Entanglement Entropy of Fermi Liquids via Multidimensional Bosonization

Wenxin Ding,[1] Alexander Seidel,[2] and Kun Yang[1]

[1]*National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306, USA*
[2]*Department of Physics and Center for Materials Innovation, Washington University, St. Louis, Missouri 63136, USA*
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The logarithmic violations of the area law, i.e., an "area law" with logarithmic correction of the form $S \sim L^{d-1} \log L$, for entanglement entropy are found in both 1D gapless fermionic systems with Fermi points and high-dimensional free fermions. This paper shows that both violations are of the same origin, and that, in the presence of Fermi-liquid interactions, such behavior persists for 2D fermion systems. In this paper, we first consider the entanglement entropy of a toy model, namely, a set of decoupled 1D chains of free spinless fermions, to relate both violations in an intuitive way. We then use multidimensional bosonization to rederive the formula by Gioev and Klich [D. Gioev and I. Klich, *Entanglement Entropy of Fermions in Any Dimension and the Widom Conjecture*, Phys. Rev. Lett. **96**, 100503 (2006).] for free fermions through a low-energy effective Hamiltonian and explicitly show that, in both cases, the logarithmic corrections to the area law share the same origin: the discontinuity at the Fermi surface (points). In the presence of Fermi-liquid (forward-scattering) interactions, the bosonized theory remains quadratic in terms of the original local degrees of freedom, and, after regularizing the theory with a mass term, we are able to calculate the entanglement entropy perturbatively up to second order in powers of the coupling parameter for a special geometry via the replica trick. We show that these interactions do not change the leading scaling behavior for the entanglement entropy of a Fermi liquid. At higher orders, we argue that this should remain true through a scaling analysis.

I. INTRODUCTION

The study of entanglement, which is one of the most fundamental aspects of quantum mechanics, has and continues to lead to much important progress and applications in several fields of modern physics such as quantum information [1], condensed matter physics [2–4], and so on. It has led to a better understanding of the density-matrix-renormalization-group (DMRG) technique [5–7], and it has also been proposed as a tool for the characterization of certain topological phases [8–10].

Among various ways of quantifying entanglement, in condensed matter or many-body physics, efforts have mainly focused on the bipartite block-entanglement entropy (von Neumann entropy) and its generalizations (Rényi or Tsallis entropy). Entanglement has become increasingly useful in characterizing phases [11] and phase transitions [12,13]. The area law [14], one of the most important results of entanglement entropy, states that the entanglement entropy is proportional to the area of the surface that separates two subsystems. Thus far, however, two important classes of systems violate the area law. (1) In gapless one-dimensional (1D) systems, a logarithmic divergence [13,15] is found where, according to the area law, the entanglement entropy should saturate as the size of the subsystem grows. (2) In higher dimensions, for free fermions, the area law is corrected by a logarithmic factor similar to the 1D case, $\log L$ [16–23], where L is the linear dimension of the subsystem.

In this work, we first show that the scaling behavior of the entanglement entropy for systems with a Fermi surface is the same as that of 1D systems with Fermi points [24–27]. We then seek for a generalization of the behavior in 1D systems with Fermi points to that of interacting fermions in the Fermi-liquid phase. We first develop an intuitive understanding via a toy model, showing that, in this model, the entanglement entropy has the same form as that given by Gioev and Klich (GK) in Ref. [16]. Next, we develop a more general and formal treatment using high-dimensional bosonization [28–32]. This approach not only leads to a reproduction of the result for free fermions obtained by GK based on Widom's conjecture [33,34], but also lends itself to the inclusion and subsequent treatment of Fermi-liquid type (forward-scattering) interactions.

This paper is organized as follows. In Sec. II, we describe the toy model for which the entanglement entropy can be written in the same form as the GK result. In Sec. III, we briefly introduce the tool box of multidimensional bosonization, and apply it to free fermions to reproduce the GK formula. The main results of this work are presented in Sec. IV, in which we calculate the entanglement entropy of a Fermi liquid for a special geometry using a combination of multidimensional bosonization and the replica trick (explained below). We summarize and discuss our results in Sec. V. Some technical details are discussed in the two appendixes.

II. THE INTUITIVE PICTURE—A TOY MODEL

Consider a set of decoupled, parallel 1D chains of non-interacting spinless fermions with spacing a, as shown in Fig. 1(a). Here we consider only $d = 2$ for simplicity, but this toy model is viable in general d dimensions. The asymptotic behavior of entanglement entropy in the large L limit of a convex subsystem A of this model can be obtained by simply counting the number of chains that intersect A, and each segment contributes $\frac{1}{3}\log L$, where L is the linear dimension of the subsystem [14,35,36]. Because of the logarithm, different shapes lead only to differences at the area-law level. Since each segment must have two intersections, we can count the intersections instead of segments, and thus automatically take care of nonconvex geometries. Although there is an additional correction for multiple intervals on a single chain [37–39], as long as only the $\log L$ behavior is concerned, that contribution is negligible. For L large enough, we can write the number of these intersections as an integral over the surface of A, projected onto the direction perpendicular to the chains, times one-half of the chain density, $1/a$. To compare with the GK result, we note that this model also has Fermi surfaces, as shown in Fig. 1(b), with a total "area" of $4\pi/a$. This enables us to replace the density of chains by an integral over the Fermi surfaces of the system,

$$\frac{1}{a} = \frac{1}{4\pi} \oint_{\partial\Gamma} dS_k,$$

where Γ indicates the occupied area in momentum space, so its boundary $\partial\Gamma$ is the Fermi surface(s). Therefore, we can write the entanglement entropy as

$$
\begin{aligned}
S(\rho_A) &= \frac{1}{2} \times \frac{1}{3} \log L \times \frac{1}{a} \oint_{\partial A} |\hat{n} \cdot dS_x| \\
&= \frac{1}{12(2\pi)^{2-1}} \log L \times \oint_{\partial A} \oint_{\partial\Gamma} |dS_x \cdot dS_k|, \quad (1)
\end{aligned}
$$

where $dS_{\mathbf{k}}$ denotes the surface element in momentum space, $\hat{n}$ is the direction along the chains that is also normal to the Fermi surface, and an overall factor of $\frac{1}{2}$ accounts for the double-counting of chain segments. In Eq. (1), we recover the GK formula in this special case but written in a slightly different way. In Ref. [16], the entanglement entropy is given as

$$
S = \frac{1}{12} \frac{L^{d-1} \log L}{(2\pi)^{d-1}} \oint_{\partial A} \oint_{\partial\Gamma} |\hat{n}_x \cdot \hat{n}_k| dS_x dS_k, \quad (2)
$$

where the real-space surface integral is carried out over the subsystem whose volume is normalized to 1. The surface area is factored out as L^{d-1}. However, in our formula, the surface area proportional to L^{d-1} is implicitly included in the integral over the surface of the subsystem.

We note that the model discussed in Ref. [26] is equivalent with our toy model but motivated from a different perspective. In Ref. [26], models are constructed from the momentum space, either with Fermi surfaces (as in our toy model) or with a square Fermi surface, and a boxlike and a spherical geometry are discussed. In contrast, our toy model is constructed from a real-space perspective, and general, single, connected geometries are discussed.

Motivated by the toy model, in this work we extend this intuitive understanding of GK's result to generic free Fermi systems and generalize it to include Fermi-liquid interactions in two dimensions (2D) via high-dimensional bosonization. When we use the method of multidimensional bosonization, the Fermi-liquid theory can be written as a tensor product of low-energy effective theories of quasi-1D systems similar to this toy model, along all directions. This method provides us with a tool to treat the entanglement entropy of fermions in high dimensions, even in the presence of *interactions*, as we will explain below.

At this point, we could also include forward scattering for each chain. From 1D bosonization, we know that, for spinless fermions, forward scattering only leads to renormalization of the Fermi velocity, and thus does not change the logarithmic scaling of the entanglement entropy for our toy model. This fact hints that the same conclusion might hold for Fermi liquids, as we can include Fermi-liquid interactions in a similar way via high-dimensional bosonization. Although, as we show later, this is indeed true at the leading order, the situation is more delicate than it
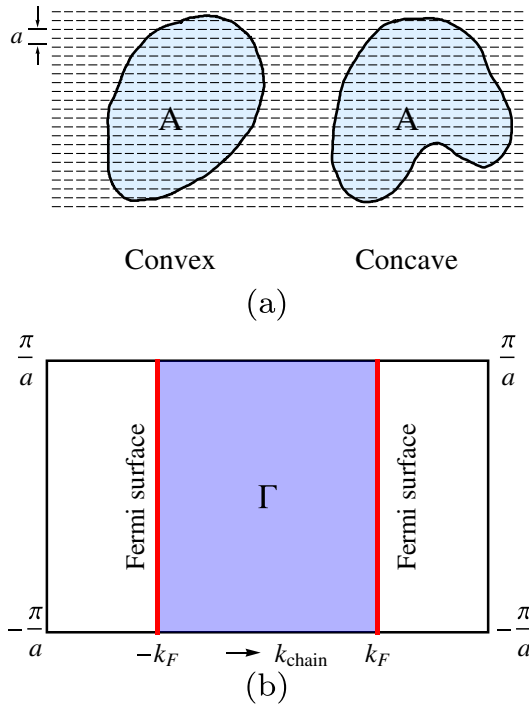

FIG. 1. The toy model both in real space and in momentum space. (a) A set of parallel decoupled $1d$ chains of spinless free fermions (dashed lines); the subsystem division is represented by the solid lines, both convex and concave geometries. (b) Fermi surfaces of the toy model.

seems. The Fermi-liquid interactions couple a family of toy models aligned along different directions in the language of high-dimensional bosonization, and lead to a correction to the entanglement entropy of approximately $\mathcal{O}(1) \times \log L$.

III. MULTI-DIMENSIONAL BOSONIZATION

The scheme of multi-dimensional bosonization was first introduced by Haldane [28], followed by others [29–32]. The basic idea is to start with a low-energy effective Hamiltonian (obtained through a renormalization group approach) restricted to within a thin shell of thickness λ around the Fermi surface, $k_F - \lambda/2 < k < k_F + \lambda/2$. Then one divides this thin shell into N patches with dimensionality $\sim \Lambda^{d-1} \times \lambda$ as shown in Fig. 2, in such a way that $\lambda \ll \Lambda \ll k_F$ and $\Lambda^2/k_F \ll \lambda$, where $d = 2, 3$ is the space dimension, and Λ is the linear dimension of the tangential extent of each patch. The condition $\lambda \ll \Lambda$ minimizes interpatch scattering; $\Lambda \ll k_F$ and $\Lambda^2/k_F \ll \lambda$ together make the curvature of the Fermi surface negligible. In the end, we shall take the limit $\Lambda/k_F \to 0$, so that the sum over all patches can be converted to an integral over the Fermi surface. In this work, we treat the free theory in general d dimensions, but we shall restrict ourselves to $d = 2$ when interactions are included. For an arbitrary patch $\mathbf{S}$, labeled by the Fermi momentum $\mathbf{k}_S$ at the center of the patch, we introduce the patch fermion-field operator

$$\psi(\mathbf{S}; \mathbf{x}) = e^{i\mathbf{k}_S \cdot \mathbf{x}} \sum_{\mathbf{p}} \theta(\mathbf{S}; \mathbf{p}) e^{i(\mathbf{p} - \mathbf{k}_S) \cdot \mathbf{x}} \psi_{\mathbf{p}}, \qquad (3)$$

where $\psi_{\mathbf{p}}$ is the usual fermion field in momentum space,

$$\theta(\mathbf{S}; \mathbf{p}) = \begin{cases} 1 & \text{if } \mathbf{p} \text{ lies in patch } \mathbf{S}, \\ 0 & \text{if } \mathbf{p} \text{ lies outside patch } \mathbf{S}. \end{cases}$$

The effective Fermi-liquid Hamiltonian can be written as

$$H[\psi^{\dagger}, \psi] = \int d^d x \sum_{\mathbf{S}} \psi^{\dagger}(\mathbf{S}; \mathbf{x}) \left(\frac{\mathbf{k}_S}{m^*} \cdot \nabla \right) \psi(\mathbf{S}; \mathbf{x})$$
$$+ \int d^d x \, d^d y \sum_{\mathbf{S}, \mathbf{T}} V(\mathbf{S}, \mathbf{T}; \mathbf{x} - \mathbf{y}) \psi^{\dagger}(\mathbf{S}; \mathbf{x})$$
$$\times \psi(\mathbf{S}; \mathbf{x}) \psi^{\dagger}(\mathbf{T}; \mathbf{y}) \psi(\mathbf{T}; \mathbf{y}), \qquad (4)$$

where m^* is the effective mass, and $V(\mathbf{S}, \mathbf{T}; \mathbf{x} - \mathbf{y})$ is the effective interaction between patch $\mathbf{S}$ and $\mathbf{T}$ in the forward-scattering channels. Even though this model is restricted to special interactions of this form, forward scattering is known to be the only marginal interaction in renormalization-group analysis [40]. As the leading-order contribution of the entanglement entropy is dominated by the low-energy modes around the Fermi surface, it is
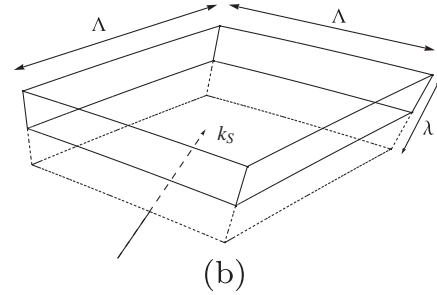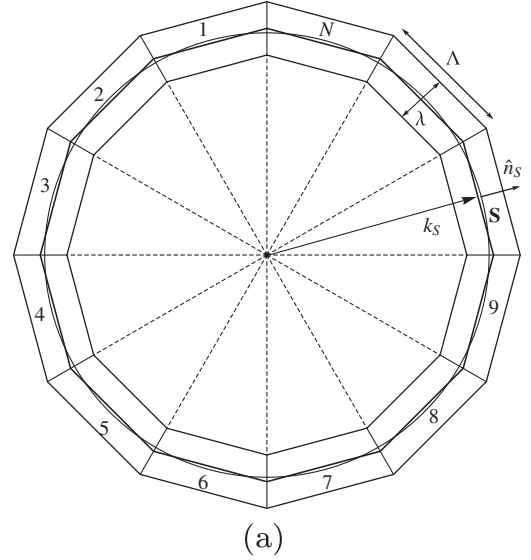

(a)

(b)

FIG. 2. Patching of the Fermi surface. The low-energy theory is restricted to within a thin shell about the Fermi surface with a thickness $\lambda \ll k_F$, in the sense of renormalization. The thin shell is further divided into N different patches; each has a transverse dimension Λ^{d-1} where $d = 2, 3$ is the space dimensions. The dimensions of the patch satisfy three conditions: (1) $\lambda \ll \Lambda$ minimizes interpatch scattering; (2) $\Lambda \ll k_F$ and $\Lambda^2/k_F \ll \lambda$ together makes the curvature of the Fermi surface negligible. (a): Division of a 2D Fermi surface into N patches. Patch $\mathbf{S}$ is characterized by the Fermi momentum $\mathbf{k}_S$. (b): A patch for $d = 3$. The patch has a thickness λ along the normal direction and a width Λ along the transverse direction(s).

sufficient to consider this model. Similar to the 1D case, the bosonic degrees of freedom are the density modes of the system. In this case, they are defined within each patch of the Fermi surface:

$$J(\mathbf{S}; \mathbf{q}) = \sum_{\mathbf{k}} \theta(\mathbf{S}; \mathbf{k} - \mathbf{q}) \theta(\mathbf{S}; \mathbf{k}) \left\{ \psi^{\dagger}_{\mathbf{k}-\mathbf{q}} \psi_{\mathbf{k}} - \delta^d_{\mathbf{q},0} \langle \psi^{\dagger}_{\mathbf{k}} \psi_{\mathbf{k}} \rangle \right\}. \tag{5}$$

Although $\mathbf{q}$ is not explicitly bounded in the above definition of the patch-density operator, its transverse components $\mathbf{q}_{S\perp} = (q^{(1)}_{S\perp}, \ldots, q^{(\alpha)}_{S\perp}, \ldots, q^{(d-1)}_{S\perp})$ (those components parallel to the Fermi surface) are limited

$q_{S\perp}^{(\alpha)} \in (-\Lambda, \Lambda)$ due to the patch confinement. Their commutation relation is

$$[J(S;\boldsymbol{q}), J(T;\boldsymbol{p})]$$
$$\simeq \delta_{S,T}\delta_{\boldsymbol{q}+\boldsymbol{p},0}^d \sum_k \theta(S;\boldsymbol{k})[\theta(S;\boldsymbol{k}-\boldsymbol{q})-\theta(S;\boldsymbol{k}+\boldsymbol{q})]n_{\boldsymbol{k}} \quad (6)$$

$$= \delta_{S,T}\delta_{\boldsymbol{q}+\boldsymbol{p},0}^d \Omega(\hat{\boldsymbol{n}}_S \cdot \boldsymbol{q})\theta_2(\boldsymbol{q}_{S\perp}) + \mathcal{O}(\lambda/\Lambda), \quad (7)$$

where

$$\theta_2(\boldsymbol{q}_{S\perp}) = \prod_\alpha (1 - q_{S\perp}^{(\alpha)}/\Lambda), \quad (8)$$

$\Omega = \Lambda^{d-1}[L_0/(2\pi)]^d$, $n_{\boldsymbol{k}} = \langle \psi_{\boldsymbol{k}}^\dagger \psi_{\boldsymbol{k}} \rangle$ is the occupation number of state with momentum $\boldsymbol{k}$, $\hat{\boldsymbol{n}}_S$ is the outward normal direction of patch S, $\boldsymbol{q}_{S\perp}$ represents all other components of $\boldsymbol{q}$ that are perpendicular to $\hat{\boldsymbol{n}}_S$, and L_0 is the linear dimension of the entire system. The appearance of $\delta_{\boldsymbol{q}+\boldsymbol{p},0}$ is a result of momentum conservation. The calculation of the commutator is reduced to computing the difference of occupied states, i.e., the area difference below the Fermi surface, between the two θ functions $(\theta(S;\boldsymbol{k}-\boldsymbol{q})-\theta(S;\boldsymbol{k}-\boldsymbol{p}))$, as indicated by Eq. (6). This is similar to 1D bosonization. If we consider both $\boldsymbol{k}$ and $\boldsymbol{q}$ to be 1D momenta, Eq. (6) would give us the 1D bosonization commutator. The 2D result Eq. (7) is similar, because the Fermi surface confined within the patch is essentially flat, and thus the dispersion is 1D. That leads to the $\hat{\boldsymbol{n}}_S \cdot \boldsymbol{q}$ dependence of the commutator as that of the 1D case, even for $\boldsymbol{q}_{S\perp} \neq 0$. The difference is that, as illustrated in Fig. 3, due to the patch confinement on the
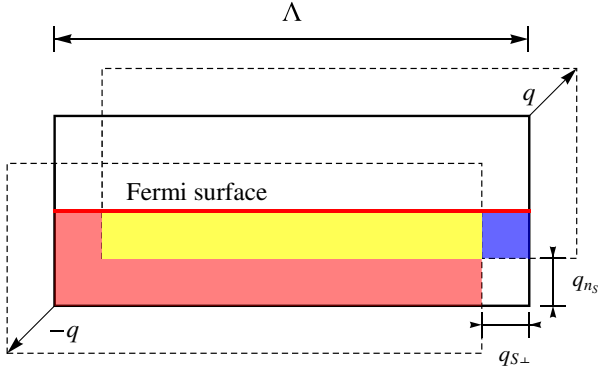

FIG. 3. Origin of the bosonic commutator of patch density operators illustrated for $d = 2$. As shown in Eq. (7), the commutator is reduced to computing the difference of occupied states, i.e., the area difference below the Fermi surface, between the two θ functions $(\theta(S;\boldsymbol{k}-\boldsymbol{q})-\theta(S;\boldsymbol{k}+\boldsymbol{p}))$. The solid box indicates the original patch, or $\theta(S;\boldsymbol{k})$. The red line shows the Fermi surface. Both $\theta(S;\boldsymbol{k}-\boldsymbol{q})$ and $\theta(S;\boldsymbol{k}+\boldsymbol{p})$ are denoted by dashed boxes. The occupied part in $\theta(S;\boldsymbol{k}+\boldsymbol{q})$ is denoted by blue, that of $\theta(S;\boldsymbol{k}-\boldsymbol{q})$ is denoted by red, and the overlapping region is denoted by yellow. Subtracting the remaining blue area from the red, we obtain that $\theta(S;\boldsymbol{k}-\boldsymbol{q})$ occupies $(\Lambda - q_{S\perp})q_{\hat{\boldsymbol{n}}_S}$ more states, which gives us the commutator.

transverse direction(s), when $\boldsymbol{q}_{S\perp}$ increases, $\boldsymbol{k} \pm \boldsymbol{q}$ would increasingly find itself outside the patch, and thus not contributing to the commutator. According to Fig. 3, one can see that this gives rise to the factor $\theta_2(\boldsymbol{q}_{S\perp})$, which diminishes the commutator at large $\boldsymbol{q}_{S\perp}$. It is usually neglected in the literature because the long-wavelength limit is taken [29–32]. However, as it is important in the present context to correctly count the number of total degrees of freedom, this $\theta_2(\boldsymbol{q}_{S\perp})$ factor cannot be neglected because it comes from counting the transverse degrees of freedom. To simplify things, we replace $\theta_2(\boldsymbol{q}_{S\perp})$ by

$$\theta_2(\boldsymbol{q}_{S\perp}) = 1 \quad \text{for } -\Lambda/2 < q_{S\perp}^{(\alpha)} < \Lambda/2, \quad (9)$$

and we also limit $\boldsymbol{q}_{S\perp}$ to this range. This approximation makes it easier to do a Fourier transform while keeping the total degrees of freedom intact. To see that, it is sufficient to consider one direction, comparing the area enclosed by the two different functions: $\theta_2(q_\perp) = 1 - q_\perp/\Lambda$ over the range $(-\Lambda, \Lambda)$ and $\theta_2(q_\perp)=1$ over the range $(-\Lambda/2, \Lambda/2)$. Both functions enclose the same area and thus the same number of states. This approximation can also be interpreted as relaxation of the hard-wall cutoff in Eq. (5) and softening of the step function $\theta(S;\boldsymbol{k})$. In Eq. (5), $\boldsymbol{q}$ is not bounded while $\boldsymbol{k}$ is bounded by $\theta(S;\boldsymbol{k})$. If we relax the restriction on $\boldsymbol{k}$ in the transverse direction, allowing $\boldsymbol{k}$ with $|k_{S\perp}^{(\alpha)}| > \Lambda/2$ in the summation, but require $q_{S\perp}^\alpha$ to be bounded within the patch, we would obtain the alternative $\theta_2(q_\perp)$.

Using the above approximation from now on, we construct the local bosonic degrees of freedom $\phi(S;\boldsymbol{x}) = \phi(S;x_S, \boldsymbol{x}_{S\perp})$ as

$$J(S;\boldsymbol{x}) = \sqrt{\Omega}\,\partial_{x_S}\phi(S;x_S, \boldsymbol{x}_{S\perp}), \quad (10)$$

where $J(S;\boldsymbol{x}) = \sum_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} J(S;\boldsymbol{q})$, $x_S = \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_S$, and $\boldsymbol{x}_{S\perp} = \boldsymbol{x} - (\boldsymbol{x}\cdot\hat{\boldsymbol{n}}_S)\hat{\boldsymbol{n}}_S$. The commutation relations for the ϕs are then

$$[\partial_{x_S}\phi(S;\boldsymbol{x}), \phi(T;\boldsymbol{y})]$$
$$= i2\pi\Omega\,\delta_{S,T}\delta(x_S - y_S)\prod_{\alpha=1}^{d-1}\left(\frac{\sin[\Lambda(x_{S\perp}^{(\alpha)} - y_{S\perp}^{(\alpha)})]}{2\pi(x_{S\perp}^{(\alpha)} - y_{S\perp}^{(\alpha)})}\right) \quad (11)$$

which is the bosonic-commutation relation we are looking for. The factor

$$\prod_\alpha \left(\frac{\sin[\Lambda(x_{S\perp}^{(\alpha)} - y_{S\perp}^{(\alpha)})]}{2\pi(x_{S\perp}^{(\alpha)} - y_{S\perp}^{(\alpha)})}\right)$$

arising from transverse directions must be treated with care in different circumstances. In most of the literature, the focus is on the physics at large-length scale $l \gg 1/\Lambda$; therefore, this factor is usually approximated by $\delta^{d-1}(\boldsymbol{x}_{S\perp} - \boldsymbol{y}_{S\perp})$, which is good in that limit without further discussion. This is also what we shall do for most of the discussion unless noted otherwise:

$$[\partial_{x_S}\phi(S;x),\,\phi(T;y)]\xrightarrow{|x_{S\perp}^{(\alpha)}-y_{S\perp}^{(\alpha)}|\gg 1/\Lambda}$$
$$\simeq i2\pi\Omega\delta_{S,T}^{d-1}\delta(x_S-y_S)\delta^{d-1}(x_{S\perp}-y_{S\perp}). \quad (12)$$

However, the more accurate expression of Eq. (11) is useful to help us understand how to count the transverse degrees of freedom correctly. It tells us that the transverse degrees are not independent of the short-length scale $l < 1/\Lambda$. More important, later on, we will need to consider the limit $\delta^{d-1}(x_{S\perp}-y_{S\perp})|_{y_{S\perp}\to x_{S\perp}}$; without Eq. (11), this limit would be ill defined.

With the above equations, the Hamiltonian $H[\psi^\dagger,\psi]$ is found to be quadratic in terms of these $J(S;q)$s:

$$H[\psi^\dagger,\psi] = \frac{1}{2}\sum_{S,T;q}\frac{v_F^*\delta_{S,T}}{\Omega}J(S;-q)J(T;q)$$
$$+ V(S,T;q)J(S;-q)J(T;q), \quad (13)$$

where $V(S,T;q)$ is the Fourier transform of $V(S,T;x-y)$, so it is also quadratic in the bosonic fields associated with the $J(S;q)$s.

A. Entanglement entropy of free fermions

The kinetic energy part of Eq. (4) or its bosonized version Eq. (13) can be written in terms of the boson fields constructed above as

$$H_0 = \frac{1}{2}\sum_{S;q}\frac{v_F^*}{\Omega}J(S;-q)J(S;q)$$
$$= \frac{2\pi v_F^*}{\Omega V}\sum_S\int d^2x[\partial_{x_S}\phi(S;x)]^2. \quad (14)$$

We see that there is no coupling between different patches. The theory is thus formally a tensor product of many independent theories, one for each patch. We can therefore calculate the entanglement entropy patch by patch and sum up contributions from each patch in the end. Within a single patch, there is no dynamics in the perpendicular direction as dictated by the Hamiltonian, and the problem is reduced to a *one-dimensional* problem. Note that transverse degrees of freedom are not completely independent. According to Eq. (11), the commutator is nonvanishing for $x_{S\perp}\neq y_{S\perp}$ up to a length scale of $2\pi/\Lambda$, as a consequence of restricting $q_{S\perp}$ to within the range $[-\Lambda/2,\Lambda/2]$. Physically, one can view this as *discretization* along the transverse direction due to a restricted momentum range, similar to the relation between a lattice and its Brillouin zone. In this view, the single-patch problem is reduced to a 1D problem with a chain density of $(\Lambda/(2\pi))^{d-1}$. Therefore, the Hamiltonian (14) becomes

$$H_0 = \frac{2\pi v_F^*}{\Omega V}\sum_{S;x_{S\perp}}\int dx_S[\partial_{x_S}\phi(S;x)]^2. \quad (15)$$

Note that the bosonized theory of a single patch is chiral. To directly make use of our toy model, we need to consider two patches having opposite $\hat{n}_S$ simultaneously. This is because, for a 1D-fermion model at nonzero filling, there are two Fermi points. Both need to be considered to construct well-defined local degrees of freedom. Once we consider two such patches together, it is more convenient to combine the two chiral theories into a nonchiral theory. We will do this for the rest of this work. Next, we introduce the nonchiral fields

$$\varphi(S;x) = \tfrac{1}{\sqrt{2}}(\phi(S;x)-\phi(-S;x)),$$
$$\chi(S;x) = \tfrac{1}{\sqrt{2}}(\phi(S;x)+\phi(-S;x)), \quad (16)$$

where $-S$ indicates the patch with normal direction opposite to that of patch S: $\hat{n}_{-S}=-\hat{n}_S$. One finds that χ and φ are mutually dual fields with S restricted to one hemisphere, but $\partial_{x_S}\varphi$ and φ now commute while χ and φ have a nontrivial commutator:

$$[\varphi(S;x),\partial_{y_S}\varphi(S;y)]=[\chi(S;x),\partial_{y_S}\chi(S;y)]=0,$$
$$[\partial_{x_S}\varphi(S;x),\chi(T;y)]=[\partial_{x_S}\chi(S;x),\varphi(T;y)]$$
$$=2i\pi\Omega\delta_{S,T}\delta(x_S-y_S)\delta^{d-1}(x_{S\perp}-y_{S\perp}). \quad (17)$$

Therefore, two patches with opposite $\hat{n}_S$ are equivalent to a set of ordinary 1D boson fields. Throughout the rest of this work, we shall assume that this chiral-to-nonchiral transformation is done, and, when we refer to patches, we always refer to the two companion patches that together form a nonchiral patch. For the nonchiral boson theory, it is known that the entanglement entropy of a single interval (with two end points) is $\frac{1}{3}\log L$.

Before we proceed further, we note that the relationship between boson fields and the original fermion fields is not completely local. However, the underlying physical quantity that matters is not the fermion fields, but the fermion density, or in other words, the fermion-number basis that one chooses in order to expand the Hilbert space of the problem. This physical basis is also what one uses to perform the partial trace. The fermion-density operator obeys a locally one-to-one corresponding relation to the boson fields. Thus, we argue that, in 1D, the nonlocal relation between the fermion and boson fields does not affect the partial trace operation, and also does not affect the calculation of entanglement entropy.

By referring to our result for the toy model, we readily obtain the contribution from a single patch,

$$S(S) = \frac{1}{12}\log L\oint_{\partial A}|\hat{n}_S\cdot d\vec{S}_x|\times\left(\frac{\Lambda}{2\pi}\right)^{d-1}, \quad (18)$$

where an additional factor of $\frac{1}{2}$ has been introduced in order to count only once each pair of patches that forms a nonchiral theory. Identifying $\hat{n}_S\Lambda^{d-1}$ as the surface element at the Fermi surface $d\vec{S}_k$ and taking the $N\to\infty$ limit, the total entanglement entropy is

$$S = \frac{1}{12(2\pi)^{d-1}} \log L \oint_{\partial A} \oint_{\partial \Gamma} |d\vec{S}_k \cdot d\vec{S}_x|. \quad (19)$$

So we recover the GK result for generic free fermions.

B. Solution for the Fermi-liquid case and nonlocality of the Bogoliubov fields

When Fermi-liquid interactions (forward scattering) are included, the full Hamiltonian will no longer be diagonal in the patch index S. But it is still quadratic in terms of the patch-density operators, i.e., the bosonic degrees of freedom, and can be diagonalized by a Bogoliubov transformation. According to Eq. (7) and ignoring terms of $\mathcal{O}(\lambda/\Lambda)$, one can define a set of boson creation/annihilation operators $\hat{a}^\dagger(q)/\hat{a}(q)$ as follows:

$$\phi(S; x) = i \sum_{q, \hat{n}_S \cdot q > 0} \frac{a^\dagger(S; q)e^{-iq \cdot x} - a(S; q)e^{iq \cdot x}}{\sqrt{\hat{n}_S \cdot q}}. \quad (20)$$

It can be shown that the full Hamiltonian is diagonal in q, and it can be diagonalized by a Bogoliubov transformation [32] independently for each q sector. In Ref. [32], only a Hubbard-U-like interaction is considered for practical reasons. However, in principle, such a Bogoliubov transformation also applies to general interactions:

$$\hat{a}_i(q) = \sum_j u_{ij}\alpha_j(q) + v_{ij}\beta_j^\dagger(q),$$
$$\hat{b}_i(q) = \sum_j u_{ij}\beta_j(q) + v_{ij}\alpha_j^\dagger(q), \quad (21)$$

where both i and j refer to the patch index, and α_j and β_j are the Bogoliubov bosonic-annihilation operators that diagonalize the Hamiltonian. With the proper choice of u's and v's, the Hamiltonian can be readily diagonalized. Reference [32] solves the Hubbard-U-like interaction and provides a successful description of Fermi liquids, even in the strong-U limit.

However, even for this simple case in which U has no dependence on q, the Bogoliubov transformation still depends on q. To be more precise, u_{ij} and v_{ij} will depend only on the angle between the normal direction $\hat{n}_S$ of the patch and q, leading to discontinuities in the derivatives at $q = 0$. Consequently, the real-space fields constructed from the Bogoliubov operators α_j and β_j are *no longer* local with respect to the original boson fields. The real-space Bogoliubov fields are constructed in a manner similar to Eq. (20):

$$\tilde{\phi}(S; x) = i \sum_{q, \hat{n}_S \cdot q > 0} \frac{\alpha^\dagger(S; q)e^{-iq \cdot x} - \alpha(S; q)e^{iq \cdot x}}{\sqrt{\hat{n}_S \cdot q}}. \quad (22)$$

Then one can show that the original local degrees of freedom $\phi(S; x)$ can be expressed in terms of the above Bogoliubov fields as

$$\phi(S; x) = \tilde{\phi}(S; x) + \int dy \sum_l f(S, l; x - y)\tilde{\phi}(l; y), \quad (23)$$

where $f(S, l; x - y)$ is typically long-range, even for the short-range Hubbard-U interaction. For more general cases, with further q dependence in the interaction, the nonlocality would only be enhanced. The loss of locality prevents us from calculating the entanglement entropy directly using those eigen modes, since it is difficult to implement the partial trace using those nonlocal degrees of freedom. Therefore, although the Bogoliubov fields have a local core, as we would expect for Fermi liquids from adiabaticity, they *do* acquire a nonlocal dressing due to interaction. Although, in principle, the partial trace can be performed with those Bogoliubov fields, such nonlocality makes it difficult, and we have not been able to do it, which further renders calculating the entanglement entropy impossible. This situation is very different from the 1D theory, where for local interactions the eigen fields remain local, since there are only two Fermi points. There, the transformation can never involve such angular q dependence due to limited dimensionality. Despite these technical difficulties, the nonlocality may suggest possible corrections to the entanglement entropy. This is indeed the case, as revealed by our calculation for Fermi-liquid interactions (shown later), although, with the Fermi-liquid interaction calculations, such extra contributions are only of $\mathcal{O}(1) \times \log L$, which is of $\mathcal{O}(1/L)$ when compared to the leading term. This shows that the mode-counting argument in Ref. [26], although correctly suggesting the $\log L$ violation of the area law for Fermi liquids, does not always fully account for all sources of entanglement entropy.

IV. ENTANGLEMENT ENTROPY FROM THE GREEN'S FUNCTION

In order to preserve locality, we need to work with the original local degrees of freedom. To do that, we adopt the approach used by Calabrese and Cardy [41] (CC) on calculating the entanglement entropy of a free-massive-1D-bosonic field theory. The calculation is done in terms of the Green's function by applying the replica trick. In our case, we find that the CC approach can be generalized in a special geometry for solving the interacting theory which is quadratic after bosonization. In this way, we avoid diagonalizing the Hamiltonian and thus also avoid the nonlocality issue. However, we do have to regularize the theory by adding a mass term by hand. In the end, we shall take the small mass limit and replace the divergent correlation length $\xi \sim 1/m$ by the subsystem size L. The regularization procedure facilitates the calculation, but it also strictly restricts us to computing the entanglement entropy only at the $\log L$ level.

In this section, by using the replica trick, we convey the calculation of entanglement entropy into computing the Green's function on an n-sheeted replica manifold. We first demonstrate the method by applying it to free-fermion theory in d-dimensional space; then, based on that demonstration, we compute the entanglement entropy

perturbatively for a simple Fermi-liquid theory in powers of the interaction strength up to the second order.

A. The replica trick and its application to 1D free-bosonic theory

In this section, we briefly describe the replica trick in $(1+1)$ space-time dimensions $[(1+1)d]$ so that, later on, we can straightforwardly generalize it to $(2+1)$ space-time dimensions $[(2+1)d]$ accordingly for our problem.

The replica trick makes use of the following identity:

$$S_A = -\text{tr}(\rho_A \ln \rho_A) = -\lim_{n\to 1} \frac{\partial}{\partial n} \text{tr}\rho_A^n. \quad (24)$$

To compute $\text{tr}\rho_A^n$, CC use a path integral to express the density matrix ρ in terms of the boson fields

$$\rho(\{\phi(x)\}|\{\phi(x')'\}) = Z^{-1}\langle\{\phi(x)\}|e^{-H}|\{\phi(x')'\}\rangle, \quad (25)$$

where $Z = \text{tr}\, e^{-\beta H}$ is the partition function, β is the inverse temperature, and $\{\phi(x)\}$ are the corresponding eigenstates of $\hat{\phi}(x)$: $\hat{\phi}(x)|\{\phi(x')\}\rangle = \phi(x')|\{\phi(x')\}\rangle$. ρ can be expressed as a (Euclidean) path integral:

$$\rho = Z^{-1}\int[d\phi(x,\tau)]\prod_x\delta[\phi(x,0)-\phi(x)']$$
$$\times\prod_x\delta[\phi(x,\beta)-\phi(x)'']e^{-S_E}, \quad (26)$$

where $S_E = \int_0^\beta L_E d\tau$, with L_E being the Euclidean Lagrangian. The normalization factor Z, i.e., the partition function, is found by setting $\{\phi(x)''\} = \{\phi(x)'\}$ and integrating over these variables. This procedure has the effect of sewing together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β, as illustrated in Fig. (4(a)).

The reduced density matrix of an interval $A = (x_i, x_f)$ can be obtained by sewing together only those points that are not in the interval A, which has the effect of leaving an open cut along the line $\tau = 0$ as shown in Fig. 4(b). To compute ρ_A^n, we make n copies (replicas) of the above setup, labeled by an integer k with $1 \le k \le n$, and sew them together cyclically along the open cut so that $\phi(x)_k' = \phi(x)_{k+1}''$ [and $\phi(x)_n' = \phi(x)_1''$] for all $x \in A$. In Fig. 5(a), we show the case for $n = 2$. Let us denote the path integral on this n-sheeted structure (known as an n-sheeted Riemann surface) by $Z_n(A)$. Then
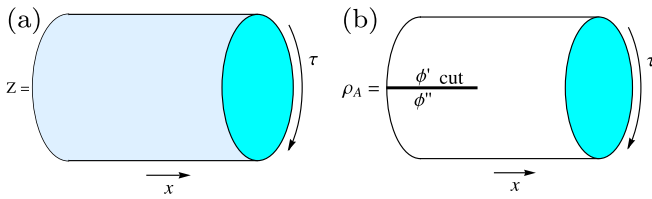

FIG. 4. Path integral representation of the reduced density matrix. (a) When we sew $\phi(x)' = \phi(x)''$ together for all x's, we get the partition function Z. (b) When we sew only $x \notin A$ together, we get ρ_A.
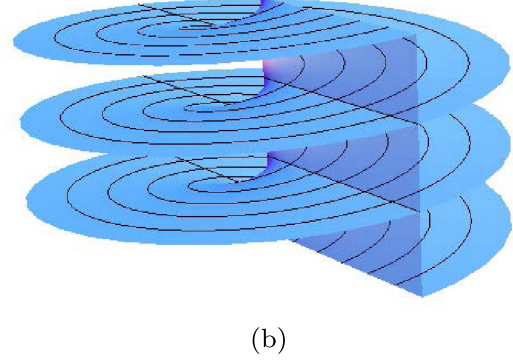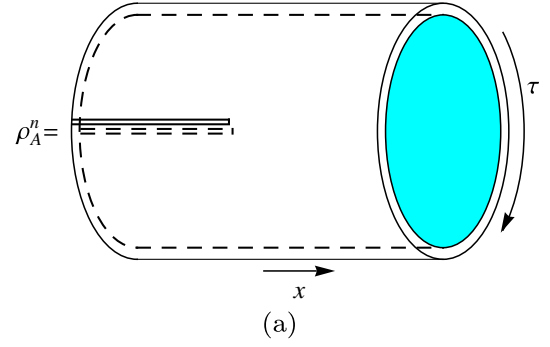

(a)

(b)

FIG. 5. Formation of the n-sheeted Riemann surface in the replica trick. By sewing n copies of the reduced density matrices together, one obtains the replica partition function Z_n. In the zero temperature limit, $\beta \to \infty$, each cylinder representing one copy of ρ_A becomes an infinite plane. Those n-planes sewed together form a n-sheeted Riemann surface in Fig. 5 which can be simply realized by enforcing a $2n\pi$ periodicity on the angular variable of the polar coordinates of the $(1+1)d$ plane instead of the usual 2π one. (a): n copies of the reduced density matrices. For clarity only $n = 2$ is shown. (b): Visualization of a n-sheeted Riemann surface.

$$\text{tr}\,\rho_A^n = \frac{Z_n(A)}{Z^n}, \quad (27)$$

so that

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{Z_n(A)}{Z^n}. \quad (28)$$

If we consider the theory as that of a single field living on this complex n-sheeted Riemann surface instead of a theory of n copies, it is possible to remove the replica index n from the fields, and instead consider a problem defined on such an n-sheeted Riemann surface that can be realized by imposing proper boundary conditions.

In Ref. [41], CC consider the entanglement entropy between the two semi-infinite 1D systems (i.e., cutting an infinite chain into two halves at $x = 0$) for massive-free-boson fields. For such geometry, as illustrated in Fig. 5, the n-sheeted-Riemann-surface constraint is realized by imposing a $2n\pi$ periodicity on the angular variable of the polar coordinates of the $(1+1)d$ plane instead of the usual

2π periodicity. In this way, the $(1+1)d$ variable $\boldsymbol{x} = (x, \tau)$ acquires n branches $\boldsymbol{x}_n$, and each branch corresponds to one copy of ϕ. Notationwise, we obtain

$$\phi(x, \tau)_k \Rightarrow \phi(\boldsymbol{x}_k) \Rightarrow \phi(\boldsymbol{x}), \qquad (29)$$

and the sewing conditions $\phi(x)'_k = \phi(x)''_{k+1}$ simply become the continuity condition for $\phi(\boldsymbol{x})$ across its consecutive branches. Here, we use a generalized polar coordinate: $\boldsymbol{x} = (r, \theta)$ with $0 < r < \infty$, and $0 \le \theta < 2n\pi$.

The massivefree-boson theory considered by CC is defined by the following action:

$$\mathcal{S} = \int \tfrac{1}{2}[(\partial_\mu \phi)^2 - m^2 \phi^2] d^2 r.$$

The $(1+1)d$ bosonic Green's function $G_{0,b}^{(n)}(\boldsymbol{r}, \boldsymbol{r}') = \langle \phi(\boldsymbol{r}) \phi(\boldsymbol{r}') \rangle$ on the n-sheeted Riemann surface satisfies the differential equation

$$(-\nabla_r^2 + m^2) G_{0,b}^{(n)} = \delta(\boldsymbol{r} - \boldsymbol{r}').$$

To compute the partition function, one can make use of the identity

$$\frac{\partial}{\partial m^2} \log Z_n = -\frac{1}{2} \int d^{d+1} x G^{(n)}(\boldsymbol{x}, \boldsymbol{x}). \qquad (30)$$

Note that here the integration is over the entire n-sheeted space. Equation (30) is applicable to general quadratic theories of bosons, and we will apply it in Sec. IV D to bosonized theories of interacting fermions. In Eq. (30) we use $G^{(n)}(\boldsymbol{x}, \boldsymbol{x}')$, a general two-point correlation function on the n-sheeted Riemann surface in d-dimensional space for later use, instead of the specific $G_{0,b}^{(n)}$ defined in the equation just above Eq. (30). Accordingly, S_A is then given as

$$S_A = -\lim_{n \to 1} \frac{\partial}{\partial n} e^{-(1/2) \int dm^2 \int d^{D+1} x [G^{(n)}(\boldsymbol{x}, \boldsymbol{x}) - n G^{(1)}(\boldsymbol{x}, \boldsymbol{x})]}. \quad (31)$$

Here and in the following, we will leave it understood that the first term in the integrand is integrated over the n-sheeted geometry, whereas the second is integrated over a one-sheeted geometry. There should be no confusion, as the superscript of G generally indicates the geometry.

The benefit of the above approach is that the two-point correlation function, or Green's function, defined in terms of certain differential equations obtained from the equation of motion, can be solved for on the n-sheeted Riemann surface, thus enabling us to compute the entanglement entropy. Although CC's work considers only massive-$(1+1)d$ boson fields, it is also applicable to our case. The price one has to pay is to introduce a mass term for regularization. At the end of the calculation, the inverse mass, which is the correlation length of the system, shall be considered to be on the same scale as L: $1/m \sim L$, where L is the characteristic length scale of the subsystem. The validity of such consideration is well-established in other

cases [35,42], where the correlation length is set by either finite temperature or mass. The only modification necessary to apply the above approach to a bosonized Fermi surface in higher dimensions is to introduce a sum over the patch index.

B. Geometry and replica boundary conditions

Throughout the remainder of this work, instead of the general geometry considered earlier, we work with a special half-cylinder geometry, as shown in Fig. 6(a): The system is infinite in the $\hat{x}$ direction while obeying the periodic boundary condition along the $\hat{y}$ direction with length L. The system is cut along the $\hat{y}$ axis so that we are computing the entanglement entropy between the two half planes. We require L to be large so that it can be considered approximately ∞ unless otherwise noted.

We choose such a simple geometry for the following reasons. Cutting the system straight along the $\hat{y}$ direction, yielding a two-half-plane geometry, is a straightforward
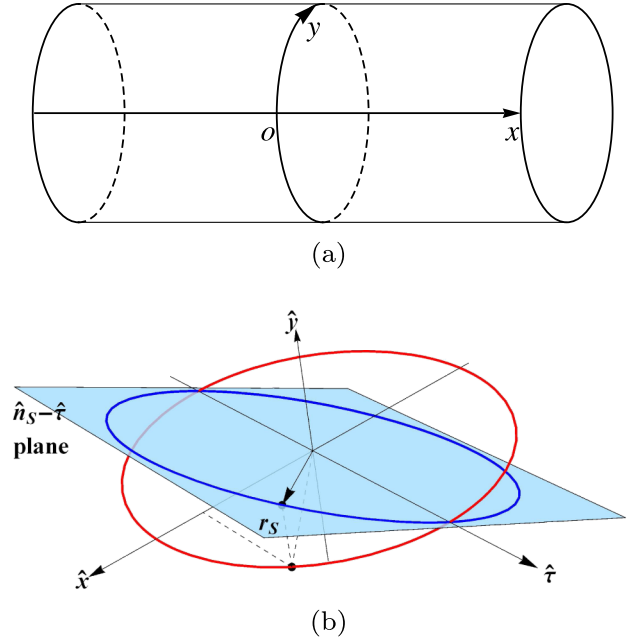

(a)

(b)

FIG. 6.   The half-cylinder geometry and equivalence of boundary conditions in $\hat{x}$-$\hat{\tau}$ and $\hat{n}_S$-$\hat{\tau}$ planes. The system is infinite in the $\hat{x}$ direction while obeys periodic boundary condition along the $\hat{y}$ direction with length L. The system is cut along the $\hat{y}$ axis so that we are computing the entanglement entropy between the two half planes. (a): The half-cylinder geometry. (b): The projection of $\boldsymbol{r}_S$ onto the $\hat{x}$-$\hat{\tau}$ plane. Consider polar coordinates of an arbitrary $\hat{n}_S$-$\hat{\tau}$ plane (the blue plane). Since the polar coordinates in the $\hat{x}$-$\hat{\tau}$ plane satisfy the $2n\pi$ periodic boundary condition, consider the one-to-one projection of the vector $\boldsymbol{r}_S$ onto the $\hat{x}$-$\hat{\tau}$ plane. Consider, if we move the vector in the $\hat{x}$-$\hat{\tau}$ plane around the origin n times (the red circle). Because of the one-to one mapping, $\boldsymbol{r}_S$ should also move around the origin n times (the blue "circle"; it is actually an eclipse), thus obeys the $2n\pi$ periodicity as well.

$(2 + 1)d$ generalization of the semi-infinite chain geometry considered by CC. It makes any straight line intersect the boundary only once, dividing the line into two semi-infinite segments, for all patch directions, as in the 1D case, except for lines parallel to the $\hat{n}_S = \hat{y}$ patch direction. The degrees of freedom associated with this special patch do not contribute to the entanglement entropy, since they are not coupled (have no dynamics) along $\hat{x}$, and are of measure zero in the large-patch-number limit anyway.

For this simple geometry, the $(2 + 1)d$ n-sheeted geometry is constructed from n identical copies,

$$S^n = \{(x, y, \tau) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}, \qquad (32)$$

sliced along "branch cuts,"

$$C^n = \{(x, y, \tau) \in \mathbb{R}^- \times \mathbb{R} \times \{0\}\}, \qquad (33)$$

and then appropriately glued together along these cuts. This happens exactly as in 1D, and the y coordinate is so far a mere spectator. This geometry defines an n-sheeted, or, in this case, more appropriately, the n-layered, replica manifold which is a simple enough generalization of the $(1 + 1)d$ case. The n-sheeted Riemann surface, as discussed in Sec. IVA and shown in Fig. 5(b), now acquires an extra direction $\hat{y}$ perpendicular to the $\hat{x}$-$\hat{\tau}$ plane. It can still be implemented by imposing the same $2n\pi$ periodicity boundary conditions on θ, the angular variable of the polar coordinates $(x, \tau) = (r\cos\theta, r\sin\theta)$ in the $\hat{x}$-$\hat{\tau}$ plane. Therefore, we can safely make use of the CC result, i.e., the solution to the Green's function on an n-sheeted Riemann surface, in the free-fermion theory, and can further use it as a starting point for treating the interacting theory. This is obviously true for the patch with $\hat{n}_S = \hat{x}$, but it also holds for general $\hat{n}_S$, as we shall validate next.

For a general patch direction $\hat{n}_S$, the noninteracting Green's function associated with this patch embodies correlations in the affine $\hat{n}_S$-$\hat{\tau}$ "planes." The geometry of each such plane is that of the n-sheeted Riemann surface of the $(1 + 1)d$ problem, as we will now argue. With each patch direction, we thus associate a different foliation of the n-sheeted $(2 + 1)d$ geometry into $(1 + 1)d$ counterparts.

To be more precise, for given patch S, instead of the Cartesian coordinates (x, y, τ), we consider a parallel and perpendicular decomposition $(x_S, x_{S\perp}, \tau)$ for each sheet via

$$(x, y) = x_S \hat{n}_S + x_{S\perp} \hat{n}_{S\perp}, \qquad (34)$$

where $\hat{n}_{S\perp}$ are the perpendicular unit vectors aligned with patch S. The natural choice of coordinates for a given patch is to choose polar coordinates within the $\hat{n}_S$-$\hat{\tau}$ plane:

$$r_S = \left(x_S + x_{S\perp}\frac{\hat{n}_{S\perp}^x}{\hat{n}_S^x}, \tau\right) = (r_S \cos\theta_S, r_S \sin\theta_S), \quad (35)$$

because these are the coordinates in which the $\hat{n}_S$-$\hat{\tau}$ planes restricted to each sheet are naturally glued together by extending the range of θ_S to $2\pi n$, as we will now show.

The shift $x_{S\perp}\hat{n}_{S\perp}^x/\hat{n}_S^x$ of x_S is necessary to ensure that $x = 0$, the location of the onset of the branch cut, corresponds to $r_S = 0$, which is what makes these coordinates so convenient. The $\hat{n}_S$-$\hat{\tau}$ planes are now defined by fixed $x_{S\perp}$.

If we can establish that the $2\pi n$ periodicity of θ is equivalent to a $2\pi n$ periodicity of θ_S, then CC's solution would be justified in the above setup so that the noninteracting Green's function $G_0(S, S;, r_S, \theta_S, r'_S, \theta'_S)$ can be expressed through CC's result. This equivalence can be achieved by establishing a one-to-one correspondence (mapping) between θ and θ_S. The mapping is intuitively constructed, as shown in Fig. 6(b), as the vertical projection from the $\hat{n}_S$-$\hat{\tau}$ plane [the blue plane in Fig. 6(b)] onto the $\hat{x}$-$\hat{\tau}$ plane along the $\hat{y}$ direction. Consider moving the projection of r_S in the $\hat{x}$-$\hat{\tau}$ plane around the origin n times (the red circle). It is clear that r_S (on the blue ellipse) follows its projection while also moving around the origin n times, always being on the same sheet. In particular, the branch cut is always traversed simultaneously for $\theta = \theta_S = \pi \mod 2\pi$. The $\hat{n}_S$-$\hat{\tau}$ planes, the leaves of our foliation, thus have the familiar $1 + 1d$ n-sheeted geometry, and θ_S obeys the same $2n\pi$ periodicity as θ.

Finally, the periodicity condition of the $\hat{y}$ direction is necessary for the total entanglement entropy to be finite; it also provides the only length scale for the subsystem, which is needed for extracting the scaling behavior of entanglement entropy. However, if we are concerned only with the integral form of the entanglement entropy as in Eq. (19), not requiring it to be finite as a whole, but rather requiring only the entanglement entropy per unit length to be finite, we may take the y direction to be infinite. This point of view will be taken here and in the following analysis in order to simplify our calculation.

C. Entanglement entropy of free fermions revisited

In order to treat the interacting theory, in this section, we rederive the free fermion result for the half-cylinder geometry via the replica trick. Later in the paper, we shall generalize the method to include interactions. Rewriting the Hamiltonian Eq. (14) in terms of the nonchiral fields, and adding the mass term by hand, we have

$$H[\phi(S; x)] = \sum_S \int d^2x \frac{2\pi v_F^*}{\Omega V}\left\{[\partial_{x_S}\varphi(S; x)]^2\right.$$

$$\left. + [\partial_{x_S}\chi(S; x)]^2\right\} + \frac{m^2}{2}\varphi(S; x)^2. \quad (36)$$

For convenience, we use the Lagrangian formalism and work with the $\varphi(S; x)$ representation throughout the rest of this work.

Switching to imaginary time $t \to i\tau$, and rescaling the coordinates in the following manner,

$$\tau \to \sqrt{\frac{V}{16\pi^3 v_F^*\Omega}}\frac{\tau}{m}, \qquad x \to \sqrt{\frac{\Omega V}{4\pi^2 v_F^*}}\frac{x}{m}, \qquad (37)$$

we obtain the following Lagrangian density in the φ representation :

$$\mathcal{L} = -\frac{m^2}{2}\left\{[\partial_\tau\varphi(S;x)]^2 + [\partial_S\varphi(S;x)]^2 + [\varphi(S;x)]^2\right\}. \quad (38)$$

Then we can work out the Euler-Lagrangian (E-L) equation of motion. Making use of the E-L equation of motion, we find the Green's function $G_0^{(n)}(S, T; x, x') = \langle T\varphi(S;x)\varphi(T;y)\rangle_0$ satisfies the following differential equation:

$$- (\partial_\tau^2 + \partial_{x_S}^2 - 1)G_0^{(n)}(S, T; x, y)$$
$$= C\delta_{S,T}\delta(\tau - \tau_y)\delta(x_S - y_S)\delta^{d-1}(x_{S\perp} - y_{S\perp}), \quad (39)$$

where $C = 2\pi\Omega m^{d-1}(\sqrt{\Omega V/(4\pi^2 v_F^*)})^{d-2}$. The rescaling makes the Green's function dimensionless, and thus easier to handle when it comes to computing $\int d^{d+1}x G^{(n)}(x, x)$. The extra factor C generated on the right-hand side will be canceled by the Jacobian of the integral over the Green's function, leaving only a factor of $1/m^2$. All that then needs to be computed is an integral over the dimensionless G. Therefore, it is legitimate to ignore this factor from now on. The δ functions originate from the commutator Eq. (11) and are *coarse grained*. After we include the patch index, perform the integral over m^2, and take the n derivative, Eq. (31) becomes

$$S_A = \frac{1}{2}\log(m^2 a_0^2)\lim_{n\to 1}\frac{\partial}{\partial n}\sum_S[C_G(S; n) - nC_G(S; 1)], \quad (40)$$

where a_0 is an ultraviolet cutoff, and

$$C_G(S; n) = \int d^{d+1}x G_0^{(n)}(S, S; x, x). \quad (41)$$

The exponential factor in Eq. (31) becomes 1 after the $n \to 1$ limit is applied. Note that $\frac{1}{2}\log(m^2 a_0^2) \sim -\log L$. Our major task is now to compute $C_G(S; n)$.

Observing that there is no $x_{S\perp}$ dependence on the left-hand side of Eq. (39), we can write

$$G_0^{(n)}(S, T; x, y) = \delta_{S,T}\delta^{d-1}(x_{S\perp} - y_{S\perp})G_{0,b}^{(n)}(S; r_x, r_y),$$

and we obtain a $(1 + 1)d$ equation

$$- (\partial_\tau^2 + \partial_{x_S}^2 + 1)G_{0,b}^{(n)}(S; r_{S,x}, r_{S,y}) = \delta(\tau - \tau_y)\delta(x_S - y_S), \quad (42)$$

in which $r_{S,x(y)}$ is as defined in Eq. (35). The same equation appears in CC. We shall also suppress the subscript S unless necessary, as it is normally already specified in the notation for $G_{0,b}$.

The transverse part of the integral in $C_G(S; n)$ can be factored out as

$$\int d^{d-1}x_{S\perp}\delta^{d-1}(x_{S\perp} - y_{S\perp})|_{y\to x}.$$

Recalling our discussion about Eq. (11), this is a coarse-grained δ function. At short distances, instead of a divergence, we should use

$$\delta^{d-1}(x_{S\perp} - y_{S\perp})|_{y\to x} = [\Lambda/(2\pi)]^{d-1}. \quad (43)$$

Therefore, the transverse direction integral becomes

$$[\Lambda/(2\pi)]^{d-1}\int d^{d-1}x_{S\perp} = [\Lambda/(2\pi)]^{d-1}\oint_{\partial A}dS_x \cdot \hat{n}_S.$$

Identifying $\Lambda^{d-1}\hat{n}_S$ as the surface element dS_k, for a given patch the integration can be rewritten as $(2\pi)^{-d+1} \times \oint_{\partial_A}|dS_x \cdot dS_k|$. This leaves us with only an integral over $G_{0,b}^{(n)}(S; r_x, r_x) - nG_{0,b}^{(1)}(S; r_x, r_x)$.

The solution for the $(1 + 1)d$ Green's function on the n-sheeted-replica manifold is given in CC:

$$G_{0,b}^{(n)}(S; r_x, r_y) = \frac{1}{2\pi n}\sum_{k=0}^{\infty}d_k\mathcal{C}_{k/n}(\theta_x - \theta_y)g_{k/n}(r_x, r_y), \quad (44)$$

where $d_0 = 1$, $d_k = 2$ for $k > 0$, $\mathcal{C}_\nu(\theta) = \cos(\nu\theta)$, $g_\nu(r, r') = \theta(r - r')I_\nu(r')K_\nu(r) + \theta(r' - r)I_\nu(r)K_\nu(r')$, and $I_\nu(r)$ and $K_\nu(r)$ are the modified Bessel functions of the first and second kind, respectively. r and θ are again the polar coordinates of the $\hat{n}_S$-$\hat{\tau}$ plane, and we have suppressed the index S of r, as only one patch direction is involved.

The integral over $G_{0,b}^{(n)}$ is

$$\int d^2r_x G_{0,b}^{(n)}(S; r_x, r_x) = \int dr_x r_x\sum_k d_k g_{k/n}(r_x, r_x). \quad (45)$$

The integral is divergent since the integrand $r_x g_{k/n}(r_x, r_x)|_{r_x\to\infty} = 1/4$, a consequence of the fact that we are calculating the partition function of an infinite system. But this divergence should be canceled in $C_G(S; n) - nC_G(S; 1)$. To regularize the divergence, we use the Euler-MacLaurin (E-M) summation formula following CC, and sum over k first:

$$\frac{1}{2}\sum_{k=0}^{\infty}d_k f(k) = \int_0^{\infty}f(k)dk - \frac{1}{12}f'(0) - \sum_{j=2}^{\infty}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(0), \quad (46)$$

where B_{2n} are the Bernoulli numbers, and $f^{(2j-1)}(0) = \partial_k^{2j-1}f(k)|_{k=0}$. Note that the first term, the integral over k, is always canceled by rescaling $k/n \to k$ in $g_{k/n}$. For the remaining terms, which contain derivatives with respect to k, we may add a constant, in this case, $-\frac{1}{4}$, under the derivative, which allows us to pull the derivative outside the integral. The integrand is now well-behaved at infinity. To be more precise, according to Eq. (46), we need to compute

$$\int dr_x r_x \partial_k^j g_{k/n}(r_x, r_x)|_{k\to 0}$$

$$= \partial_k^j \int dr_x (r_x g_{k/n}(r_x, r_x) - \frac{1}{4})|_{k\to 0} = \partial_k^j \left(-\frac{k}{2n}\right). \quad (47)$$

So we have

$$C_G(\boldsymbol{S}; n) - nC_G(\boldsymbol{S}; 1) = \frac{1 - n^2}{24n}. \quad (48)$$

Combining the above results into Eq. (40) and converting the sum over $\boldsymbol{S}$ into an integral around the Fermi surface, we obtain Eq. (19) for this geometry.

D. Differential equations of the Green's functions and an iterative solution

In this section, we derive the differential equations of the Green's functions for the quadratic boson theory with interpatch coupling and provide an iterative solution.

Including the Fermi-liquid interaction $V(\boldsymbol{S}, \boldsymbol{T}; \boldsymbol{x} - \boldsymbol{y}) = U_{\boldsymbol{S},\boldsymbol{T}}$, the Hamiltonian becomes

$$H[\phi(\boldsymbol{S}; \boldsymbol{x})] = \frac{2\pi v_F^*}{\Omega V} \int d^2 x \left\{ \sum_{\boldsymbol{S}} [\partial_{\boldsymbol{S}} \phi(\boldsymbol{S}; \boldsymbol{x})]^2 \right.$$

$$\left. + \sum_{\boldsymbol{S},\boldsymbol{T}} g_{\boldsymbol{S},\boldsymbol{T}} \partial_{\boldsymbol{S}} \phi(\boldsymbol{S}; \boldsymbol{x}) \partial_{\boldsymbol{T}} \phi(\boldsymbol{T}; \boldsymbol{x}) \right\}, \quad (49)$$

where $g_{\boldsymbol{S},\boldsymbol{T}} = \frac{U_{\boldsymbol{S},\boldsymbol{T}}\Omega}{2\pi v_F^*}$ is order $1/N$. This Hamiltonian can be written in terms of the nonchiral fields as

$$H = \frac{2\pi v_F^*}{\Omega V} \int d^2 x \left\{ \sum_{\boldsymbol{S}} \{[\partial_{x_S} \varphi(\boldsymbol{S}; \boldsymbol{x})]^2 + [\partial_{x_S} \chi(\boldsymbol{S}; \boldsymbol{x})]^2\} \right.$$

$$+ \sum_{\boldsymbol{S},\boldsymbol{T}} [g_{\boldsymbol{S},\boldsymbol{T}} \partial_{x_S} \chi(\boldsymbol{S}; \boldsymbol{x}) \partial_{x_T} \chi(\boldsymbol{T}; \boldsymbol{x})$$

$$\left. + \partial_{x_S} \varphi(\boldsymbol{S}; \boldsymbol{x}) \partial_{x_T} \varphi(\boldsymbol{T}; \boldsymbol{x})] \right\}, \quad (50)$$

where we have made use of the fact that $g_{\boldsymbol{S},\boldsymbol{T}} = g_{-\boldsymbol{S},-\boldsymbol{T}}$, which is required by time-reversal symmetry. Here, the summation over $\boldsymbol{S}$ is restricted to a semicircle. This Hamiltonian contains generalized kinetic terms (interpatch coupling due to interaction) that are not diagonal. To obtain the corresponding Lagrangian, we need to invoke the general Legendre transformation [43], and we obtain the following Lagrangian densities, respectively, in terms of φ or χ:

$$\mathcal{L}_\varphi = \frac{1}{2} \left\{ \sum_{\boldsymbol{S}} \{[\partial_t \varphi(\boldsymbol{S}; \boldsymbol{x})]^2 - [\partial_{\boldsymbol{S}} \varphi(\boldsymbol{S}; \boldsymbol{x})^2\} \right.$$

$$+ \sum_{\boldsymbol{S},\boldsymbol{T}} \{h_2(\boldsymbol{S}, \boldsymbol{T}) \partial_t \varphi(\boldsymbol{S}; \boldsymbol{x}) \partial_t \varphi(\boldsymbol{T}; \boldsymbol{x})$$

$$\left. - f_1(\boldsymbol{S}, \boldsymbol{T}) \partial_{\boldsymbol{S}} \varphi(\boldsymbol{S}; \boldsymbol{x}) \partial_{\boldsymbol{T}} \varphi(\boldsymbol{T}; \boldsymbol{x})\} \right\},$$

$$\mathcal{L}_\chi = \frac{1}{2} \left\{ \sum_{\boldsymbol{S}} \{[\partial_t \chi(\boldsymbol{S}; \boldsymbol{x})]^2 - [\partial_{\boldsymbol{S}} \chi(\boldsymbol{S}; \boldsymbol{x})]^2\} \right. \quad (51)$$

$$+ \sum_{\boldsymbol{S},\boldsymbol{T}} [h_1(\boldsymbol{S}, \boldsymbol{T}) \partial_t \chi(\boldsymbol{S}; \boldsymbol{x}) \partial_t \chi(\boldsymbol{T}; \boldsymbol{x})$$

$$\left. - f_2(\boldsymbol{S}, \boldsymbol{T}) \partial_{\boldsymbol{S}} \chi(\boldsymbol{S}; \boldsymbol{x}) \partial_{\boldsymbol{T}} \chi(\boldsymbol{T}; \boldsymbol{x})] \right\},$$

where

$$f_1(\boldsymbol{S}, \boldsymbol{T}) = g_{\boldsymbol{S},\boldsymbol{T}} + g_{-\boldsymbol{S},-\boldsymbol{T}} - g_{\boldsymbol{S},-\boldsymbol{T}} - g_{-\boldsymbol{S},\boldsymbol{T}},$$

$$f_2(\boldsymbol{S}, \boldsymbol{T}) = g_{\boldsymbol{S},\boldsymbol{T}} + g_{-\boldsymbol{S},-\boldsymbol{T}} + g_{-\boldsymbol{S},\boldsymbol{T}} + g_{\boldsymbol{S},-\boldsymbol{T}},$$

and $h_{1(2)}(\boldsymbol{S}, \boldsymbol{T})$ is defined through

$$\{I + [f_{1(2)}(\boldsymbol{S}, \boldsymbol{T})]\}^{-1} = I + [h_{1(2)}(\boldsymbol{S}, \boldsymbol{T})]. \quad (52)$$

Here, I is the identity matrix, and $[f(h)_i(\boldsymbol{S}, \boldsymbol{T})]$ is the matrix formed by $f(h)_i(\boldsymbol{S}, \boldsymbol{T})$, $i = 1, 2$. Applying this result and making use of equations of motion obtained from the Hamiltonian, we obtain the Lagrangians $\mathcal{L}_\varphi$ or $\mathcal{L}_\chi$. Here, we arbitrarily choose to work with $\mathcal{L}_\varphi$. Then, by making use of the E-L equation of motion, applying the same rescaling Eq. (37), and letting $t = i\tau$, we obtain the differential equations that are satisfied by the interacting Green's function $G^{(n)} = \langle \varphi(\boldsymbol{S}; \boldsymbol{x}) \varphi(\boldsymbol{T}; \boldsymbol{x}') \rangle$:

$$-(\partial_\tau^2 + \partial_{\boldsymbol{S}}^2 - 1)G^{(n)}(\boldsymbol{S}, \boldsymbol{T}; \boldsymbol{x}, \boldsymbol{x}') + \sum_l (h_2(l, \boldsymbol{T}) \partial_\tau^2$$

$$+ f_1(l, \boldsymbol{T}) \partial_l \partial_{\boldsymbol{T}}) G^{(n)}(l, \boldsymbol{T}; \boldsymbol{x}, \boldsymbol{x}')$$

$$= C\delta_{\boldsymbol{S},\boldsymbol{T}} \delta(\tau - \tau') \delta(x_{\boldsymbol{S}} - x_{\boldsymbol{S}}') \delta(x_{\boldsymbol{S}\perp} - x_{\boldsymbol{S}\perp}'). \quad (53)$$

Here, the Jacobian due to the change of variables is the same as in the free-fermion case. The entanglement entropy is still given by Eq. (40), but we replace $G_0^{(n)}$ with $G^{(n)}$ in $C_G(\boldsymbol{S}; n)$. In the following, we omit the replica index n in the Green's function unless different values of n are involved in a single equation.

As is well known, differential equations such as Eq. (53) can be converted to an integral form [44] that relates the full Green's function to the noninteracting one. This leads to an iterative (perturbative) definition of the former in terms of the latter. In the present case, this integral equation reads

$$G(\mathbf{S}, \mathbf{T}; \mathbf{x}, \mathbf{y}) = G_0(\mathbf{S}, \mathbf{T}; \mathbf{x}, \mathbf{y})$$

$$+ \int d^3 z G_0(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{z}) \left\{ \sum_l [h_2(l, \mathbf{T}) \partial_\tau^2 \right.$$

$$\left. + f_1(l, \mathbf{T}) \partial_l \partial_{\mathbf{T}}] G(l, \mathbf{T}; \mathbf{z}, \mathbf{y}) \right\}$$

$$= G_0(\mathbf{S}, \mathbf{T}; \mathbf{x}, \mathbf{y}) + \delta G(\mathbf{S}, \mathbf{T}; \mathbf{x}, \mathbf{y}). \qquad (54)$$

Given this equation, we can now compute the Green's function and thus the entanglement entropy perturbatively in powers of U.

E. Entanglement entropy from the iterative solution

In Eq. (54), the G_0 term is the same as that of the free fermions, and thus yields the same contribution to entanglement entropy. To study how the correction term $\delta G(\mathbf{S}, \mathbf{T}; \mathbf{x}, \mathbf{y})$ affects the entanglement entropy, we need to study

$$\int d^3 x \delta G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x}) = \sum_{M=1}^{\infty} \int d^3 x \delta^{(M)} G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x}), \quad (55)$$

where $\delta^{(M)} G$ denotes the Mth-order correction. There are two distinctive types of terms in the perturbative expansion of δG, as we explain next. In general, at order M, we have in total $3(M + 1)$ integrals. Let us examine one of the many terms contributing to the Mth-order correction, to be summed over patch indices:

$$\int d^3 x \delta^{(M)} G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x})$$

$$\sim \int d^3 x \prod_{i=0}^{M-1} (d^3 z_i) G_0(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{z}_0) \partial_{\tau_0}^2 G_0(l_0, l_0; \mathbf{z}_0, \mathbf{z}_1) \dots$$

$$\times \partial_{\tau_i}^2 G_0(l_i, l_i; \mathbf{z}_i, \mathbf{z}_{i+1}) \dots \times \partial_{\tau_{M-1}}^2 G_0(\mathbf{S}, \mathbf{S}; \mathbf{z}_{M-1}, \mathbf{x}). \qquad (56)$$

Here, we include only the τ derivatives. In general, we would also have spatial ($\hat{\mathbf{n}}_{l_i}$) derivative terms, as well as terms with mixed derivatives. But the $\hat{\tau}$ and $\hat{\mathbf{n}}_{\mathbf{S}}$ directions are equivalent. Using rotational symmetry and the fact that the two different derivatives in each term are with respect to independent variables that are integrated out, one can see that all terms are identical except for S-dependent prefactors. The two categories of terms are defined by the set $\{l_i\}$: (1) $l_i = \mathbf{S} \ \forall \ i$, i.e., with *intrapatch* coupling only; and (2) $\exists l_i \neq \mathbf{S}$, containing *interpatch* coupling. We shall label the two categories as

$$\delta^{(M)} G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x}) = \delta_{\text{intra}}^{(M)} G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x}) + \delta_{\text{inter}}^{(M)} G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x}). \qquad (57)$$

1. Intrapatch coupling and comparison with 1D

Setting $l_i = \mathbf{S}$ for all i's in Eq. (56), first we consider the transverse direction

$$G_0(\mathbf{S}, \mathbf{S}; \mathbf{z}_i, \mathbf{z}_{i+1}) \sim \delta(z_{\perp,i}^{(S)} - z_{\perp,i+1}^{(S)}).$$

We can immediately integrate out the transverse component of all $\mathbf{z}_i$'s and obtain

$$\int d^3 x \delta_{\text{intra}}^{(M)} G(\mathbf{S}, \mathbf{S}; \mathbf{x}, \mathbf{x})$$

$$\sim \int dx_{\mathbf{S}\perp} \delta(x_{\mathbf{S}\perp} - z_{\mathbf{S}\perp,0}) \delta(z_{\mathbf{S}\perp,M-1} - y_{\mathbf{S}\perp})|_{\mathbf{y}\to\mathbf{x}}$$

$$\times \int \prod_i dz_{\mathbf{S}\perp,i} \prod_i \delta(z_{\mathbf{S}\perp,i} - z_{\mathbf{S}\perp,i+1})$$

$$= \int dx_{\mathbf{S}\perp} \delta(0) = L^{d-1} [\Lambda/(2\pi)]^{d-1}. \qquad (58)$$

In the last line we use again the fact that the transverse δ function is a coarse-grained one [Eq. (43)].

The rest of $\delta_{\text{intra}}^{(M)} G$ is obtained by substituting $G_0(\mathbf{S}, \mathbf{S}; \mathbf{z}_i, \mathbf{z}_{i+1})$ with the $(1 + 1)d$ Green's function $G_{0,b}(\mathbf{S}; z_{\mathbf{S},i}, z_{\mathbf{S},i+1})$. Although a direct computation is possible, we first give a general argument that, for any M, the contribution to entanglement entropy from $\delta_{\text{intra}}^M G$ vanishes. We confirm this argument by making a comparison with the 1D case, for which a rigorous solution is available.

For the 1D Luttinger liquid with only forward scattering, the entanglement entropy can be calculated directly via bosonization, and the result remains at $\frac{1}{3} \log L$ in the presence of interactions. The calculation is possible because, in our language, there are only two patches, so the transformation which diagonalizes the Hamiltonian is not plagued by the nonlocality issue we encounter in the 2D theory. However, we can also treat the 1D case with our perturbative approach. The resulting series of integrals turns out to be *identical* to the one obtained from the intrapatch contributions in the higher-dimensional case except for the transverse δ function. Therefore, we argue that, at all orders, the intrapatch coupling terms have a vanishing contribution to the entanglement entropy. Later, we shall demonstrate such behavior explicitly up to second order in U.

2. Scaling analysis of interpatch coupling

For terms with interpatch coupling, we find that they are of order $\mathcal{O}(1/L)$ when we compare them to the leading term according a scaling argument. The crucial observation here is that, as long as $\exists l_i \neq \mathbf{S}$, we do not encounter the factor $\delta^{D-1}(0) = L^{D-1}(\Lambda)^{D-1}$, Eq. (43), because, for $l \neq \mathbf{S}$,

$$\delta(z_{1\perp}^{(S)} - z_\perp^{(S)}) \delta(z_\perp^{(l)} - z_{1\perp}^{(l)}) = \frac{\delta^2(z_1 - z)}{|\sin(\theta_l - \theta_S)|}, \qquad (59)$$

where $\theta_{\mathbf{S}}$ (θ_l) is the angle between $\hat{\mathbf{n}}_{\mathbf{S}}$ ($\hat{\mathbf{n}}_l$) and the $\hat{x}$ axis in the $\hat{x}$-$\hat{y}$ plane. Therefore, when we integrate out the $(M + 1)$ transverse δ functions, the factor $\delta^{D-1}(0) = L^{D-1}(\Lambda)^{D-1}$ would be suppressed by even a single $l_i \neq \mathbf{S}$.

To examine the remaining integral, we can ignore the angular part as it cannot affect the scaling behavior. The asymptotic expansion of $K_\nu(r)$ and $I_\nu(r)$ for real r at large value is [45]

$$K_\nu(r) \simeq \sqrt{\frac{\pi}{2r}} e^{-r}\left[1 + \sum_{n=1}^{\infty} \frac{(\nu, n)}{(2r)^n}\right],$$

$$I_\nu(r) \simeq \frac{e^r}{\sqrt{2\pi r}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n(\nu, n)}{(2r)^n}\right],$$

where $(\nu, n) = \frac{\Gamma(1/2+\nu+n)}{n!\Gamma(1/2+\nu-n)}$. By using the above asymptotic expansion of Bessel functions, the leading term for $\partial_{\tau_0}^2 G_0(l_i, l; z_i, z_{i+1})$ behaves as $\sim \theta(z_i - z_{i+1})e^{-(z_i - z_{i+1})}/(z_i - z_{i+1}) + \theta(z_{i+1} - z_i)e^{-(z_{i+1}-z_i)}/(z_{i+1} - z_i)$. All of these terms peak around $z_{i+1} = z_i$ and are otherwise exponentially suppressed. We may therefore again estimate this integral by letting $x = z_0 = z_1 = \ldots = z_{M-1}$ and removing $(M + 1)$ of the integrals. The remaining integrals yield, at the leading order, $\int d^{M+1}z(1/z^M) \sim \int dz z^M z^{-M}$. However, at the leading order, there is no ν dependence. According to the formalism in Sec. IV C, such terms have no contribution to the entanglement entropy. Therefore, the term that contributes to the entanglement entropy is the next order, which behaves as $\int dz \frac{1}{z}$ and is of order $\mathcal{O}(\log L)$, leading only to a correction of approximately $\mathcal{O}(\log L) \times \log L$ to the entanglement entropy.

Next, we shall demonstrate in detail our above analysis for both interpatch and intrapatch coupling terms by explicit calculation up to the second order.

3. First-order correction

The first-order term correction to $\int dx G(S, S; x, x)$ is

$$\delta^{(1)}C_G(S; n) = \int d^3x d^3z G_0(S, S; x, z)[h_2(S, S)\partial_{\tau_z}^2$$
$$+ f_1(S, S)\partial_{z_S}^2]G_0(S, S; z, x). \quad (60)$$

As we have pointed out, it is sufficient to calculate either piece of the two terms due to the equivalence of the imaginary time direction and the real space direction. The other piece should be just the same except for the coefficient. Here, we choose to compute the first term.

The transverse degrees of freedom provide an overall factor, counting the total degrees of freedom as discussed in the general case. We can also integrate out the angular degrees of freedom in the $\hat{x}_S$-$\hat{\tau}$ plane, both θ_x and θ_z as defined in Eq. (44), after which one obtains

$$\delta^{(1)}C_G(S; n) \sim \sum_k \frac{d_k}{2} \delta^{(1)}G_{k/n} \oint_{\partial A} |dS_x \cdot dS_k|, \quad (61)$$

where

$$\delta^{(1)}G_{k/n} = \int dr_x dr_z r_x r_z g_{k/n}(r_x, r_z)\left(\partial_{r_z}^2 - \frac{k^2}{r_z^2 n^2}\right)g_{k/n}(r_x, r_z). \quad (62)$$

The two-k summations are reduced to one due to the orthogonality of the angular function $\mathcal{C}_{k/n}(\theta)$. By employing the E-M formula and properties of the Bessel functions, we show in Appendix A that the sum over k values in Eq. (61) can be converted into an integral, which cancels in Eq. (40) for the same scaling reasons discussed above following Eq. (46). Therefore, we find that the contribution of Eq. (60) to the entanglement entropy vanishes.

4. Second-order correction

The second-order correction is

$$\delta^{(2)}C_G(S; n) = \int d^3x d^3z d^3z_1 G_0(S, S; x, z)\sum_l (h_2(l, S)\partial_{\tau_z}^2$$
$$+ f_1(l, S)\partial_{z_l}\partial_{z_S})G_0(l, l; z, z_1)(h_2(S, S)\partial_{\tau_{z_1}}^2$$
$$+ f_1(S, S)\partial_{z_{1S}}^2)G_0(S, S; z_1, x). \quad (63)$$

(i) For $l = S$:

$$\int d^3x d^3z d^3z_1 G_0(S, S; x, z)[h_2(S, S)\partial_{\tau_z}^2$$
$$+ f_1(S, S)\partial_{z_S}^2]G_0(S, S; z, z_1)[h_2(S, S)\partial_{\tau_{z_1}}^2$$
$$+ f_1(S, S)\partial_{z_{1S}}^2]G_0(S, S; z_1, x). \quad (64)$$

According to our general discussion, we need to consider only the following piece:

$$\int d^3x d^3z d^3z_1 G_0(S, S; x, z)\partial_{\tau_z}^2 G_0(S, S; z, z_1)$$
$$\times \partial_{\tau_{z_1}}^2 G_0(S, S; z_1, x)$$
$$= (2\pi)^{-1} \oint_{\partial A} |dS_x \cdot dS_k| \sum_k \frac{d_k}{4} \delta^{(2)}G_{k/n}, \quad (65)$$

where

$$\delta^{(2)}G_{k/n} = \int dr_x dr_z dr_1 r_x r_z r_1 g_{k/n}(r_x, r_z)$$
$$\times [\partial_{r_z}^2 - k^2/(r_z n)^2]g_{k/n}(r_z, r_1)$$
$$\times [\partial_{r_1}^2 - k^2/(r_1 n)^2]g_{k/n}(r_1, r_x). \quad (66)$$

In the above calculation, we have proceeded as in the first-order calculation, integrating out the angular part first to obtain the expression for $\delta^{(2)}G_{k/n}$.

After a lengthy but similar calculation as for the first order (see Appendix B), we find, using the E-M formula:

$$\frac{1}{2}\sum_k d_k \delta^{(2)} G_{k/n} = \int dr_x dr_z dr_1 r_x r_z r_1$$

$$\times \int_0^\infty dk p_{k/n}(r_x, r_z, r_1)$$

$$+ \left[\frac{1}{12}\partial_k + \sum_{j=2}^\infty \frac{B_{2j}}{(2j)!}\partial_k^{(2j-1)}\right]\frac{n}{16k},$$

(67)

where $p_{k/n}(r_x, r_z, r_1)$ is the product of $g_{k/n}$ dependent terms in Eq. (66). The usual scaling argument for the integral shows that the entire expression is proportional to n, and thus cancels the second ($n = 1$) term in Eq. (40):

$$S(\mathbf{S}) \sim -\frac{\partial}{\partial n}\int d^2 x(G_n - nG_1))|_{n=1}.$$

Therefore, at the second-order level for the $l = \mathbf{S}$ piece, we still have no correction to the scaling law of entanglement entropy.

(ii) For $l \neq \mathbf{S}$:

The integrand we need to consider is

$$G_0(\mathbf{S}, \mathbf{S}; \mathbf{x}, z)[h_2(l, \mathbf{S})\partial_{\tau_z}^2 + f_1(l, \mathbf{S})\partial_{z_l}\partial_{z_S}]$$

$$\times G_0(l, l; z, z_1)[h_2(\mathbf{S}, \mathbf{S})\partial_{\tau_{z_1}}^2 + f_1(\mathbf{S}, \mathbf{S})\partial_{z_{1S}}^2]$$

$$\times G_0(\mathbf{S}, \mathbf{S}; z_1, \mathbf{x}).$$

(68)

First, we notice in Eq. (68) that we have derivatives along directions different from the normal direction $\hat{\mathbf{n}}_{\mathbf{S}}$ of the patch acting on the noninteracting Green's function. We expand this term as

$$\partial_{z_l}\partial_{z_S}G_0(l, l; z, z_1)$$

$$= \delta(z_\perp^{(l)} - z_{1\perp}^{(l)})\partial_{z_l}\partial_{z_S}G_{0,b}(l; z, z_1)$$

$$+ \partial_{z_S}\delta(z_\perp^{(l)} - z_{1\perp}^{(l)})\partial_{z_l}G_{0,b}(l; z, z_1).$$

(69)

For the first term, we can decompose the derivative ∂_{z_S} into terms that act along $\hat{\mathbf{n}}_l$ and along its transverse direction, respectively. The noninteracting

Green's function depends only on the transverse coordinates via $G_0(l, l; \mathbf{x}, \mathbf{y}) \sim \delta(x_\perp^{(l)} - y_\perp^{(l)})$, which indicates that those derivative terms vanish. Thus, it is proportional to $\delta(z_\perp^{(l)} - z_{1\perp}^{(l)})\partial_{z_l}^2 G_{0,b}(l; z, z_1)$. For the second term, we integrate by parts with respect to z_S, which leads to (now including the first G_0 factor, which depends on z)

$$- \delta(x_\perp^{(S)} - z_{1\perp}^{(S)})\delta(z_\perp^{(l)} - z_{1\perp}^{(l)})\partial_{z_S}G_{0,b}(\mathbf{S}; \mathbf{x}, z)\partial_{z_l}$$

$$\times G_{0,b}(l; z, z_1).$$

Therefore, the overall integrand is proportional to $\delta(x_\perp^{(S)} - z_{1\perp}^{(S)})\delta(z_{1\perp}^{(S)} - x_\perp^{(S)})\delta(z_\perp^{(l)} - z_{1\perp}^{(l)})$. Note that the $x_\perp^{(S)}$ dependence appears only in these δ functions. We can integrate it out, leaving only $\delta(z_\perp^{(S)} - z_{1\perp}^{(S)})\delta(z_\perp^{(l)} - z_{1\perp}^{(l)}) \sim \delta(z_1 - z)$.

Second, it is sufficient to focus on the following terms in the integrand,

$$[G_{0,b}(\mathbf{S}; \mathbf{x}, z)\partial_{\tau_z}^2 G_{0,b}(l; z, z_1)$$

$$+ \partial_{z_S}G_{0,b}(\mathbf{S}; \mathbf{x}, z)\partial_{z_l}G_{0,b}(l; z, z_1)]\partial_{\tau_{z_1}}^2 G_{0,b}(\mathbf{S}; z_1, \mathbf{x}),$$

to ease the presentation. For other combinations, the rest of this section is equally applicable, with minor modifications that lead only to different coefficients and do not affect the scaling analysis. We first perform the intrapatch integration

$$\int d^3 x G_{0,b}(\mathbf{S}; \mathbf{x}, z)G_{0,b}(\mathbf{S}; z_1, \mathbf{x}) = H(\mathbf{S}; z, z_1),$$

(70)

where

$$H(\mathbf{S}; z, z_1) = \sum_k \frac{d_k}{2\pi n}\mathcal{C}_{k/n}(\theta_z, \theta_{z_1})[\theta(r_z - r_1)$$

$$\times (r_z K_{z+}I_1 - r_1 I_{1-}K_z) + \theta(r_1 - r_z)$$

$$\times (r_1 K_{1+}I_z - r_z I_{z-}K_1)].$$

So, for a given $\mathbf{S}$, the contribution to entanglement entropy due to coupling with patch l can be written as

$$\left|\frac{1}{\sin(\theta_l - \theta_\mathbf{S})}\right|\left|\int d^3 z d^3 z_1 \delta^2(z_1 - z)[\partial_{\tau_{z_1}}^2 H(\mathbf{S}; z, z_1)\partial_\tau^2 G_{0,b}(l; z_1, z) + \partial_{z_S}\partial_{\tau_{z_1}}^2 H(\mathbf{S}; z, z_1)\partial_{z_l}G_{0,b}(l; z_1, z)]\right.$$

$$= \left|\frac{1}{\sin(\theta_l - \theta_\mathbf{S})}\right|\left|\int d^3 z d\tau_1[\partial_{\tau_{z_1}}^2 H(\mathbf{S}; z, z_1)\partial_\tau^2 G_{0,b}(l; z_1, z) + \partial_{z_S}\partial_{\tau_{z_1}}^2 H(\mathbf{S}; z, z_1)\partial_{z_l}G_{0,b}(l; z_1, z)]\right|_{z_{1,x}=z_{,x}, z_{1,y}=z_{,y}},$$

(71)

where $z_{,x}$, $z_{,y}$ indicate the two spatial components of z.

As we argued in Sec. IV E 3, to extract the order of magnitude of the result, it is sufficient to set $\tau_1 = \tau$ in the final line of Eq. (71) and remove the integral over τ_1. We also note that the derivatives do not alter the leading power of r, owing to the presence of the

exponential function. Therefore, it is sufficient to examine

$$\int d^3 z[H(\mathbf{S}; z, z_1)G_{0,b}(l; z_1, z)]|_{z_1=z}.$$

(72)

At the lowest order in $1/r$, we have

$$G_{0,b}(\mathbf{S}; r, r) \sim I_\nu(r) K_\nu(r) \sim 1/r, \qquad (73)$$

$$
\begin{aligned}
H(\mathbf{S}; r, r) &\sim r I_\nu(r) K_{\nu+}(r) - r K_\nu(r) I_{\nu-}(r) \\
&= \left(1 + \frac{(\nu+1, 1)}{2r} + \cdots\right)\left(1 - \frac{(\nu, 1)}{2r} + \cdots\right) \\
&\quad - \left(1 - \frac{(\nu-1, 1)}{2r} + \cdots\right)\left(1 + \frac{(\nu, 1)}{2r} + \cdots\right) \\
&= \frac{1}{r} + \mathcal{O}\!\left(\frac{1}{r^2}\right).
\end{aligned}
\qquad (74)
$$

Since the τ derivative does not alter the leading powers, we extract the leading term to be

$$[H(\mathbf{S}; z, z_1)\partial_\tau^2 G_{0,b}(\mathbf{S}; z_1, z)]|_{z_1=z} \sim \frac{1}{z^2}. \qquad (75)$$

For a triple integral over $1/z^2$, one would get a linear divergence, i.e., the result would be $\sim L$. This is indeed the case as we have already seen in previous calculation. However, at the lowest order, everything is independent on $\nu = k/n$. Actually what finally appears in the entanglement entropy are the k-derivatives of these terms appearing in the E-L summation formula. This means the leading term has vanishing contribution to the entanglement entropy. The first term contributing to entanglement entropy is then $\sim \int d^3 z \frac{1}{z^3}$, the upper limit of which is order $\mathcal{O}(\log L)$ and only leads to a correction of up to $\sim \mathcal{O}(\log L) \times \log L$ to the free fermion entanglement entropy.

V. SUMMARY AND CONCLUDING REMARKS

In this paper, we developed an intuitive understanding of the logarithmic correction to the area law for the entanglement entropy of free fermions in one and higher dimensions on equal footing—the criticality associated with the Fermi surface (or points). Then we used the tool of high-dimensional bosonization to compute the entanglement entropy, and generalized this procedure to include Fermi-liquid interactions. In the presence of such interactions we calculated the entanglement entropy for a special geometry perturbatively in powers of the interaction strength up to the second order, and found no correction to the leading scaling behavior. We also point out that the situation is the same at higher orders. Our results thus strongly suggest that the leading scaling behavior of the block entanglement entropy of a Fermi liquid is the *same* as that of a free Fermi gas with the same Fermi surface, not only for the special block geometry studied in this paper, but for arbitrary geometries. Explicit demonstration of the latter is an obvious direction for future work.

In the special geometry in which we performed explicit calculations using the replica trick, a masslike term is introduced to regularize the theory at long distance, as is done in closely related contexts [35,42]. For a Fermi liquid (which is quantum-critical) the corresponding length scale

$\xi \sim v_F/m$ must be identified with the block size L, and is thus *not* an independent length scale. On the other hand, such a masslike term can also describe a superconducting gap due to pairing. In particular, for a weak-coupling superconductor, ξ, the superconducting coherence length, is much longer than all microscopic length scales, but finite nevertheless. In this case it is *independent* of L, and the interplay between the two is interesting. For $L < \xi$, the Fermi-liquid result (1) still holds. But for $L > \xi$, the logarithmic factor in the entanglement entropy *saturates* at $\log \xi$, and we expect

$$S(\rho_A) = \frac{1}{12(2\pi)^{2-1}} \log \xi \times \oint_{\partial A} \oint_{\partial \Gamma} dS_x \cdot dS_k, \qquad (76)$$

which agrees with the conjecture made in Ref. [26].

More generally, Fermi liquids are (perhaps the best understood) examples of quantum-critical phases (or points) in high dimensions. Unlike in 1D where conformal symmetry powerfully constrains the behavior of entanglement entropy, our understanding of entanglement properties of such high-dimensional quantum-critical phases or points (many of them have Fermi surfaces but are *not* Fermi liquids) is very limited. Our work can be viewed as a step in that general direction. Furthermore, the formalism developed in this work has potential applicability to systems with composite or emergent fermions with Fermi surfaces as well, or more generally, *non-Fermi-liquid* phases with Fermi surfaces. The system studied in Ref. [22], where there is an emergent spinon Fermi surface, is a potential example.

APPENDIX A: CALCULATION OF $\delta^{(1)}G_{k/n}$

Throughout this Appendix, we shall denote the modified Bessel functions $K_\nu(r_i)$, $I_\nu(r_i)$ as K_i, I_i for simplicity with $\nu = k/n$. We also have $K(I)_{\nu\pm1}(r_i)$ which shall be shortened as $K(I)_{i,\pm}$.

$$\delta^{(1)}G_{k/n} = \int dr_x dr_z r_x r_z g_{k/n}(r_x, r_z)\left(\partial_{r_z}^2 - \frac{k^2}{r_z^2 n^2}\right) g_{k/n}(r_z, r_x). \qquad (A1)$$

Expanding $(\partial_{r_z}^2 - \frac{k^2}{r_z^2 n^2})g_{k/n}(r_z, r_x)$, and noting the identities $I'K - K'I = 1/x$, $X'' - (\nu^2/x^2)X = X - (1/x)X'$ where $X = K, I$, we get

$$\left(\partial_{r_z}^2 - \frac{k^2}{r_z^2 n^2}\right)g_{k/n}(r_z, r_x)$$

$$= -\frac{\delta(r_x - r_z)}{r_x} + \theta(r_x - r_z)\left(1 - \frac{1}{r_z}\partial_{r_z}\right)I_z K_x$$

$$+ \theta(r_z - r_x)\left(1 - \frac{1}{r_z}\partial_{r_z}\right)I_x K_z. \tag{A2}$$

Then integrating over r_x first, and making use of the following formula

$$\int dx x X_\nu^2(x) = \frac{1}{2}x^2[X_\nu^2(x) - X_{\nu-1}(x)X_{\nu+1}(x)], \tag{A3}$$

where $X_\nu(x)$ can be the first or second kind of modified Bessel function, I_ν or K_ν, and the identities $I_-K + IK_- = I_+K + IK_+ = 1/x$ in addition to those given above, $\delta^{(1)}G_{k/n}$ is reduced to

$$\delta^{(1)}G_{k/n} = \int dr r[-IK + r^2(I^2 - I_+I_-)K^2]. \tag{A4}$$

We apply the same strategy as in Sec. IV C, making use of the E-M formula to do the sum over the k-index in Eq. (61). This converts the sum into a divergent integral over k which cancels in Eq. (40) as before, and a sum over terms of the form $\partial_k^j\delta^{(1)}G_{k/n}|_{k\to0}$ that turn out to vanish, as we will now show. Again, we can include proper constants under the derivative into the integrand. These derivatives then act on well-defined integrals. The first term has been discussed in Sec. IV C:

$$\int dr\partial_k^j(-rI_{k/n}K_{k/n}) = \partial_k^j\int dr\left(-rIK + \frac{1}{4}\right) = \partial_k^j\left(\frac{k}{2n}\right). \tag{A5}$$

The second term can be shown to be

$$\int dr\partial_k^j[r^3(I^2 - I_+I_-)K^2]$$

$$= \partial_k^j\int dr\left(r^3(I^2 - I_+I_-)K^2 - \frac{1}{4}\right)$$

$$= \partial_k^j\left(-\frac{1}{16} - \frac{k}{2n}\right). \tag{A6}$$

Summing the two terms together, we find

$$(\partial_k^j\delta^{(1)}G_{k/n})|_{k\to0} = \partial_k^j\left(-\frac{1}{16}\right) = 0, \tag{A7}$$

for all $j > 0$.

APPENDIX B: CALCULATION OF $\delta^{(2)}G_{k/n}$

Let us first compute the integral:

$$\int dr_2 r_2 g_\nu(r, r_2)\left(\partial_{r_2}^2 g_\nu(r_2, r_1) - \left(\frac{\nu}{r_2}\right)^2 g_{k/n}(r_2, r_1)\right)$$

$$= \theta(r - r_1)h(r, r_1) + \theta(r_1 - r)h(r_1, r) \tag{B1}$$

with

$$h(r, r_1) = -I_1 K + \frac{1}{2}\left(f_1(r, r_1) + KI_1 \ln\frac{r}{r_1}\right),$$

$$f_1(r, r_1) = KK_1[r_1^2(I_1^2 - I_{1,+}I_{1,-}) - I_1^2]$$

$$- II_1[r^2(K^2 - K_+K_-) - K^2]$$

$$+ KI_1(F(r) - F(r_1) - IK + I_1K_1),$$

$$F(r) = 2\int dr rIK = r^2 IK + r^2 I_+ K_-.$$

The IK_1 term in $h(r, r_1)$ results from δ-functions (derivatives of δ-functions) coming from the derivative applied on the step function ($\theta(r)$). The remaining part comes from terms involving a product of two θ-functions. Here one needs to distinguish between $r > r_1$ and $r < r_1$, which gives rise to the terms in $\theta(r - r_1)$ and $\theta(r_1 - r)$, respectively. The remaining integral

$$\int dr dr_1 r r_1[\theta(r - r_1)h(r, r_1) + \theta(r_1 - r)h(r_1, r)]$$

$$\times\left(-\frac{\delta(r_1 - r)}{r} + \theta(r_1 - r)\left(1 - \frac{1}{r_1}\partial_{r_1}\right)K_1 I + \theta(r - r_1)\right.$$

$$\times\left.\left(1 - \frac{1}{r_1}\partial_{r_1}\right)KI_1\right) \tag{B2}$$

can be carried out by applying the identities of Bessel functions I and K used in Appendix A. In applying the E-M formula to the sum over k in (65), we again arrive at a divergent k-integral that can be rescaled and subsequently canceled (see (67) and below), and a sum over derivative terms that are well-behaved. In the latter terms, we always add proper constants under the derivatives to regularize the integrand at infinity, as before. We divide (B2) into two terms. The first is the one containing the δ-function. After integrating out r_1, this term becomes

$$\partial_k^j\left(-\int drr h(r, r)\right) = \partial_k^j\left(-\int dr\left(rh(r, r) + \frac{1}{4}\right)\right)$$

$$= \partial_k^j\left(-\int dr\left(r\left(-IK + \frac{r^2}{2}\right.\right.\right.$$

$$\times(-K^2 I_+ I_- + I^2 K_+ K_-)\right) + \frac{1}{4}\right)\right)$$

$$= \partial_k^j(0). \tag{B3}$$

The second term is expanded to

$$\int dr dr_1 r r_1[\theta(r - r_1)h(r, r_1)K(I_1 - I_1'/r_1)$$

$$+ \theta(r_1 - r)h(r_1, r)I(K_1 - K_1'/r_1)]. \tag{B4}$$

Because of the complexity of $h(r, r_1)$, we examine each of the three terms of $h(r, r_1)$ separately. The first term is simple. Applying those identities of K's and I's and including the proper constant, we get

$$\partial_k^j\left(\int dr dr_1\left(r r_1\left(\theta(r-r_1)(-I_1 K)K(I_1-I_1'/r_1)\right.\right.\right.$$
$$\left.\left.\left.+\theta(r_1-r)(-IK_1)I(K_1-K_1'/r_1))-\frac{1}{4}\right)\right)=\partial_k^j\left(\frac{k}{2n}\right).$$
(B5)

The second term of $h(r, r_1)$ contributes

$$\partial_k^j\left(\frac{1}{2}\int dr dr_1 r r_1 \theta(r-r_1)f1(r,r_1)K\left(I_1-\frac{1}{r_1}I_1'\right)\right.$$
$$\left.+\frac{1}{2}\int dr dr_1 r r_1 \theta(r_1-r)f1(r_1,r)\left(K_1-\frac{1}{r_1}K_1'\right)I\right). \quad (B6)$$

By interchanging the dummy variables r and r_1, employing the properties of the modified Bessel functions with care, and including the regularization constant, we arrive at

$$\partial_k^j\left(\frac{1}{2}\int dr dr_1 (r r_1 \theta(r-r_1)f_1(r,r_1)\left(2KI_1-KI_1'/r_1\right.\right.$$
$$\left.\left.-K'I_1/r\right)-\frac{1}{8}\right)\right)=\partial_k^j\left(-\frac{k}{2n}\right). \quad (B7)$$

The last part of $h(r, r_1)$ contributes as

$$\int dr dr_1 \frac{r r_1}{2}\theta(r-r_1)KI_1 \ln\frac{r}{r_1}(K'I_1/r-KI_1'/r_1)$$
$$=-\frac{1}{16\nu}. \quad (B8)$$

Summing all the above terms together we get

$$\partial_k^j(\delta^{(2)}G_{k/n})=\partial_k^j\left(-\frac{n}{16k}\right). \quad (B9)$$

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