phase by considering the stroboscopic Floquet operator $F = F_2F_1$, with

$$\begin{aligned} F_1 &= e^{-i(\sum_{j=1}^{L-1}g_{B,j}g_{A,j+1}+\mathrm{H.c.})}, \\ F_2 &= \prod_{j=1}^{L}\sigma^+_{A,j}\sigma^-_{B,j} = \sigma^+_{A,0}\left(\prod_{j=1}^{L-1}\sigma^-_{B,j}\sigma^+_{A,j+1}\right)\sigma^-_{B,L} \\ &\equiv \sigma^+_{A,0}W\sigma^-_{B,L}. \end{aligned} \tag{15}$$

Note that F_2, being the product of local unitary operators $\sigma^+_{A,j}$, $\sigma^-_{B,j}$, can be realized by time evolution with a local Hamiltonian. Furthermore, note that W commutes with F_1, gives nontrivial phases to all bulk degrees of freedom, and does not involve subsites A_1 or B_L. On the ends, g acts like $g_{A,1}$ and $g_{B,L}$, respectively, and F acts like $\sigma^+_{A,1}$ and $\sigma^-_{B,L}$, respectively. Hence, we see that on, say, the left end, $F^\dagger_Lg_LF_Lg^\dagger_L = e^{2\pi i/N}$, time-translation and the $\mathbb{Z}_N$ symmetry are represented projectively. Moreover, we see that the π modes of the fermionic models are generalized to quasienergy $2\pi/N$ modes for generic n, where the $\mathbb{Z}_N$ symmetry protects the quantization of quasienergy to multiples of $2\pi/N$. As for the AKLT chain, this projective action is preserved under the local projection of each two-spin "site" onto the degrees of freedom of a single non-projective $\mathbb{Z}_N$ spin.

Moreover, we can consider a sequence of related phases with $F_2 = \prod_{j=1}^{L}(\sigma^+_{A,j}\sigma^-_{B,j})^n$, for $n = 0, 1, \ldots, N-1$, which result in bosonic edge modes with quasienergy fixed at $e^{2\pi in/N}$, protected by a projective interplay of Floquet evolution and Z_N symmetry at the edge: $F_L^\dagger g_L F_L g_L^\dagger = e^{2\pi in/N}$. These phases exhaust all projective representations of the group $\mathbb{Z}_N \times \mathbb{Z}$, in agreement with the conjectured classification of 1D bosonic Floquet SPTs: $C_F[\mathbb{Z}_N] = \mathcal{H}^2(\mathbb{Z}_N \times \mathbb{Z}) = \mathbb{Z}_N$.

C. Generalizations

We can repeat the above construction for $\mathbb{Z}_N$ symmetry bosonic Floquet SPTs in more general terms for an arbitrary symmetry group. For a given projective representation $\mathcal{PR}$ of $\mathbb{Z} \times G$, we can construct a solvable lattice model that realizes the corresponding Floquet SPT phase. Namely, consider a model whose physical sites are considered to be composite sites of an A subsite DOF that transforms under $\mathcal{PR}$ and a B subsite DOF that transforms under the conjugate representation $\overline{\mathcal{PR}}$ (where the projective phases are complex conjugates of those in $\mathcal{PR}$). Since each site contains degrees of freedom transforming as $\mathcal{PR} \times \overline{\mathcal{PR}}$, the symmetry action is overall ordinary (nonprojective). However, we will arrange the Floquet evolution in a way that exposes free projective degrees of freedom on each end, *à la* the AKLT model.

We take the static symmetry of this model to be defined as the product of $U(g) = \prod_i U_{A,i}(g) \otimes U_{b,i}(g)$, where $U_{A,i}$ ($U_{B,i} = U^\dagger_{A,i}$) form a projective (conjugate projective) representation of $g \in G$ and act only on the A (B) sublattices. For the Floquet evolution operator, we again consider a two-stage stroboscopic evolution,

$$F = F_2 F_1,$$
$$F_1 = \exp\left[-i \sum_{\mathrm{irreps},I} \lambda_I \sum_{j=1}^{L-1} P^I_{B,j;A,j+1}\right],$$
$$F_2 = \prod_{j=1}^{L} \mathcal{F}_{A,j}\mathcal{F}^\dagger_{B,j}, \tag{16}$$

where $\mathcal{F}_{A,j} = \mathcal{F}$ is the generator of the $\mathbb{Z}$ factor associated with time translation in the projective representation $\mathcal{PR}$, and similarly $\mathcal{F}_{B,j} = \mathcal{F}^\dagger$. Moreover, $P^I_{i;j}$ is the projection operator of sites i and j onto the Ith irreducible representation (irrep) R^I of G. If the irreps of G are all singlets (i.e., have dimension one), then F_1 gives a different random quasienergy to all bulk degrees of freedom, resulting in many-body localization. This construction fails for non-Abelian groups with irreps of dimension higher than one, for which there are extensive local degeneracies in the quasienergy spectrum of F_1. Below, we will show that this obstacle is fundamental and that MBL is possible only for Abelian groups with irreps of dimension one. Hence, this classification works for all relevant symmetry groups.

Since $U_{B,j}(g)U^\dagger_{A,j+1}(g)$ can be block diagonalized in R^I and commutes with $\mathcal{F}_{B,j}\mathcal{F}^\dagger_{A,j+1}$ (since A and B transform under conjugate projective representations of $\mathbb{Z} \times G$), such projectors will commute with F_2 terms in the Floquet operator.

As with the case for $\mathbb{Z}_N$ symmetry above, this construction results in a projective implementation of $G \times \mathbb{Z}$ at the edge. Namely, at the left edge, symmetry acts like $g_{A,1}$ and Floquet time evolution acts like $U_{A,j}$, which by construction satisfies a projective realization of $G \times \mathbb{Z}$.

While this model is constructed at a highly fine-tuned point with zero-correlation length, the results are robust to small perturbations that do not result in a phase transition. So long as the perturbation is sufficiently weak that Floquet eigenstates retain their area-law entanglement structure (in the nonequilibrium Floquet setting, a phase transition is defined as a breakdown of the area-law entanglement structure of Floquet eigenstates), then there is a well defined sense of the local action of symmetry on the edge states of the system, and hence the edge states form a projective local representation of symmetry. Moreover, since projective representations are discrete, different projective representations cannot be continuously deformed into each other, and small perturbations cannot continuously alter the realized projective representation.

V. FORMAL CLASSIFICATION

In the previous sections, we built a family of zero-correlation length (fixed-point) models that realize various fermionic and bosonic Floquet SPT phases and support the hypothesis that the classification of these phases is given by projective representations of the symmetry group enhanced by an extra factor of $\mathbb{Z}$ to account for time-translation symmetry. In this section, we formalize these ideas, making extensive use of the ideas behind the related classification of equilibrium SPT ground states [32,41,42].

Our strategy will be to construct a precise definition of the local action of symmetry in order to sharpen the notion of projective interplay of on-site and time-translation symmetries. To this end, consider a system with localized Floquet eigenstates protected by symmetry group G; we may construct an operator that commutes with the Floquet evolution and has the same action as locally applying a symmetry element, $g \in G$, on a large but finite interval $I = [x_l, x_r]$ whose size greatly exceeds the localization length $|x_r - x_l| \gg \xi$:

$$g_I = U_{l,g}\left(\prod_{j\in I} g_j\right)U_{r,g}. \tag{17}$$

The middle term represents the symmetry operator restricted to sites within the interval. This term has an exponentially small effect on the quasilocal quantum numbers n_α residing deep in the bulk of the interval I (as these commute with the unrestricted action of $g=\prod_i g_i$) and, similarly, an exponentially small effect on quasilocal quantum numbers far away from I. On the other hand, this term strongly disturbs those quantum numbers near the boundaries of the interval and, hence, does not by itself commute with the Floquet evolution. However, we may repair the disturbance by acting with a pair of quasilocal unitary operators $U_{l,g}$ and $U_{r,g}$ that are exponentially well localized to the left and right ends of the interval, respectively, which restores the state of the conserved DOF that were altered by $\prod_{j\in I} g_j$.

Paralleling Ref. [25], we can first construct explicit formal expressions for $U_{l/r}$ for the special case of strictly localized "zero-correlation length" Floquet Hamiltonians, whose conserved quantities $\{n_\alpha\}$ have bounded support on a finite number of sites. All of the models we have constructed so far take this form. Subsequently, we adapt these ideas to the more generic case of only exponentially well-localized Floquet operators.

For zero-correlation length Floquet Hamiltonians, $\prod_{j\in I} g_j$ preserves all n_α whose support is fully contained inside I or resides completely outside of I and disturbs only a finite number N_l (N_r) of n_α on the left (right) boundary. We can divide the integrals of motion into four groups: those strictly in the interval I, those strictly in the complement of the interval, I^c, those intersecting the left boundary, ∂I_l, and those intersecting the right boundary, ∂I_r. We can compute the matrix elements

$$(U^\dagger_{l,g})^{n'_{\beta_1}\ldots n'_{\beta_{N_l}}}_{n_{\beta_1}\ldots n_{\beta_{N_l}}}(U^\dagger_{r,g})^{n'_{\delta_1}\ldots n'_{\delta_{N_r}}}_{n_{\delta_1}\ldots n_{\delta_{N_l}}} \equiv \langle \{n_{\alpha_i\in I^c}\},\{n'_{\beta_i\in\partial I_l}\},\{n_{\gamma_i\in I}\},\{n'_{\delta_i\in\partial I_r}\}|\prod_{j\in I} g_j|\{n_{\alpha_i\in I^c}\},\{n_{\beta_i\in\partial I_l}\},\{n_{\gamma_i\in I}\},\{n_{\delta_i\in\partial I_r}\}\rangle, \tag{18}$$

which defines $U_{l/r,g}$ up to an overall phase.

For the more generic case of exponentially well-localized Floquet Hamiltonians, whose conserved quantities are quasi-local, the above construction is only approximate as all integrals of motion have some nonzero (albeit exponentially small) overlap with the boundaries of I. However, we may approximately break the n_α into the same groups by using an arbitrary cutoff to decide which n_α belong to the boundary regions $\partial I_{l/r}$. This approximation is exponentially accurate in the number of integrals of motion $N_{l/r}$ taken to be in the boundary region, allowing for a well-defined limiting procedure where we take the size of I to infinity first and then take $N_{l/r}$ to infinity. In this order of limits, the above construction becomes exact even for only exponentially well-localized systems. In practice, the approximation will become accurate once the subinterval and boundary sizes are both taken to be much larger than the localization length ξ.

Having defined a precise notion of the local action of symmetry, we would also like to sharply define the local action of the Floquet drive near the ends of the interval I. To this end, we first note that a generic localized Floquet Hamiltonian of the form Eq. (3) may be deformed by a finite-depth local unitary transformation (to exponential-in-depth accuracy) to a simpler form for which the Floquet Hamiltonian decomposes into a sum of independent terms for each n_α:

$$\tilde{F}=\prod_\alpha e^{-i\sum_{n_\alpha}\lambda_\alpha(\Pi_{n_\alpha})}. \tag{19}$$

Such a finite-depth unitary circuit preserves the area-law structure of entanglement in the Floquet eigenstates and hence cannot change the underlying phase (which would require a phase transition accompanied by a singularity in the entanglement entropy). Therefore, we may, without loss of generality, consider the Floquet Hamiltonian to decompose in this way. For such decomposable Floquet evolutions, we can divide the Floquet evolution operator into four independent pieces: $F\equiv F_{I^c}F_{\partial I_l}F_I F_{\partial I_r}$, and focus on the action at the left and right boundaries of I: $F_{\partial I_{l/r}}$.

By construction, g_I commutes with the Floquet evolution F and forms a unitary representation of the symmetry group G [i.e., $g_I g'_I=(gg')_I$]. However, the quasilocal operators $U_{l,g}$ and $U_{r,g}$ need not separately form a representation but rather need only satisfy the group composition rules up to an overall phase that cancels between the l and r end points: $U_{l/r,g}U_{l/r,g'}=e^{\pm i\phi(g,g')}U_{l/r,gg'}$, where the $+$ $(-)$ sign in the exponential applies to the l (r) label. Thus, the edge operators $U_{l/r,g}$ need only form a projective representation of the symmetry group. As there are only a discrete set of such projective representations, the particular projective representation realized cannot be continuously altered by arbitrary perturbations, barring a phase transition that spoils the locality of the above constructions. Consequently, in the absence of periodic driving, such projective representations fully characterize the set of ground-state SPT phases [32,41,42].

In the Floquet system, we know that the entire object g_I commutes with the Floquet evolution operator F. However, separately, the local operators $\tilde{U}_{l/r,g}=U_{l/r,g}\prod_{j\in I\cap\partial I_{l/r}} g_I$ need only commute with the action of the Floquet evolution near the interval boundaries, $F_{\partial I_{l/r}}$ up to a phase $F_{\partial I_{l/r}}\tilde{U}_{l/r,g}=e^{\pm i\phi(F,g)}\tilde{U}_{l/r,g}F_{\partial I_{l/r}}$. Having opposite projective phases, $\pm\phi(F,g)$, for the left and right edges, respectively, ensures that the global symmetry operators for the full system commute.

In this way, the local action of symmetry and Floquet evolution near the edge of the interval I is implemented projectively. To understand the group structure involved, we note that the Floquet evolution implements a unitary representation of the group of integers, $\mathbb{Z}$, where positive (negative) integers $N > 0$ are represented by forward (backward) time evolution by N periods: F^N $((F^\dagger)^N)$. Thus, we see that, together with the symmetry group action restricted to one end of the interval, say, $U_{l,g}$, the Floquet evolution forms a projective representation of $G\times\mathbb{Z}$, confirming our hypothesized classification. Moreover, since this projective representation cannot be continuously altered by perturbations that preserve the locality of the Floquet eigenstates, the projective representations correspond to distinct dynamical phases. We also note that, when the full Floquet spectrum is localized, various Floquet eigenstates of a given system differ only by bulk excitations that do not change the projective action of symmetry at the edges, implying that all eigenstates must belong to the same Floquet SPT phase. Since the time evolution of an arbitrary initial state is governed by the Floquet eigenstates, the Floquet SPT order is also imprinted on the dynamics starting from a noneigenstate.

VI. SIGNATURES IN ENTANGLEMENT SPECTRUM

In this section, we describe signatures of intrinsic Floquet SPT order (i.e., Floquet SPT order which cannot occur in undriven systems or, equivalently, does not survive to the infinite frequency limit) in the entanglement spectrum of Floquet eigenstates. These arguments provide an alternative phrasing of the general classification presented in the previous section.

The entanglement spectrum of the Floquet eigenstate $|\Psi\rangle$ can be obtained by performing a Schmidt decomposition: $|\Psi\rangle = \sum_n (e^{-\epsilon_n/2}/\sqrt{Z})|\Psi_{n,L}\rangle|\Psi_{n,R}\rangle$, where $|\Psi_{L/R}\rangle$ are states living on the left and right of the entanglement cut, respectively, such that the reduced density matrix for the left half of the system, $\rho_L = \sum_n (e^{-\epsilon_n}/Z)|\Psi_{n,L}\rangle\langle\Psi_{n,L}|$, takes the form of a thermal density matrix with entanglement Hamiltonian $H = \sum_n \epsilon_n |\Psi_{n,L}\rangle\langle\Psi_{n,L}|$. Here, $Z = \sum_n e^{-\epsilon_n}$ normalizes the trace of the reduced density matrix. Note that, unlike the quasienergy spectrum, the entanglement spectrum is noncompact and is not periodic modulo 2π. Consequently, $\epsilon = 0$ is a special entanglement energy dividing positive and negative states, unlike quasienergies whose absolute value has no meaning.

For equilibrium 1D SPTs, the entanglement spectrum exhibits degenerate zero modes that permit one to diagnose the SPT order. For systems with on-site symmetries, the entanglement spectrum exhibits zero modes ($\epsilon_n = 0$) with multiplicity equal to the edge-state degeneracies of a system with open boundary conditions [32,42]. For 1D crystalline SPTs (e.g., protected by inversion), the entanglement spectrum contains zero modes indicative of the SPT order, even in cases where a physical edge would break the protecting symmetry and fail to exhibit edge states [42]. Acting with the symmetry operations on either side of the entanglement cut reveals the projective action of symmetry on the edge states. This bulk-edge correspondence provides a numerically testable probe of the equilibrium SPT order in a system without boundaries.

However, for a state with intrinsic Floquet SPT order but no equilibrium SPT order, i.e., projective edge action of F and G but nonprojective edge action of symmetry alone, there is no universal signature in the static entanglement spectrum of a Floquet eigenstate. This is manifestly seen by considering the special case of a zero-correlation Floquet drive that realizes the FSPT order with $\mathbb{Z}_2$ symmetry. Rather than considering the AKLT-like model described above, which allows us to construct states with either equilibrium or driven SPT order on the same footing, we may consider the simpler model introduced in Ref. [26], which realizes the nontrivial FSPT phase. This model consists of a spin-$1/2$ chain with Ising symmetry generated by $g = \prod_i \sigma_i^x$, and stroboscopic Floquet drive:

$$\begin{aligned} F_{\mathbb{Z}_2} &= e^{-i\pi/2\sum_{i=1}^{L-1}\sigma_i^z\sigma_{i+1}^z} e^{-i\sum_{i=1}^{L} h_i\sigma_i^x} \\ &= \tilde{\sigma}_1^z\tilde{\sigma}_L^z e^{-i\sum_{i=2}^{L-1} h_i\sigma_i^x}, \end{aligned} \tag{20}$$

where $\tilde{\sigma}_{1/L}^z = (e^{-ih_{1/L}/2\sigma_{1/L}^x}\sigma_{1/L}^z e^{ih_{1/L}/2\sigma_{1/L}^x})$ are rotated Pauli matrices with a quantization access tilted along the $\cos(h_{1/L})\hat{z} + \sin(h_{1/L})\hat{y}$ direction in the xy plane. We can readily verify that $F_{\mathbb{Z}_2}$ realizes the projective edge action of $\{F,g\}_{\text{edge}} = 0$ since $F_{\text{edge}} = \tilde{\sigma}^z$ and $g_{\text{edge}} = \sigma^x = \tilde{\sigma}^x$, which anticommute. Moreover, since $F_{\mathbb{Z}_2}$ is a product of on-site unitary operators, its eigenstates are just product states and hence have a trivial entanglement spectrum.

Thus, the static entanglement spectrum does not contain information about the topological edge states, unlike the equilibrium case. However, the SPT order does manifest itself if we examine the full time-dependent micro-motion of the entanglement-spectrum Floquet eigenstates for times $0 \le t \le T$. To see this, let us continue working with the special zero-correlation length Hamiltonian of Eq. (20); however, to avoid issues with edge states or periodic boundary conditions, we work directly in the infinite system limit where the site index i can take any integer value. For convenience, we place our entanglement cut between sites $i = 0$ and $i = 1$. Let us consider the entanglement spectrum of a particular Floquet eigenstate, $|\Psi\rangle = \otimes_i |s_i\rangle$, where $|s_i\rangle$ are σ_i^x eigenstates with eigenvalue $s_i = \pm 1$. The first phase of the Floquet evolution, $U(t,0) = e^{-it/T_1\sum_i h_i\sigma_i^x}$, just generates an overall phase for $|\Psi\rangle$ and does not effect the entanglement spectrum. The second stage, $U_2(t+T_1,T_1)=e^{-it/T_2\sum_i\sigma_i^z\sigma_{i+1}^z}=U_{2,L}U_{2,R}U_{2,\mathrm{cut}}$, can be decomposed into pieces that act only on the left and right halves, and one term that acts across the cut: $U_2=U_{2,L}U_{2,R}U_{2,\mathrm{cut}}$, with $U_{2,L}=e^{-i\pi t/2T_2\sum_{i\le -1}\sigma_i^z\sigma_{i+1}^z}$, $U_{2,R}=e^{-i\pi t/2T_2\sum_{i\ge 1}\sigma_i^z\sigma_{i+1}^z}$, and $U_{2,\mathrm{cut}}=e^{-i\pi t/2T_2\sigma_0^z\sigma_1^z}$. Since only $U_{2,\mathrm{cut}}$ generates entanglement, we may equivalently consider the simplified problem of finding the entanglement spectrum of the two-spin system straddling the cut, evolving according to $U_{2,\mathrm{cut}}$. Explicitly, we have $e^{-i\pi t/2T_2\sigma_0^z\sigma_1^z}|s_0s_1\rangle=\cos(\pi t/2T_2)|s_0s_1\rangle+\sin(\pi t/2T_2)|-s_0,-s_1\rangle$; i.e., the reduced density matrix on the left side is $\rho_L(t+T_1)=\cos^2(\pi t/2T_2)|s_0\rangle\langle s_0|+\sin^2(\pi t/2T_2)|-s_0\rangle\langle -s_0|=(e^{-h(t)\sigma_0^x}/Z)$, where $h(t)=\mp\tanh^{-1}[\cos(\pi t/T_2)]$ for $s_0=\pm 1$.

We see that the entanglement spectrum contains two eigenvalues: $\epsilon=\pm h(t)$ (see Fig. 4), whose corresponding Schmidt states have opposite $\mathbb{Z}_2$ eigenvalues and which are initially at $\pm\infty$ at the beginning of the Floquet period ($t=0$). During the second stage of the Floquet evolution, $h(t)$ decreases $+\infty$ towards $-\infty$, crossing zero at time $t_*=T_1+T_2/2$, at which point the entanglement spectrum becomes degenerate and the two $|\pm s_0\rangle$ branches cross each other. Continuing the evolution, the entanglement spectrum returns towards $\epsilon=\pm\infty$ but with the $|\pm s_0\rangle$ branches exchanged. Fixing $s_0=+1$ for concreteness, we see that the $\mathbb{Z}_2$ symmetry charge of the negative entanglement energy bands ($\epsilon<0$) changes by one unit, as the $|-\rangle$ branch of the spectrum exchanges with the $|+\rangle$ branch. This pumping of symmetry charge provides a bulk probe of the projective action of Floquet evolution and symmetry at the edge $[(FgF^{-1}g)_{\mathrm{edge}}=-1]$, as we will explain in more detail below. We note that taking into account $U_{2,L}$ does not affect this pumping of symmetry charge because $U_{2,L}$ commutes with $U_{2,\mathrm{cut}}$ and the symmetry operation g restricted to the left half of the chain. Note also that, to diagnose the Floquet SPT order, we need the full micromotion of the entanglement spectrum, rather than just the spectrum at any single time cut.

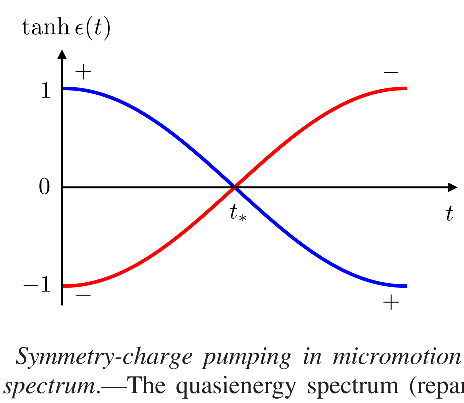

FIG. 4. *Symmetry-charge pumping in micromotion of entanglement spectrum.*—The quasienergy spectrum (reparametrized by tanh to fall between ± 1 rather than $\pm\infty$) of the $\mathbb{Z}_2$-symmetry protected bosonic FSPT exhibits a quantized pumping of $\mathbb{Z}_2$ symmetry charge, in each Floquet cycle.

While we have worked out this structure for a particular example, the pumping of symmetry charge in the entanglement spectrum turns out to be a robust way to characterize the Floquet SPT order and is equivalent to the projective action of symmetry and time translation obtained at the edges of a finite system with open boundaries. To see this, first note that the $t=0$ and $t=T$ end points of the evolution have identical gapped, zero-dimensional entanglement spectra. However, if we look at the Schmidt states corresponding to each entanglement energy at $t=0$ and $t=T$, they need not be identical. Specifically, we may classify the zero-dimensional SPT properties of the Schmidt state corresponding to a particular entanglement energy E_0 at $t=0$. For a bosonic zero-dimensional system with symmetry G, the SPT invariant is given by one-dimensional representations of the group, $\mathcal{H}_1(G,U(1))$, i.e., the possible symmetry charges. In the above example, we have seen that, over the course of a Floquet period, the entanglement spectrum returns to itself at $t=T$; however, if we examine the Schmidt state corresponding to E_0 at $t=T$, we find that there is a robust change in its symmetry charge, as this state has interchanged with another state with opposite charge. This difference between initial and final entanglement symmetry charges cannot be changed without closing the entanglement gap at $t=0$, T, i.e., by driving a phase transition into a different phase, and is hence a robust characteristic of the FSPT phase. For generic symmetry groups G, such one-dimensional group representations coincide exactly with the extra factor of $\mathcal{H}_1(G,U(1))$ appearing in the Kunneth formula for classifications of $G\times\mathbb{Z}$, in precise agreement with the interpretation of 1D FSPT phases as having projective action of edge symmetries, but it provides an alternative perspective that is testable in systems without boundaries. A closely related picture holds for the fermionic FSPT states, though here fermion parity plays a key role in the pumping (e.g., one must keep track of the pumping of fermion parity and symmetry charges across the entanglement cut).

VII. SYMMETRY CONSTRAINTS ON LOCALIZABILITY

Localization is crucial to avoid heating and obtain Floquet eigenstates with area-law entanglement entropy, which permits sharp distinctions between dynamical phases—e.g., with entanglement entropy serving as a free energy, whose singularities in the limit of infinite system size represent phase transitions. In this section, we show that the requirement of localization places strong constraints on the type of symmetry groups that we may consider. We will review the general criterion based on the representation theory of the symmetry group, namely, that symmetry-preserving localization is only possible for groups whose irreducible representations (irreps) all have dimension one—i.e., only for Abelian symmetry groups [48]. This constraint applies to both static MBL Hamiltonians and periodic Floquet MBL systems alike.

In a symmetry-preserving MBL system, the symmetry generators g must commute with the projection operators Π_{n_α} onto the states of the local conserved quantities. Thus, the different values of n_α label one-dimensional representations of G. For a generic symmetric Hamiltonian, the different values of n_α will label irreps of G (as reducible representations can be subdivided into irreps by the application of infinitesimal local perturbations) [48]. Moreover, since the microscopic degrees of freedom must form a faithful representation of the symmetry group (otherwise the true symmetry group should be regarded as a subset of G), tensor products of lattice-scale degrees of freedom will generate all possible irreps of G.

When G is Abelian, all irreps are one dimensional, and the state labeled by $|n_1 n_2 \ldots n_L\rangle$ is unique and well defined. On the other hand, when G is non-Abelian, some values of n_α necessarily correspond to irreps with dimension $D_{n_\alpha} > 1$. In this case, the quantum numbers n_α can, at most, label local irreps of the symmetry group, each of which must be augmented with some additional quantum numbers $q_\alpha = 1, \ldots, D_{n_\alpha}$ to specify a quantum state. In a generic state, local excitations that transform under such multidimensional irreps will be present at finite density in a generic state, such that "$|n_1 n_2 \ldots n_L\rangle$" actually corresponds to a collection of extensively degenerate $\prod_\alpha D_{n_\alpha} \sim e^{AL}$ states $|(n_1, q_1); (n_2, q_2) \ldots\rangle$ for some constant $A = \sum_{\mathrm{irreps},I} \log D_I \rho_I$, where ρ_I is the density of excitations in the Ith irrep.

Such an extensive degeneracy will be inherently unstable to arbitrarily small perturbations, which will lead to interactions among the locally degenerate excitations, resulting in resonant quantum fluctuations that will resolve the extensive local degeneracy. However, regardless of the details of this degeneracy lifting by fluctuations, there is no possible localized state that respects the symmetry. Instead, we see three conceivable alternative outcomes:

(1) Quantum fluctuations among the highly degenerate states can lead to thermalization and a breakdown of MBL.

(2) The state may spontaneously lift the degeneracy by choosing a product state of quantum numbers q_α; however, this necessarily corresponds to a spontaneous breaking of symmetry G down to an Abelian subgroup since the auxiliary quantum numbers q_α transform nontrivially under G.

(3) If the residual interactions between pairs of (n_α, q_α) with $D_{n_\alpha} > 1$ are strongly random and local, the system may form a quantum critical state that is neither thermal nor strictly localized. This state can be viewed as a generalized random singlet phase, such as those recently identified in loosely related systems of random anyonic chains [51].

Options 1 and 2 were both recently observed in renormalization-group and numerical studies [52] of 1D topological chains of fermions with random hopping, whose ground states form a SPT protected by $G = U(1) \times \mathbb{Z}_2^T$, where $\mathbb{Z}_2^T$ corresponds to antiunitary time-reversal symmetry. For this system, there is one singlet [with zero $U(1)$ charge] and an infinite number of $D = 2$ irreps with integer nonzero $U(1)$ charge $\pm n$ that are interchanged by time reversal. At weak disorder, the symmetry-ensured local degeneracies correspond to strongly overlapping degrees of freedom that lead to thermalization. At strong disorder, the excited states of this model were found to inevitably spontaneously break the $\mathbb{Z}_2^T$ reversal. In the strong-disorder renormalization-group treatment, this spontaneous symmetry breaking arises because of the accumulation of clusters with increasingly large charge q, strongly suppressing quantum fluctuations and leaving essentially dominantly classical interactions that lead to symmetry breaking. The strong-disorder physics of this model is potentially special to the presence of an infinite number of irreps. The third, critical option described above is likely only a possibility for discrete non-Abelian groups, with a finite number of irreps; however, we leave the precise properties of this for future study.

While we presented results for unitary symmetry groups, we note that this construction can be readily generalized to antiunitary time-reversal symmetry by using the ideas of Ref. [53] to define the local action of complex conjugation on tensor-product states. For example, in a fermionic system, there must be local fermionic excitations; i.e., different values of n_α must label states with either even or odd local fermion parity. In a system where time reversal squares to (-1) in the odd fermion sector ($\mathcal{T}^2 = P_f$), there will be a local Kramers degeneracy in the odd local fermion parity sectors of each subsystem α. Consequently, these arguments also rule out MBL SPT phases protected by time-reversal symmetry with degrees of freedom with Kramers doublet fermions ($\mathcal{T}^2 = P_f$), such as the familiar 2D and 3D electronic time-reversal symmetric topological insulator materials realized in solid-state materials.

VIII. DISCUSSION

We have shown that the classification of topological phases in 1D Floquet systems can be understood by generalizing the equilibrium classification to include an extra time-translation symmetry, and we have provided explicit model constructions of a large class of Floquet SPT phases. A simple generalization of our arguments to higher dimensions d would suggest that the bosonic Floquet SPT classification with symmetry group G is given by higher cohomology groups $\mathcal{H}^{d+1}(\tilde{G}, U(1))$, where again $\tilde{G}$ consists of G enhanced by time-translation symmetry (e.g.,

$\tilde{G} = G \times \mathbb{Z}$ for unitary G or $G \rtimes \mathbb{Z}$ for antiunitary G, as appropriate) [54,55]. Characterizing the phenomenology of these phases and understanding the classification of higher-dimensional interacting fermion phases are interesting challenges for future work.

Having obtained a systematic theoretical understanding of the structure of topological phases in 1D Floquet systems, a natural next step is to investigate potential experimental realizations of these phases in cold-atom or other quantum-optics-based systems. To this end, the most promising candidate seems to be spin systems, such as the dynamical analog of the Haldane chain, since the nontrivial fermionic phases do not permit localization, either because of the non-Abelian nature of their symmetry group or because they occur in explicitly particle-number nonconserving systems, i.e., superfluids, which in cold-atom contexts possess extended Goldstone modes that will act as a thermalizing bath.

If realized, the Floquet Haldane phase may be diagnosed experimentally by the absence of decoherence for the edge spins. Namely, the time scale for decoherence of an initially prepared quantum state of the Floquet topological edge states comes only from the interaction between edge states on opposite sides of the system, and the decoherence time diverges exponentially in the length of the system. This coherent storage is also present for nondriven MBL topological phases; however, the Floquet topological phases may be distinguished by noting that the edge-state spin coherently flips over the course of each Floquet period. One can probe the symmetry-protected nature of the Floquet SPT edge states by intentionally introducing a symmetry-breaking field [23] to induce decoherence of the edge-state information, which can subsequently be reversed by applying the opposite symmetry-breaking field.

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Note added.—During the completion of this paper, we became aware of related independent works by C. von Keyserlingk and S. Sondhi [43], and D. Else and C. Nayak [44], whose results are consistent with our own, where they overlap.

APPENDIX A: BILINEAR COUPLINGS BETWEEN MZM AND MPM

In this section, we show that a bilinear coupling between a MZM and a MPM does not change the quasienergies of the Majorana end states for the specific model of superconducting chains of spinless fermions subject to the stroboscopic periodic drive defined in Eq. (4). We consider two superconducting chains denoted by Majorana fermions a_i, b_i and a'_i, b'_i that support a MPM and a MZM, respectively. Specifically, we focus on the Hamiltonian given by

$$H(t) = \begin{cases} H_1 = \frac{i\lambda_1}{4}\sum_{j=1}^{L}(a_j b_j + a'_j b'_j) & 0 \le t < T_1 \\ H_2 = \frac{i\lambda_2}{4}\sum_{j=1}^{L-1}(b_j a_{j+1} + b'_j a'_{j+1}) & T_1 \le t < T_1 + T_2, \end{cases} \tag{A1}$$

with $\lambda_1 T_1/4 = \pi/2$. In this case, the Floquet operator reads

$$F = a_1 F' b_L, \tag{A2}$$

$$F' = e^{-i\tilde{T}_2 H_2} \prod_{j=1}^{L-1} a'_j b'_j. \tag{A3}$$

Here, we note that F' commutes with a_1 and a'_1. Since $F a_1 F^\dagger = -a_1$ and $F a'_1 F^\dagger = a'_1$ hold, a_1 and a'_1 are a MPM and a MZM, respectively.

Now we add a coupling between the MPM and the MZM by adding the bilinear term $i(\delta/T_2) a_1 a'_1$ to H_2. This modifies the Floquet operator as

$$F = a_1 e^{-\delta a_1 a'_1} F' b_L, \tag{A4}$$

and a_1 and a'_1 no longer describe eigenstates of F; the operator a_1 (a'_1) does not satisfy $F a_1 F^\dagger = \epsilon a_1$ ($F a_1 F^\dagger = \epsilon' a_1$) with quasienergy ϵ (ϵ'). Instead, eigenstates are given by superpositions of a_1 and a'_1 as

$$\tilde{a}_1 = a_1 e^{-\delta a_1 a'_1} = a_1 \cos\delta - a'_1 \sin\delta, \tag{A5}$$

$$\tilde{a}'_1 = a'_1 e^{-\delta a_1 a'_1} = a_1 \sin\delta + a'_1 \cos\delta. \tag{A6}$$

These Majorana fermions $\tilde{a}_1$ and $\tilde{a}'_1$ satisfy

$$F\tilde{a}_1 F^\dagger = -\tilde{a}_1, \qquad F\tilde{a}'_1 F^\dagger = \tilde{a}'_1, \tag{A7}$$

and correspond to a new MPM and a new MZM, respectively. Thus, the bilinear coupling for a MPM and a MZM does not change the quasienergies. It only modifies associated Majorana operators.

APPENDIX B: DERIVATION OF $H^2(\prod_i \mathbb{Z}_{n_i}, U(1))$ FROM THE KUNNETH FORMULA

In this section, we derive the second cohomology group $H^2(\prod_i \mathbb{Z}_{n_i}, U(1))$ that appears in Table III. This involves the universal coefficient theorem and the Kunneth formula for cohomology and homology groups [54–57]. First, the

TABLE III. *Classification of 1D bosonic Floquet SPTs.*—Group structure of nontrivial topological classes for 1D bosonic systems with discrete, Abelian on-site symmetries. Non-Abelian symmetry groups and symmetry groups with antiunitary symmetries with irreducible representations of dimension larger than one do not permit symmetry-preserving many-body localization and are unstable to heating. The last entry represents the most general finite Abelian symmetry group; a derivation of the Floquet classification for this general case is present in Appendix B

Symmetry group (G)	Static classification ($C[G]$)	Floquet classification ($C_F[G]$)
None	None	None
$\mathbb{Z}_2^T$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$\mathbb{Z}_n$	None	$\mathbb{Z}_n$
$\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\ldots\mathbb{Z}_{n_p}$	$\prod_{i\neq j=1}^{p}\mathbb{Z}_{\gcd(n_i,n_j)}$	$C[G]\times\prod_{i=1}^{p}\mathbb{Z}_{\gcd(n_i)}$

universal coefficient theorem relates the cohomology group to the homology group as

$$H^2\left(\prod_i \mathbb{Z}_{n_i}, U(1)\right) = H_2\left(\prod_i \mathbb{Z}_{n_i}, \mathbb{Z}\right). \tag{B1}$$

For this homology group, we apply the Kunneth formula

$$\begin{aligned} H_2(G_1\times G_2,\mathbb{Z}) &= \prod_{i=0}^{2} H^i(G_1,\mathbb{Z})\otimes H^{2-i}(G_2,\mathbb{Z}) \\ &\quad\times \prod_{i=0}^{1}\mathrm{Tor}_1^{\mathbb{Z}}[H^i(G_1,\mathbb{Z}),H^{1-i}(G_2,\mathbb{Z})] \end{aligned} \tag{B2}$$

by using the following equations [55]:

$$\mathbb{Z}_{n_1}\otimes\mathbb{Z}_{n_2} = \mathbb{Z}_{\gcd(n_1,n_2)}, \tag{B3}$$

$$G_1\otimes(G_2\times G_3) = (G_1\otimes G_2)\times(G_1\otimes G_3), \tag{B4}$$

$$H^0(\mathbb{Z}_n, U(1)) = H_0(\mathbb{Z}_n,\mathbb{Z}) = \mathbb{Z}, \tag{B5}$$

$$H^1\left(\prod_i \mathbb{Z}_{n_i}, U(1)\right) = H_1\left(\prod_i \mathbb{Z}_{n_i},\mathbb{Z}\right) = \prod_i \mathbb{Z}_{n_i}, \tag{B6}$$

$$H^2(\mathbb{Z}_n, U(1)) = H_2(\mathbb{Z}_n,\mathbb{Z}) = 0. \tag{B7}$$

Now, the second cohomology group $H^2(\prod_i \mathbb{Z}_{n_i}, U(1))$ is obtained by successively applying the Kunneth formula as

$$\begin{aligned} H^2\left(\prod_{i=0}^{p}\mathbb{Z}_{n_i}, U(1)\right) &= H_2\left(\prod_{i=0}^{p}\mathbb{Z}_{n_i},\mathbb{Z}\right) \\ &= \left[H_0(\mathbb{Z}_{n_1},\mathbb{Z})\otimes H_2\left(\prod_{i=1}^{p}\mathbb{Z}_{n_i},\mathbb{Z}\right)\right] \\ &\quad\times\left[H_1(\mathbb{Z}_{n_1},\mathbb{Z})\otimes H_1\left(\prod_{i=1}^{p}\mathbb{Z}_{n_i},\mathbb{Z}\right)\right] \\ &\quad\times\left[H_2(\mathbb{Z}_{n_1},\mathbb{Z})\otimes H_0\left(\prod_{i=1}^{p}\mathbb{Z}_{n_i},\mathbb{Z}\right)\right] \\ &= H_2\left(\prod_{i=1}^{p}\mathbb{Z}_{n_i},\mathbb{Z}\right)\times\prod_{i=1}^{p}\mathbb{Z}_{\gcd(n_1,n_i)} \\ &= \ldots = \prod_{i<j}\mathbb{Z}_{\gcd(n_i,n_j)}. \end{aligned} \tag{B8}$$

We note that the Tor functor part in Eq. (B2) vanishes because of $\mathrm{Tor}_1^{\mathbb{Z}}[\mathbb{Z},\prod_i\mathbb{Z}_{n_i}] = \mathrm{Tor}_1^{\mathbb{Z}}[\mathbb{Z}_{n_i},\mathbb{Z}] = 0$.

In a similar manner, $H^2(G\times\mathbb{Z}, U(1))$ with $G = \prod_{i=0}^{p}\mathbb{Z}_{n_i}$ is obtained by applying the above procedure for $G\times\mathbb{Z}$ and using $\mathbb{Z}_n\otimes\mathbb{Z} = \mathbb{Z}_n$ [crudely speaking, $\gcd(n,\infty) = n$] as

$$H^2(G\times Z, U(1)) = \left(\prod_{0\le i<j\le p}\mathbb{Z}_{\gcd(n_i,n_j)}\right)\times G. \tag{B9}$$

This cohomology group gives the Floquet classification of 1D bosonic systems with the symmetry group $G = \prod_{i=0}^{p}\mathbb{Z}_{n_i}$ in Table III.

APPENDIX C: PROJECTIVE REPRESENTATIONS OF $\mathbb{Z}\rtimes\mathbb{Z}_2^T$

In this appendix, we derive the projective representations of $\mathbb{Z}\rtimes\mathbb{Z}_2^T$, corresponding to the classification of time-reversal–invariant bosonic FSPTs. The results do not follow from the Kunneth formula explained in the previous section because of the semidirect product structure, but it can be obtained directly.

1. Bosonic systems

Denoting the generator of $\mathbb{Z}$ by F (Floquet evolution) and the generator of time reversal, $\mathbb{Z}_2^T$, as $\mathcal{T}$, the group relations are $(\mathcal{T}^2)_{\text{group}} = 1$ and $(\mathcal{T}F\mathcal{T}^{-1}F)_{\text{group}} = 1$. At the edge of a Floquet SPT, these can be implemented projectively as $\mathcal{T}^2 = \omega_T$ and $\mathcal{T}F\mathcal{T}^{-1}F = \omega_{T,F}$, where $\omega_{T,F}\in U(1)$ are phases. Because $\mathcal{T}$ is antiunitary, $\omega_{T,F}$ cannot be altered by a simple redefinition of F or $\mathcal{T}$ by an overall phase, and hence, if consistent, different values of $\omega_{T,F}$ correspond to distinct projective representations.

The possible consistent values of ω_T can be identified as follows. Since $\mathcal{T}$ is antiunitary, associativity requires $\mathcal{T}^3 = \mathcal{T}(\mathcal{T}^2) = \mathcal{T}(\omega_T) = \omega_T^*\mathcal{T} = (\mathcal{T}^2)\mathcal{T} = \omega_T\mathcal{T}$, i.e., that ω_T is real, allowing for two solutions: $\omega_T = \pm 1$. To fix the possible values of $\omega_{T,F}$, we first note that $(\mathcal{T}F\mathcal{T}^{-1})^{-1} = \mathcal{T}F^{-1}\mathcal{T}^{-1} = \omega^*_{T,F}F$. Using this relation, we see that $\mathcal{T}^2F\mathcal{T}^{-2} = \mathcal{T}\omega_{T,F}F^{-1}\mathcal{T}^{-1} = \omega^*_{T,F}\mathcal{T}F^{-1}\mathcal{T} = (\omega^*_{T,F})^2F$, but, on the other hand, $\mathcal{T}^2F\mathcal{T}^{-2} = |\omega_T|^2F = F$. Together, these relations require $\omega_{T,F} = \pm 1$.

In total, there are four projective representations of $\mathbb{Z}\rtimes\mathbb{Z}_2^T$ corresponding to $\omega_{T,F} = \pm 1$, analogous to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group structure.

2. Fermionic systems

For fermion systems, there is an additional $\mathbb{Z}_2$ fermion parity "symmetry" P_f. This gives an additional pair of gauge-invariant group relations: $(TP_fT^{-1}P_f)_{\text{group}} = 1$ and $(FP_fF^{-1}P_f^{-1})_{\text{group}} = 1$, which can be modified to projective relations $\mathcal{T}P_f\mathcal{T}^{-1}P_f = \omega_{T,P}$ and $FP_fF^{-1}P_f = \omega_{F,P}$. Consistency between $(\mathcal{T}P_f\mathcal{T})^{-1} = \mathcal{T}P_f\mathcal{T}^{-1} = \omega_{T,P}P_f$ and $(\mathcal{T}P_f\mathcal{T})^{-1} = (\omega_{T,P}P_f)^{-1} = \omega^*_{T,P}P_f$ implies that $\omega_{T,P}$ must be real: $\omega_{T,P} = \pm 1$. Repeating the same line of reasoning with $\mathcal{T} \leftrightarrow P_f$ requires $\omega_{F,P} = \pm 1$.

For just $\mathcal{T}$ and P_f alone, there are four distinct projective representations corresponding to $\omega_{T,P} = \pm 1$ and $\omega_{T,F} = \pm 1$. These representations correspond with the even entries of the $\mathbb{Z}_8$ classification of $\mathcal{T}^2 = 1$ fermions in 1D (the odd entries have unpaired Majorana zero modes, corresponding to fractional fermion parity, and do not fit into the language of projective representations of symmetry). Adding the Floquet drive to the mix gives an additional four possibilities: $\omega_{T,F} = \pm 1$ and $\omega_{P,F} = \pm 1$. These correspond to the additional factor of $\mathbb{Z}_4$ in the BDI classification (note that we have shown in the main text that two copies of the phase with $\omega_{P,F} = -1$ have $\omega_{T,F} = -1$).

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