From Cartesian coordinates to Hilbert space: Supporting student understanding of basis in quantum mechanics

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We conducted a multiyear project across three institutions to develop an instructional tutorial that supports student understanding of change of basis in quantum mechanics. Building from our previous work, we identified learning goals to guide activity development. The tutorial makes an analogy between spin-1/2 states and a Cartesian coordinate system. This paper details the iterative development process including reports of observations from classroom implementations and the resulting modifications to the activity. Further, we report preliminary findings on the success of the activity in improving students' ability to correctly change basis and their articulation that change of basis is a choice of representation, not a change to the physical system.

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I. INTRODUCTION

As part of a larger project focused on research-based curricula for quantum mechanics [1], we previously explored student understanding related to basis [2]. Subsequently, we have created a quantum basis tutorial (QBT) to support students in learning about basis. The tutorial activity draws on analogies to two-dimensional vectors that students have used in many of their prior physics and mathematics courses. Quantum mechanics courses are often described as challenging because they are abstract [3]. The tutorial is designed to counteract the abstract nature of quantum mechanical states by grounding the concept of basis in a more familiar and tangible vector context. In this paper, we present the process through which we developed the QBT, and discuss evidence for its effectiveness.

Basis is a fundamental aspect of quantum mechanics in that it is tied to the mathematical representation of quantum states used for reasoning about measurement and probability. Furthermore, there is often a preferred basis for solving a given problem (e.g., to express a time evolved state, the state needs to be expressed in the energy eigenbasis). Our previous research investigated students' interpretations of changing basis with respect to the physical nature and mathematical representation of the state (e.g., its effect, or lack thereof, on the probabilities of different measurement outcomes) and also reported on the various methods students used to change basis. In Sec. II, we briefly summarize previous findings [2] that motivated the development of the QBT.

The QBT is designed to help students develop a coherent model of basis in quantum mechanics, and is based on our research into how students understand basis [2]. We use an analogy to a familiar, two-dimensional Cartesian coordinate system to solidify concepts related to basis in the abstract setting of quantum mechanics. This analogy has been used by an activity written by Oregon State University when looking at the effects of operators [4] and by Tutorials in Physics: Quantum Mechanics [5] and the Quantum Interactive Leaning Tutorials [6] when looking at introducing Dirac notation. Our learning goals (LG) for this tutorial are for students to be able to

LG1. Recognize that changing basis does not change the state or the probabilities of any measurement on a state,

LG2. Use projection (inner products) as a method to change basis with and without prompting,

LG3. Identify the coefficients in a basis expansion (a) physically as probability amplitudes and (b) mathematically as inner products, and

LG4. Recognize that a reason for changing basis is making desired information more readily accessible.

The initial development of the tutorial was motivated by LG1 and LG2. A preliminary implementation, discussed in Sec. V B, highlighted other facets related to changing basis that we wanted to target and motivated the addition of LG3 and LG4. Various questions and activities within the tutorial were designed to target each of these goals. In Sec. III, we detail the different parts of the QBT and discuss how they align with our learning goals.

The development of the QBT fits within a growing effort to support the use of interactive engagement [7] and active learning [8] in quantum mechanics courses, building on years of instructional reform in introductory physics. Other researchers have explored the effectiveness of interactive pedagogy in quantum mechanics, including Peer Instruction and Just-In-Time Teaching [9,10]. As part of this transition away from a traditional lecture format, a wide range of activities from tutorials to simulations have been developed to support learning in quantum mechanics [1,4,5,11–13]. Many of these activities have been the subject of research aiming to evaluate their impact on student understanding (e.g., Refs. [14–21]). However, little has been published detailing the process of creation and refinement of these activities.

The goals of this paper are twofold. First, we aim to detail our process for developing the tutorial, motivate the choices we made, and describe the interplay between development, implementation, and further research. Second, like other projects of this nature, we set out to present the QBT and provide preliminary evidence of its effectiveness. After presenting the design of the QBT in Sec. III, we outline the contexts for the administration of the tutorial, and describe the methods for data collection and study design in Sec. IV. Subsequently, in Sec. V we summarize results from the two implementations of the activity, and discuss improvements made following these implementations. Finally, we present initial findings related to the effectiveness of the tutorial in Sec. VI.

II. PREVIOUS WORK AND MOTIVATION

There is a growing base of physics education research in the topical area of quantum mechanics (see Ref. [3] for an overview). These investigations have explored student understanding in several areas, including but not limited to notation [21–27], expectation values [20,27–30], time dependence [14,15,31–33], and measurement and probability [14,30,34–37]. Physics education research in other areas of quantum mechanics has occasionally involved discussion of basis, but basis itself has rarely been the main focus, with the exception of a recent paper investigating student and expert discourse related to basis [38]. One investigation reported that students do not recognize that a relative phase affects the probability of measurement in different bases [37] (student errors related to relative phase have been reported elsewhere as well [39]), while another found that students are not always cued to change basis when expressing a time evolved state [32]. In the mathematics education community, research has focused mainly on how students associate basis with the properties of span and linear independence [40–42], since these are two key properties of a generalized set of basis vectors.

Our recent work, discussed in a separate paper [2], explicitly investigated student ideas about basis and change of basis in quantum mechanics within the context of spin-½ systems. The four learning goals above were refined through this research, which ran concurrently with the process of tutorial development described in this paper. Here we present a brief summary of our previous findings to elucidate the subsequent discussion of the QBT and motivate the learning goals. The study centered on three points: students' interpretations of change of basis; what methods students use when changing basis; and how students interpret basis expansion coefficients and relate them to inner products.

Surveys and interviews revealed that students may believe that changing basis in a spins context affects a quantum state beyond merely altering its mathematical representation. Some students associated physical effects with basis change; for example, on surveys, some said that measurement probabilities were altered when a spin-½ state was converted from the S_z basis to the S_x basis. We also found that students may choose to change the notation for a ket when changing its basis (e.g., if changing to the S_x basis, students may change a ket's label from $|\psi\rangle$ to $|\psi\rangle_x$). In some cases, students opting to relabel a ket nonetheless did *not* associate physical significance with basis change. Overall, these findings inform LG1.

Interviews and responses to quiz and exam questions asking students to change basis revealed that students employ a variety of mathematical methods when doing so. The *projection* method involves computing inner products to find the new basis expansion coefficients; for example, to find the coefficient associated with the $|+\rangle_x$ basis state for the state $|\psi\rangle$, one would compute ${}_x\langle+|\psi\rangle$. Projection is often the most generalizable and computationally simple method. However, other valid methods were also common. For example, the *system of equations* method involves inverting the equations defining the new basis states (e.g., $|\pm\rangle_x$) in terms of the original basis states (e.g., $|\pm\rangle$) to find how the original basis states can be written in the new basis. Other methods, which were generally incorrect, were also used. These findings inform LG2. In Secs. V and VI, we report the frequency of each of these methods on exams and quizzes.

Interviews and surveys probing students' interpretation of basis expansion coefficients revealed that students may not spontaneously associate these coefficients with probability amplitudes or with inner products. The association with inner products is particularly important to facilitate the use of the projection when changing basis. Students have also been found to have difficulties with recognizing the connection between expansion coefficients and inner products in a wave functions context [36]. Inner products are relevant to basis representation because they give probability amplitudes for measurements in a given basis. These findings inform LG3, and the association between coefficients and probability amplitudes is a prerequisite for LG4.

To support these four learning goals, the QBT builds on an analogy between vectors in Hilbert space and real-valued, two-dimensional vectors, and includes graphing elements to support this analogy. The use of this strategy is supported by the literature. The first time many students encounter basis is in the context of two-dimensional Cartesian vectors, as used in introductory physics. Research has shown that addition and subtractions of vectors using their graphical representations is challenging for students in introductory physics [43–46] and that there are several difficulties related to student understanding of vector components and multiplication [44,45]. However, by the time students reach upper-division courses, they have been shown to fall back on Cartesian representations in situations where newer, non-Cartesian coordinate systems are more appropriate [47–50], suggesting that Cartesian coordinates had become more familiar after years of use. Furthermore, research on one tutorial activity has shown that the use of an analogy to 3D Cartesian coordinates has supported student understanding of Dirac notation [6]. Research in mathematics education has shown that the use of geometric and graphical representation bolsters students' learning of abstract concepts in linear algebra including span and linear independence [40,51]. Related research shows that using graphical representations provided physics students with helpful context for the eigenvalue-eigenvector equation [52]. Prior quantum mechanics instructional activities have incorporated Cartesian representations to exhibit the role of operators [5] or to support understanding of Dirac notation [6].

III. OVERVIEW OF THE QUANTUM BASIS TUTORIAL

We use the word *tutorials* to refer to worksheets intended to guide students in constructing conceptual physics knowledge for themselves. Often, tutorials are done in small groups in a classroom setting with a higher instructor-to-student ratio. The worksheets are not typically collected or graded.

The QBT discussed in this paper is designed around an analogy to a two-dimensional Cartesian coordinate system, with influence from the University of Washington's quantum tutorials [5]. Leveraging the familiarity juniors and seniors have with a Cartesian coordinate system, the activity uses a vector, expressed in Dirac notation, to introduce a way of thinking about basis. By drawing and labeling components of the vectors using different sets of basis vectors as axes, students discover that a basis is a representational choice that conveys a certain set of information.

TABLE I. This table presents a summary of each part of the QBT, as well as which learning goals the part targets. Each part fills one printed page, so students have room to write their answers and show their work when needed. The language for the questions or tasks in this table is significantly condensed from the real QBT, and some explanatory text and hints are missing.

Part	Abbreviated questions or tasks	Associated learning goals for each part
I	What do the coefficients in $\vert\psi_A\rangle = 3/5\vert+\rangle - 4/5\vert-\rangle$ tell you? Create histograms for the probability amplitudes and probabilities. Label the histograms with Dirac expressions. How do probability and probability amplitude relate?	LG3a: Coefficients and probability amplitudes LG3b: Coefficients and inner products LG4: Accessible information
II	Do the Cartesian unit vectors $\vert i\rangle$ and $\vert j\rangle$ form a basis? Plot the vector $\vert u\rangle = 1/\sqrt{5}\vert i\rangle + 2/\sqrt{5}\vert j\rangle$ on Cartesian axes. Express $\vert u\rangle$ as a column vector using values and inner products for the components. Label the inner products on the graph, and discuss a conceptual meaning for inner products.	LG3b: Coefficients and inner products
III	Write $\vert u\rangle$ in the basis of $\vert v_1\rangle$ and $\vert v_2\rangle$ (two new vectors). Graph the vector using a set of axes for the new basis vectors. Write the state as a column vector of the form $(\square\square)_V$. Why is there a v subscript on the column vector?	LG2: Using projection to change basis LG3b: Coefficients and inner products
IV	Plot original $\vert u\rangle$, new basis vectors/axes, and $\vert u\rangle$ in new basis. Graphically, compare the vector in the original basis with the vector in the new basis. Using the analogy, explain what changing the basis means. Does the new vector need a new name or label? Can we write $\vert\psi\rangle = a\vert+\rangle + b\vert-\rangle = c\vert+\rangle_x + d\vert-\rangle_x$?	LG1: State unaffected by change of basis LG2: Using projection to change basis
V	Draw axes for measuring position and potential energy on an incline plane. Does the coordinate system affect the physical scenario? Why might you rewrite a state in a new basis? Which basis is useful for finding probabilities of quantum measurements of S_z/S_x ? Does changing basis affect the state? Why change basis?	LG1: State unaffected by change of basis LG3a: Coefficients and probability amplitudes LG4: Accessible information

We designed the tutorial over several iterations which included two main versions—alpha and beta—that were implemented over a full lecture in the classroom. In its current form, the tutorial is divided into five parts, each with a specific purpose. Parts II–IV form the core of the tutorial activity and were given as a preliminary version, or alpha-version. Parts I and V provide priming and wrap-up discussion, respectively. These parts were added based on observations of the preliminary alpha implementation and further research, which are described in Sec. V C. The beta version of the activity was administered with all five parts outlined below. An overview of each part of the activity, as well as the learning goals each part targets, is provided in Table I. The full tutorial is available online as part of the "tutorial collection" under adaptive curricular exercises for quantum mechanics [1].

Part I of the activity was designed to address LG3 by drawing out the connections between (a) basis expansion coefficients, (b) probability amplitudes associated with a measurement, and (c) the Dirac expressions for an inner product. The first question gives a generic state, $|\psi_A\rangle = \frac{3}{5}|+\rangle - \frac{4}{5}|-\rangle$, in both Dirac and matrix representations. Students are told that the coefficients are called *probability amplitudes* and asked to explain what information they reveal about the state. Then students are instructed to complete two blank histograms labeled "probability amplitude" and "probability." After drawing the histograms for $|\psi_A\rangle$, students are asked to label the height of each bar on the histogram with Dirac notation. Lastly, students are asked about the relationship between probability and probability amplitude.

Part II begins the analogy to a Cartesian system by presenting Cartesian unit vectors in Dirac notation: $|i\rangle$ and $|j\rangle$ are given as alternative notation for $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Students first answer whether these unit vectors could be used as a basis. Then they are given a vector, $|u\rangle = \frac{1}{\sqrt{5}}|i\rangle + \frac{2}{\sqrt{5}}|j\rangle$, which they are asked to graph using a set of Cartesian axes (see student example in Fig. 1). To promote accuracy in graphing (especially important for part IV) and save students time, an unlabeled coordinate grid was added to the beta-version of the activity. Students are then asked to represent $|u\rangle$ as a two-dimensional column vector and to represent the two components as inner products. Lastly, they are asked to identify these inner products on the graph (shown in Fig. 1), and discuss the conceptual meaning of the inner product based on the graph. This final question is designed to help connect the idea of an inner product in quantum mechanics to finding a component in a specific coordinate direction, associated with learning goal LG3b.

Part III is where students calculate a change of basis. They are first given new basis states, $|v_1\rangle$ and $|v_2\rangle$, associated with column vectors in the Cartesian basis. Students are asked to write $|u\rangle$ in the new basis, such that it has the form $a|v_1\rangle + b|v_2\rangle$. They are then asked to graph the vector using $|v_1\rangle$ and $|v_2\rangle$ as the basis vectors. Since changing the basis changes the coefficients of the state, treating $|v_1\rangle$ and $|v_2\rangle$ as the horizontal and vertical axes results in the vector $|u\rangle$ being oriented differently than in part II. Mirroring the questions in part II, students write the state $|u\rangle$ as a column vector in the new basis, express each component as an inner product, and represent the inner products on their graph. Since the column vector here uses a different basis, students are given the form

$$\begin{pmatrix} a \\ b \end{pmatrix}_v$$

and are later asked about the purpose of putting the subscript on the column vector.

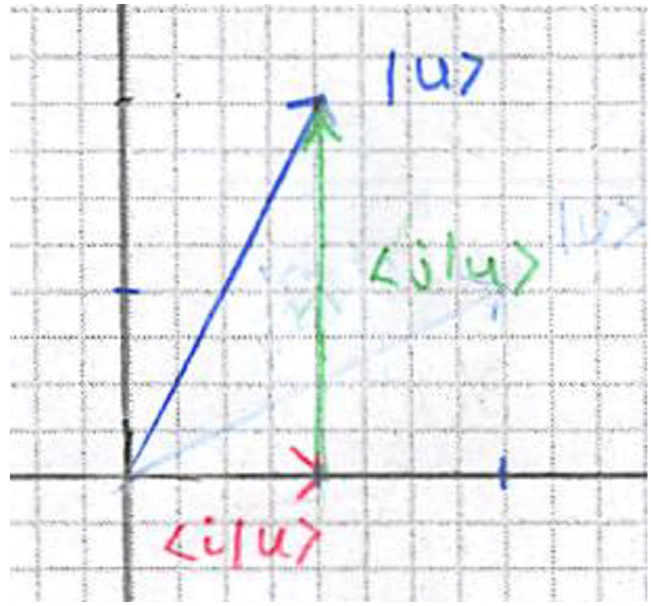

FIG. 1. Student work depicting $|u\rangle$ as a vector on a Cartesian coordinate plane. Here the individual components are labeled with inner products in Dirac notation.

The questions in part III continue to address the connection between the coefficient and the inner product (LG3b). The questions about identifying the coefficients in each basis also deliberately lay the groundwork about what information is accessible from a basis representation (LG4) which is addressed directly in part V.

In part IV, students make connections between the basis representations. They are told that they will temporarily relabel the vector represented in the basis of $|v_1\rangle$ and $|v_2\rangle$ as $|k\rangle$. Students are told, "We will decide later whether or not we needed to do this," since this relabeling is unnecessary. This allows for students to discover that the vectors are indeed the same, targeting LG1. To discover the relationship between $|u\rangle$ and $|k\rangle$, students are instructed to draw all vectors on the same set of $|i\rangle$ and $|j\rangle$ axes from the beginning of the activity. This is structured so students first redraw $|u\rangle$, add the new basis vectors, and then find where $|k\rangle$ would be placed using the new set of basis vectors (see the example in Fig. 2). Careful plotting of these vectors results in $|u\rangle$ and $|k\rangle$ overlapping.

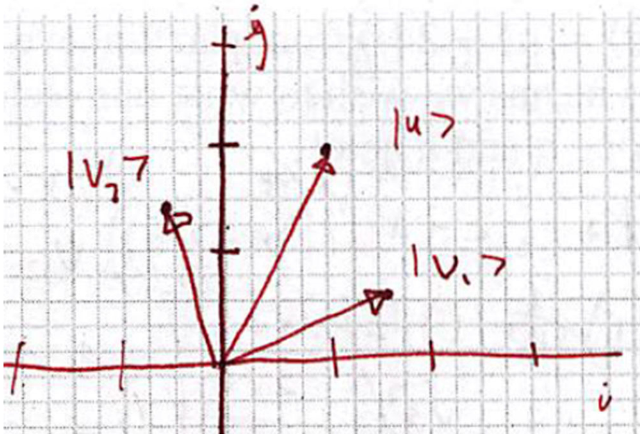

FIG. 2. Student work depicting all vectors on the same graph. The vectors $|i\rangle$ and $|j\rangle$ are represented by the horizontal and vertical axes, respectively. The new basis vectors, $|v_1\rangle$ and $|v_2\rangle$, effectively form a rotated set of axes. Students confirm that the vector $|u\rangle$ remains unchanged regardless of which set of axes it is plotted against (so long as the appropriate components are used for each basis).

Follow-up questions in part IV ask students about the relationship between $|u\rangle$ and $|k\rangle$, what their inner product would be, and to synthesize that changing the basis of a state does not affect the physical information (LG1). Lastly, students are asked two questions about labeling states. The first of the questions addresses whether relabeling the state as $|k\rangle$ was necessary. For the second question, they are given a state written out as $|\psi\rangle = a|+\rangle + b|-\rangle = c|+\rangle_x + d|-\rangle_x$ and asked "why do we not need a subscript on the state $|\psi\rangle$?" Both the z-basis and x-basis spin states are well known by this point in the course. These follow-up questions also target LG1, and are meant to encourage students to extend their conclusions to the quantum mechanics context. Additionally, they can conclude, based on their work, that a different basis representation in quantum mechanics is analogous to a projection (LG2, LG3b) to different sets of axes that, in turn, provides different sets of probabilities (LG3a, LG4).

The final portion of the activity, part V, draws out explicit discussion aligned with LG4 about what information can be *read from* or *provided by* a state. While this idea has been hinted at throughout the previous portions of the tutorial, part V directly reinforces it. Students are first prompted to think about an inclined plane from classical mechanics, draw a coordinate system for tracking the position of the car, draw another coordinate system for measuring gravitational potential energy, and discuss whether changing the coordinate system affects the physical scenario. Following the same theme, students are then asked several questions within the context of quantum mechanics; for example, "when they might prefer using a state written in the x basis as compared to the z basis," and "does a change of basis affect the physical system?"

IV. METHODOLOGY, CONTEXT FOR TUTORIAL DEVELOPMENT, AND EVALUATION

Development of the quantum basis tutorial took place in five stages across three universities (A, B, and C) and spanned three years. Preliminary investigations took place over the first two years. Interviews and assessment data revealed the need for a tutorial about basis, and allowed us to write the initial alpha version of the QBT. In year 2, we administered the alpha version at university A, conducted further investigations, and made improvements to the tutorial activity. In year 3, we administered the beta version and began to evaluate the tutorial's effectiveness. Figure 3 presents a timeline of these stages, including all the instances where we collected data in assessments, surveys, and interviews. In this section, we describe the courses in which we collected data and administered the tutorial, as well as our methods for data collection and analysis.

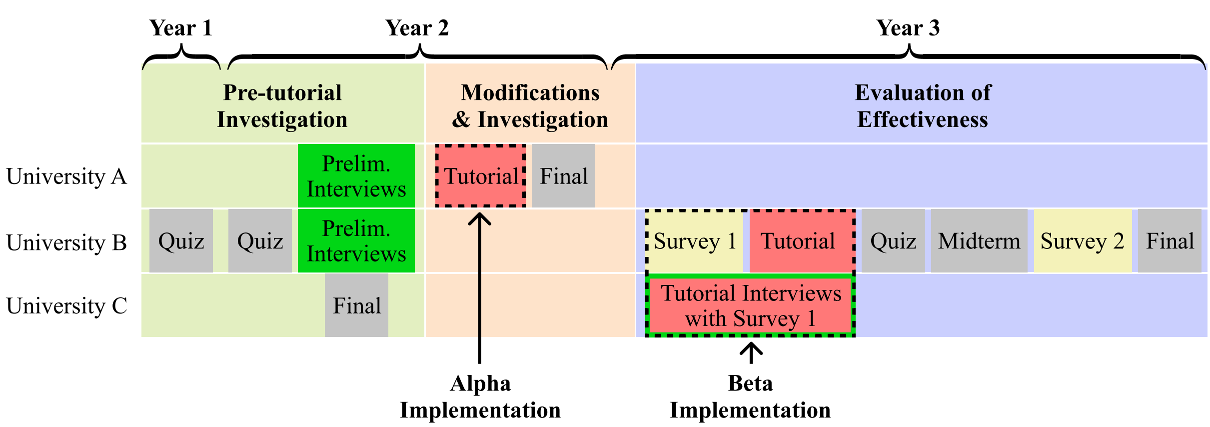

FIG. 3. Timeline for research, tutorial development, and evaluation. The three main phases of the project are identified in the column header and the different years of the project are shown with brackets above the table. Each quiz, midterm, and final included a change-of-basis question (similar to the one in Fig. 4). The tutorial was not repeated at University A in year 3 due to a change in instructor. All data collection and tutorial implementation was completed prior to the 2020 pandemic.

A. Courses included in this study

We collected data on student understanding of basis at three large, public universities in upper-division spins-first quantum mechanics courses. University A and University B are large, public, primarily undergraduate, Hispanic-serving institutions. University C is an R1, PhD-granting institution. "Spins-first" refers to one of two common curricular paradigms in quantum mechanics [53]. In a spins-first instructional paradigm, students are initially introduced to the Stern-Gerlach experiment and learn about the postulates of quantum mechanics in the context of a spin-½ system. This is in contrast to the position-first paradigm, which begins by solving Schrödinger's energy eigenvalue equation for continuous wave functions.

All the quantum mechanics courses in this study followed a similar content trajectory, aligned with a common course textbook [54]. They used interactive instructional strategies (such as peer instruction [55]), and drew from the same set of tutorials, clicker questions, and homework assignments from *Adaptable Curricular Exercises for Quantum Mechanics* (ACEQM) [1]. Universities A and B have similar pacing and cover the postulates of quantum mechanics in spin-½ systems including time evolution, entanglement, and transition to wave functions in the context of infinite and finite square well potentials. University C progresses at a faster pace and additionally covers the free particle, orbital angular momentum, and the hydrogen atom.

The topics of basis and procedures for changing basis are introduced early in each course as part of the discussion of probability and measurement of spin operators. Students are introduced to basis using both Dirac and matrix notation and are given the eigenstates of the S_y and S_x operators (y- and x-basis states) in terms of the conventional z basis. Students use these basis states to reason about measurement and make calculations. As part of instruction, students are shown the formula for a state's basis expansion coefficients in terms of inner products.

Linear algebra, as offered by the math department, is not required as part of the physics curriculum at Universities A and B, but is taken by some students. At these two institutions, the linear algebra that students learn prior to quantum mechanics courses is from a mathematical methods for physics course taught by the physics department. At University C, a one-semester linear algebra course is a required prerequisite for quantum mechanics.

B. Alpha implementations of tutorial

The first version of the QBT was developed and piloted by two of the authors in a required recitation section during week 11 of the upper-division quantum mechanics course at University A in Year 2 (see timeline, Fig. 3). By this point in the semester, all spins content had been presented and the course had transitioned to topics related to wave functions for the infinite square well. The tutorial is presented in the context of a spin-½ system and was intentionally designed for use earlier in the semester. However, the tutorial was developed as an immediate response to observations from this course (see Sec. V A). The late administration served to test the activity's potential utility, solicit feedback from students, and also readdress student understanding of basis before the end of the semester.

At the time of the tutorial's implementation, 37 students were enrolled across two recitation sections of the upper-division quantum mechanics course. Students worked in groups of 3–4. This alpha version of the tutorial consisted only of the core tasks and questions (Parts II–IV from Table I). Students were also asked to provide open-ended feedback about the tutorial. students' worksheets were collected and scanned.

C. Beta implementation of tutorial

During year 3, as part of the continued development of the QBT, the expanded beta version was administered at University B and was coupled with various pre- and postevaluations as part of a larger study. The beta version was also administered in year 3 in an interview setting at University C. Data related to the beta version was not collected from University A because of a change in instructor. The beta version contained all parts I–V.

At University B, the tutorial was administered in lecture by two of the authors during the fourth week as part of regular course instruction after change of basis had been covered. This quantum mechanics class had 26 students, and students worked on the activity in groups of 3–5. While most groups completed the activity during the 75 min class period, a couple groups needed to complete Part V outside of class. At the beginning of the following lecture, there was a whole-class discussion about students' findings from the last two parts of the activity. Students' work on the tutorial was not graded, but was collected and scanned after all groups were given time to complete part V. Students were later asked for feedback on the activity on an optional, prelecture assignment.

At University C, the tutorial was run in three semi-structured hour-long interviews with three pairs of paid student volunteers (six total students). Students were currently enrolled in the upper-division, spin-first quantum mechanics course and had received all relevant instruction on basis in a spin-½ system. Interviewees individually completed an on-paper version of the survey 1 administered at University B before working on the tutorial (see Fig. 5 below). Interviews allowed for focused investigation into how students interacted with the tutorial, leading to refinements of the beta version as well as providing a new context for studying student ideas. The interviewees were the only students at University C who completed the QBT.

D. Explorations of content understanding

In addition to observing students during the alpha- and beta-implementations of the QBT, we also investigated

Consider a spin-1/2 particle prepared in a the state $|\psi\rangle$, written in the z-basis as $|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$. Write the state in the n-basis assuming

$$|+\rangle_n = \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle \text{ and } |-\rangle_n = \frac{\sqrt{2}}{\sqrt{3}}|+\rangle - i\frac{1}{\sqrt{3}}|-\rangle$$

Hint: Solve for the values of a and b such that $|\psi\rangle = a|+\rangle_n + b|-\rangle_n$.

FIG. 4. One version of the quiz and exam change-of-basis question. Versions of this question were administered to students at each of the three institutions on several for-credit quizzes or exams in multiple semesters. Different versions of the question used different numerical coefficients and/or used varied language for the hint.

student understanding of basis and change of basis on exam questions, in interviews, and through conceptual surveys. The quizzes and midterm (only given at University B) and final exams (given at all universities) shown in the timeline (Fig. 3) all included a similar change-of-basis question. The question, presented in Fig. 4, asked students to rewrite a spin-½ state (originally provided in the z-basis) in terms of new basis states $|\pm\rangle_n$. Different iterations of the question included different numerical coefficients in the definitions of $|\pm\rangle_n$. The language of the hint varied but always referenced writing the state in terms of the n-basis states.

The change-of-basis question was coded for correctness (i.e., did students get the answer $|\psi\rangle = a|+\rangle_n + b|-\rangle_n$, with the correct numerical values for a and b?). In addition, we categorized the methods students used to change basis. Descriptions of the methods we observed are provided in Sec. V A as well as in Ref. [2]. The categorization scheme, as well as the category assigned to each sample of student work, was agreed upon by multiple authors.

A similar change-of-basis question was also given to students in interviews (the "prelim. interviews" on the timeline in Fig. 3). The interviews were designed to assess whether students attempted or were aware of projection. Students who attempted other methods were asked if there were additional methods that could be used for changing basis. Interviewees' methods for changing basis were categorized using the same scheme as used for the exam questions. In addition, interviewees were asked whether the state written in the new basis represented the same state as when it was written in the old basis.

As shown in the timeline (Fig. 3), the change-of-basis question was given at University B on a quiz in years 1 and 2 without the QBT intervention, and was given again following the QBT implementation in year 3, this time on a final exam. This allowed us to compare the performance (as well as the methods used to change basis) of a similar population of students with and without the QBT. The final exams given in year 2 at Universities A and C allowed us to assess the extent to which methods for change of basis existed at other institutions with different student populations.

Consider a spin-1/2 electron prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|+\rangle + \frac{2}{\sqrt{3}}|-\rangle$$

1. Which expression correctly converts $|\psi\rangle$ into the x-basis? **Circle any correct options.** $|+\rangle_x$ and $|-\rangle_x$ refer to the spin-up and spin-down states along the x-direction.
   a. $\frac{1}{\sqrt{3}}|+\rangle_x + \frac{\sqrt{2}}{\sqrt{3}}|-\rangle_x$
   ***b.** ${}_x\langle+|\psi\rangle\,|+\rangle_x + {}_x\langle-|\psi\rangle\,|-\rangle_x$
   c. $|{}_x\langle+|\psi\rangle|^2\,|+\rangle_x + |{}_x\langle-|\psi\rangle|^2\,|-\rangle_x$
   d. ${}_x\langle+|+\rangle\,|+\rangle_x + {}_x\langle-|-\rangle\,|-\rangle_x$
   e. $\frac{1}{\sqrt{3}}{}_x\langle+|+\rangle + \frac{\sqrt{2}}{\sqrt{3}}{}_x\langle-|-\rangle$
2. By writing the state $|\psi\rangle$ in the x-basis, we've changed the probabilities for measuring along the z-direction. Choose one and *explain your response*: True ***False**
3. By representing the state $|\psi\rangle$ in the x-basis, we've created a new quantum state. Choose one and *explain your response*: True ***False**

FIG. 5. Survey 1: Three of the questions given to students as a pre-test for the quantum basis tutorial. Questions are numbered here for reference. Correct answers are bolded and starred. In year 3, question 2 was used again on the quiz, and question 3 was incorporated into survey 2.

Also in year 3 at University B, students were given an online prelecture survey (survey 1) to complete as a pretest prior to the implementation of the beta version of the QBT. Relevant survey questions are shown in Fig. 5: students were asked about the coefficients in the Dirac representation of the state and given true-false questions that asked for explanations (Fig. 5). Students were not given answers to any of the survey questions. As mentioned above, survey 1 was also given to interviewees at University C, before they completed the tutorial during interviews.

Question 2 from survey 1 (Fig. 5) was used again on the quiz at University B in year 3. This quiz came after the tutorial, and thus served as a post-test for question 2. Similarly, question 3 was repeated at the end of the semester as part of another online prelecture survey (survey 2 on the timeline), serving as a post-test for question 3.

V. DEVELOPMENT OF THE QUANTUM BASIS TUTORIAL

This section aims to explicate the process of developing a tutorial, along with the research findings that are used to inform the development. The following Sec. VI then provides the results on the impact of the tutorial using the data collected in the larger study around the beta implementation. The four subsections here map to the

TABLE II. A list of the different methods students used to change basis and the frequency of each method at Universities B and C. The quiz was given at University B for credit within 2 weeks following all related instruction. For the three methods that can lead to correct answers, the number in parentheses indicates the percentage of all students who both applied the method and also arrived at the correct answer. These data were collected before any tutorial implementation.

	Quiz	Quiz	Final
	$N = 29$	$N = 34$	$N = 63$
Applied method	University B Year 1	University B Year 2	University C Year 2
Projection (correct/N)	31% (10%)	30% (18%)	57% (42%)
Substitution (correct/N)	48% (3%)	29% (3%)	19%(6%)
System of Eqs. (correct/N)	3% (3%)	6% (0)	6% (2%)
Other	10%	33%	16%
Blank	7%	3%	2%

pretutorial investigation, alpha implementation, modifications, and beta implementation from Fig. 3, respectively.

A. Results of pretutorial investigations

We analyze the methods students use to change basis from two years of quiz data from University B and one year of final exam data from University C, from before the implementation of the tutorial [2]. Students were given a state written in the z basis and were asked to write the state in terms of new basis states $|\pm\rangle_n$ (Fig. 4). Table II highlights the different methods that students used when changing basis and the percentage of all responses that were correct and used that method.

Analysis of these assessments (and several interviews) including a change-of-basis question revealed several common methods that students used when attempting this change of basis: *projection*, *substitution*, and *system of equations*. The *projection* method involves calculating inner products, ${}_n\langle\pm|\psi\rangle$, to determine the expansion coefficients in the new basis (Fig. 6). The *substitution* and *system of equations* methods are more algebraically demanding. The *system of equations* method treats the new basis states as a system of equations that can be solved for the $|\pm\rangle$ states expressed in the n basis. Alternatively, the *substitution* method begins with inserting the $|\pm\rangle_n$ representations into the generic state, $|\psi\rangle = a|+\rangle_n + b|-\rangle_n$, and then creating a system of equations from comparison with the original state (Fig. 7). All three of these methods can be used to correctly find the new basis representation.

The "Other" category contains methods that were used less than 10% of the time. These other methods resulted in incorrect answers, such as constructing a matrix for S_n or deriving the spherical coordinate angles associated with the orientation of a Stern-Gerlach apparatus. One incorrect method drew on probability: students used the probabilities for measuring the corresponding outcomes as the coefficients for the new basis states, or they used the square root of these probabilities without regard for phase. Another incorrect method involves students swapping subscripts (Fig. 8), replacing the given $|+\rangle_n = \frac{1}{\sqrt{3}}|+\rangle + \frac{\sqrt{2}}{\sqrt{3}}|-\rangle$ with $|+\rangle = \frac{1}{\sqrt{3}}|+\rangle_n + \frac{\sqrt{2}}{\sqrt{3}}|-\rangle_n$. Although this method works for a few special cases, it does not work in general; in this example it fails when the basis state includes complex coefficients.

Applied by approximately 30% of students at University B, *projection* was not a particularly common method. Furthermore, it only resulted in the correct answer for

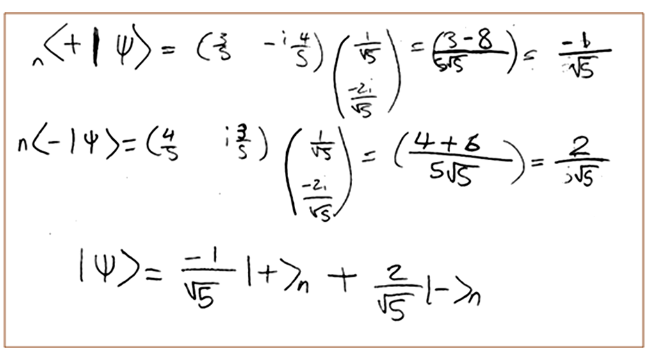

FIG. 6. Student solution to a standard change-of-basis problem. This student uses the *projection* method by taking the inner products of the state with the new basis vectors. The short calculation correctly and efficiently provides the coefficients in the new basis representation.

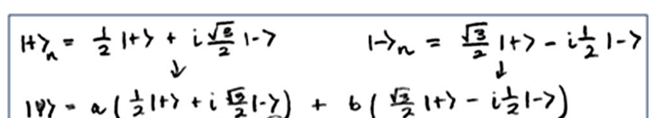

FIG. 7. The start of a student's solution to a change-of-basis question. This student employs the *substitution* method, where they insert the basis conversions into the generic basis representation. The subsequent steps (not shown) involve rearranging terms, setting each coefficient equal to the coefficient of the given state, and solving a system of equations for a and b. Reproduced from Ref. [2].

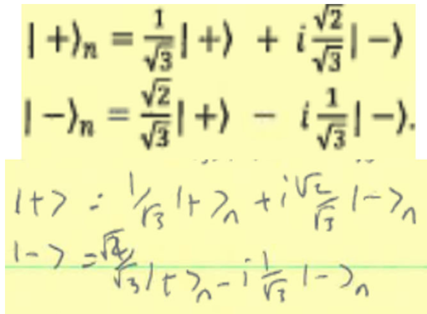

FIG. 8. An image of the given basis states (top) and a student incorrectly determining the z-basis representation by *swapping subscripts* (bottom). The student then substitutes these into the given state to change basis.

10%–18% of all University B students. *Projection* is not the only correct method, but it often requires the least algebra and is the most closely related to the meaning of basis, because it selects the probability amplitude associated with the measurement of a given basis state. Moreover, Table II shows that students using projection more often arrived at the correct answer. On the two University B quizzes, 33% and 60% of the students who applied projection arrived at a correct answer. This contrasts with the low percentages associated with the *substitution* and *system of equations* methods. The most common alternative method was *substitution*. Many students using this method did not finish the calculation or made an algebraic mistake, which resulted in its low rate of correctness.

The poor performance on University B's quiz question—especially the small number of students using the preferred method to change basis—was a key motivation behind the development of the QBT, along with classroom and interview observations at Universities A and B. More specifically, these data directly informed LG2. At University C, fewer than half of the students correctly used the projection method to change basis on a final exam, suggesting that the QBT could be beneficial to students at varied institutions.

B. Observations from the alpha-version implementation

For the initial implementation of the alpha version, students were given the core components of the activity (parts II–IV) that cover the analogy to Cartesian coordinates: plotting a vector $|u\rangle$ using $|i\rangle$ and $|j\rangle$ basis vectors, a change-of-basis representation and plotting using the new $|v_1\rangle$ and $|v_2\rangle$ basis, and the final comparison of graphical representations in the two different bases.

In the first part of the alpha version (part II), students successfully recognized $|i\rangle$ and $|j\rangle$ as basis vectors and subsequently graphed the vector $|u\rangle$ (Fig. 1). Following the graphing aspect, students then translated the state to a column vector representation and wrote the appropriate inner products without instructor intervention. Almost all students went on to define the inner products using the language of "projection" as shown in the following written response:

> *The inner product is the projection of $|u\rangle$ onto either the $|i\rangle$ or $|j\rangle$ axes. (like a dot product).*

Overall, our observations suggested that students could easily adopt Dirac notation in the Cartesian context.

We found that the next part of the tutorial (part III), in which students were asked to convert the state to the basis of $|v_1\rangle$ and $|v_2\rangle$ was more difficult for most students, as many attempted the more rigorous substitution methods. Instructor intervention was needed for almost all groups to cue the calculation of the coefficients using the projection method. Once the state was written, students quickly graphed the state using the v_1 and v_2 axes and correctly labeled the inner products. Students were asked to temporarily label the state in the v basis as $|k\rangle$. The tutorial explicitly states that students will later decide whether this relabeling was necessary.

In part IV, the majority of students drew $|v_1\rangle$ and $|v_2\rangle$ correctly when graphing all the vectors using $|i\rangle$ and $|j\rangle$ as the basis. Students who plotted carefully arrived at a representation showing the vectors $|u\rangle$ and $|k\rangle$ were indeed the same regardless of basis representation (Fig. 2). Because of difficulty plotting irrational values without graph paper, a few students erroneously drew $|u\rangle$ and $|k\rangle$ as different vectors. Students were able to fix this error after being encouraged to plot carefully.

By this point in the activity, all groups correctly answered that $\langle u|k\rangle = 1$. Most groups drew this conclusion based on their graph, but the few students who had difficulty graphing were able to explicitly compute the inner product. It was common for students in each group to express surprise and excitement by this discovery. When first changing the basis, no student objected to the initial relabeling. When asked to explain what changing basis means for a quantum state, most students wrote something explicitly related to reference frames and acknowledged that the vectors, $|u\rangle$ and $|k\rangle$, were the same, as shown in the two written responses below.

> *It's the same vector but with a new reference frame. Changing the basis is just another way of writing the vector. It does not physically change the state.*

Responses of this nature indicate that the activity was successful in supporting students' understanding of basis representation, and that students were able to recognize that change of basis does not physically change the state but is a choice of representation (LG1).

In written feedback, students described the activity as helpful and said they felt it allowed them to relate mathematics and physics concepts. Two representative student responses are shown below.

TABLE III. A list of the different methods students used to change basis and the percentage of each method on the final exams at University A. For the three methods that can lead to correct answers, the percentage of students would applied that method and arrived at the correct answer are shown for each assessment.

	$N = 37$
Applied method	Post-tutorial
Projection (correct/N)	38% (32%)
Substitution (correct/N)	14% (5%)
System of Eqs. (correct/N)	24% (5%)
Swap subscripts	14%
Other	11%
Blank	0

> *The [tutorial] generalizes the procedure for changing basis and highlights that changing basis does not affect the overall value of the vector, only its components. It was nice to relate the math to the concepts.*

Consistent with the above, the remaining feedback was overwhelming positive.

Whereas the alpha version of the QBT successfully achieved LG1 (recognition that the state is unchanged), results from a change-of-basis question given on the University A final shows it was not as successful with LG2 (use projection to change basis). The exam was administered several weeks after the QBT, and we found that approximately a third of the students used projection to change basis (see Table III). Without any pretutorial data for University A, we could not yet comment on the effectiveness of the activity. The limited use of projection could be related to several factors. First, the activity was given later in the semester, well after the spin portion of the course had concluded. Second, the alpha version of the tutorial did not emphasize the direct connection between projection and the probability amplitudes.

C. Modifications to the QBT based on research and observation

The observations above, specifically the difficulty of part III, motivated the creation of two additional sections to the tutorial (see Table I for an overview of the QBT). One section at the beginning was developed to reinforce the idea of probability amplitude and how it relates to basis (which became part I), and one section at the end provided physical and problem-solving contexts to help solidify students' understanding (which became part V).

Part I intentionally reinforces the connection between probability calculations and the use of inner products to represent the coefficients (e.g., $|\langle +|\psi\rangle|^2 = |a|^2$ versus $\langle +|\psi\rangle = a$) by asking students to make, label, and discuss histogram representations for probability and probability amplitude. This change coincides with the addition of LG3 and was partly inspired by a small number of responses ($< 10\%$) from the University C final exam where students used a probability calculation, ($|_n\langle +|\psi\rangle|^2$) and used the result inside the modulus square for the new expansion coefficients Fig. 9. This suggested that students are more comfortable with finding probabilities and could use a refresher on the connection between probability and probability amplitude.

Part V was added to the tutorial to emphasize that changing basis can be a useful tool (LG4) and does not indicate a change to the physical system (LG1). This addition asks students about which basis is best for solving a given problem and were meant to reinforce the reasons why we change basis representation, specifically in regards to being able to easily access a given set of information (LG4).

Parts II, III, and IV were modified slightly to improve how students engaged with the tutorial but the substance of these sections remained unchanged. We added grid templates to the graphing questions to assist students in the accuracy of graphing the vectors. We also changed wording and reordered text based on observation of student struggles. For example, the portion where students were asked to temporarily label the vector in the new basis as $|k\rangle$ was moved to just before the student graphed the vector rather than after the initial basis change. The text was changed to indicate that the relabeling was temporary and the meaning would be immediately discussed.

D. Observation of the beta implementation

With the additions of Part I and Part V, as well as other minor changes, a beta version of the tutorial was implemented in the classroom at University B and in interviews at University C. In this section we discuss observations of the new groups of students working through the tutorial. In the beta implementation, we saw that students were able to use Part I as a touchstone to engage with the unchanged parts of the tutorial. This may be due to the improvements made to the tutorial and/or the administration of the tutorial during the spin portion of the course right after the introduction of bases.

The newly added part I was intended to provide students with the scaffolding to successfully engage with probability amplitudes in Secs. II–IV, and begins by asking students what the coefficients for the state $|\psi\rangle = \frac{3}{5}|+\rangle - \frac{4}{5}|-\rangle$ are called. We found that a number of students who were unfamiliar with the language of the intended answer: "probability amplitudes." Most interviewed students were likewise unfamiliar with that terminology, and even expressed surprise that a probability *amplitude* was permitted to be negative. As a result, the language was changed following the beta implementation to introduce the coefficients as being called probability amplitudes and then ask "what do they tell you?"

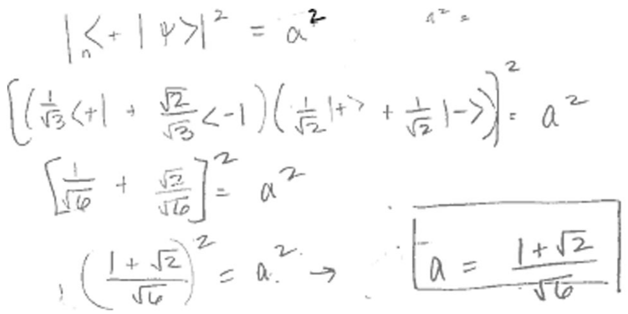

FIG. 9. Portion of a student's solution where they used probability calculations to find the expansion coefficients in a new basis representation. They started a probability calculation, $|_x\langle+|\psi\rangle|^2$, but rather than complete the complex square, they just took the result of the inner product from inside the complex square.

When students were asked to plot a histogram of probability amplitudes, many did not initially account for the negative sign associated with the second coefficient, but corrected the mistake following group discussion. In the moment, all students labeled the Dirac expression for probability and determined the appropriate expression for probability amplitude on the two histograms. Students also correctly articulated the relationship between the probability amplitude and the probability.

With the addition of the grids to the graphing questions, students more easily plotted the vectors than in the alpha implementation (Fig. 10). In both versions, students easily labeled and identified the projections for the vector given in terms of $|i\rangle$ and $|j\rangle$.

The first question of part III asks students to change basis. During the in-class tutorial implementation, we observed that many students did not recognize that they could use projection as a method for changing basis. In response, instructors were able to direct students' attention to part I of the tutorial without explicitly telling students to calculate inner products. Of the three interviewed pairs, one pair did recognize that they could use projection unprompted when the tutorial asked them to perform a change of basis. Following some guidance from the interviewer, all three discussed the strategy in detail before carrying it through, such as debating which inner products needed to be calculated to find the two coefficients. When asked to reflect at the end of the interviews, several interviewees named this basis-change question as one of the most useful. They said they appreciated the practice changing basis and computing inner products.

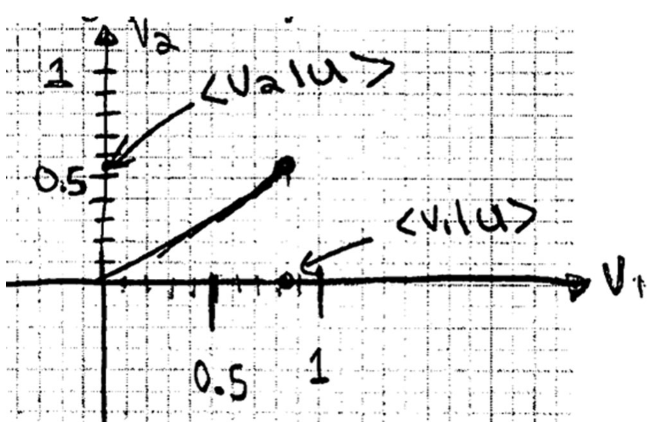

FIG. 10. Student's graph of the vector in the new basis. Inclusion of the grid allowed students to create their own scale and graph the vectors more accurately.

As described in Sec. III, part IV of the activity is designed to have students recognize that a vector points in the same direction, even when it is written in different bases. None of the interviewees initially expressed the expectation that the vector would be the same, and some appeared excited to discover that they were after plotting the two vectors. One student interviewed from University C exclaimed

Ohh, what?! [Laughs] That's cool!

This section benefited the most from the grid template since it allowed students to more accurately plot their vectors and see that vector, $|u\rangle$, was unchanged (Fig. 2). However one student treated the two sets of basis vectors as the same set of horizontal and vertical axes (Fig. 11). This error was fixed with instructor intervention, when instructors pointed out that the column vectors for the $|v_1\rangle$ and $|v_2\rangle$ were written in the $|i\rangle$ and $|j\rangle$ basis.

The remaining questions in part IV were the same as the alpha version. These questions prompt students to use the analogy of basis in Cartesian coordinates and to explain the reasons for changing basis (LG4). In response to these questions, we found that students had rich discussions regarding the role of notation in communicating the basis used to represent a vector. Example written student responses to the question "Explain what changing basis means for a quantum state" are given below.

We are reorienting the axes of our system.
It means to change to a new perspective.

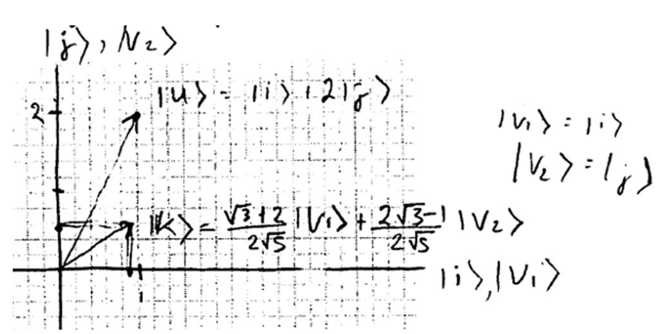

FIG. 11. Student work where both sets of basis vectors are drawn as the horizontal and vertical axes. Here the student incorrectly equates the unit vectors.

Changing the basis vectors…gives the same vector written in a different basis.
It [the state] doesn't change. It's expressed in another form.

As intended, the QBT allowed students to make sense of basis representation in quantum mechanics. In particular, the use of the Cartesian analogy provided a foothold for students, as shown in the first quote, where the student references "reorienting the axes" as they did in part III.

The last follow-up question in part IV expressed the state $|\psi\rangle$ in two different bases ($|\psi\rangle = a|+\rangle + b|-\rangle = c|+\rangle_x + d|-\rangle_x$) and asked why a subscript was not needed on the ket $|\psi\rangle$. Most students agreed the subscripts were not needed on $|\psi\rangle$ because "the subscripts on the kets tell us what basis it is in and $|\psi\rangle$ does not have to stay in one particular basis." However, not all students reached that conclusion. A few students argued that a subscript would be useful to indicate the direction because without it the z direction would be implied.

Written responses from the in class implementation were consistent with interviews at University C. One pair of interviewees debated the idea that relabeling the vector when expressing it in a new basis was useful, "just so you don't confuse it." Meanwhile, when considering the role of subscripts on a ket, one student incorrectly concluded that, if adding an x subscript was not necessary when expressing a ket in the x basis, then the subscripts on $|\rangle_x$ and S_x were optional.

In Part V of the tutorial, the focus is on why it is useful to change basis. We found this section straightforward for students after they had completed the other portions of the activity. While engaging with the inclined plane example, students acknowledged that rotating the coordinate system made it easier to solve for the position of the car, while using the more standard horizontal or vertical axes made solving for potential energy easier, but that neither choice affected the physical scenario. Students then associated these responses with quantum mechanics, noting that it was easier to solve for probabilities from measurements of S_z when the state was represented in the z basis. Students' correct responses to this last question about identifying what information is easily accessible, show the success of the tutorial in addressing LG4.

The interview protocol concluded with a brief sequence of open-ended questions prompting students to reflect on the activity. We found their responses were similar to student feedback on the alpha version. When asked, students identified a variety of things that they felt they had learned and that they found useful (or not) about the activity. First, all interviewees named practice with, and improved understanding of, the change-of-basis procedure as a key takeaway of the activity. For example, one student said that she

knew [the projection strategy for changing basis], but not like on the same level that I do now…I didn't fully understand it…I knew the equations but not what they meant.

Students also appreciated how, "all the graphing," allowed them to visualize the process.

Second, all pairs of interviewees considered the idea that basis change does not affect the vector as another key lesson. When asked what his main takeaways were, one student remarked

Changing basis does not change the vector. I did not know that prior to this.

Another student said the activity helped her specifically by "distinguishing change of basis from measurement."

Multiple students also discussed how the tutorial helped them solidify their understanding of the new notation introduced in the course. One student said

We learned many notations in the QM class and this thing helped me clear up the things we learned,

while another remarked, "I like bringing it back to 2D," because she found that all the new terminology in QM made it difficult to recognize relationships to concepts she already knew. The takeaways that students identified aligned with our intended learning goals for the activity.

VI. EVALUATING EFFECTIVENESS OF THE QBT

Our evaluation of the QBT explicitly assessed how the tutorial addresses our first two learning goals. We assessed LG1 by analyzing student responses to a pair of pre and post-test questions aimed at the effect of changing basis on a state given before and after the implementation of the tutorial. To study the effect on the methods students use to change basis, we compare the results of the quiz results to years without the QBT and to responses to question 1 on survey 1 (Fig. 5). LG3 and LG4 were the main catalysts for the addition of part I and part V of the tutorial and while our observations suggest the tutorial is successful, we leave more detailed analysis for future work.

To discuss effectiveness, we report statistical significance (p value) and effect size. A p value less than 0.05 is taken to be a measure of statistical significance for comparison between two sets of data. Given the small size of our sample, we additionally report p values within the range of 0.05–0.1, which could be interpreted as potentially significant for a small sample size. To report effect size, we apply Cramer's V, which measures how strongly two categorical fields (counts of true or false responses) are associated [56]. The effect size for Cramer's V depends on the degrees of freedom: number of choices minus 1. For our true or false questions the degree of freedom is 1, so a small effect size is given around $V = 0.10$, a medium effect size is given around 0.30 and a large effect size is 0.50 or higher.

A. Pre and post comparison related to LG1: Effects on the state

In year 3, two questions (Q2 and Q3 in Fig. 5) were given at University B around the beta implementation of the tutorial in order to probe the tutorials effectiveness at improving student understanding about the effects of changing basis (LG1). Both questions were true or false (and asked for explanations) and referenced a state given in the z basis and asked if the following were true: (Q2) "by writing the state $|\psi\rangle$ in the x basis, we have changed the probabilities for measuring along the z direction," and (Q3) "by representing the state $|\psi\rangle$ in the x basis, we've created a new quantum state." Both questions were given before the tutorial on survey 1. After the tutorial, Q2 was given on the quiz three class periods (1.5 weeks) following the tutorial and Q3 was given at the end of the semester on the ungraded survey 2. Tables IV and V show the percentage of correct responses for each questions using matched data.

When answering whether the probabilities were affected (Q2), 75% of matched students selected the correct answer on the pretest but only 54% of all students provided a correct explanation. Roughly half of the explanations considered correct included a statement about basis being a choice of representation of "the same physics" information. The remaining correct explanations noted that the change of basis was a reversible process and that the same probabilities would be found. Students answering incorrectly most often associated changing the basis with making a measurement. The explanations given were consistent with our preliminary interview findings.

TABLE IV. Percentage of students answering that changing the state to the x basis does not change the probabilities for measurements in the z direction (Q2 in Fig. 5). Pretest and post-test results are shown with 23 matched students, representing 92% of the students enrolled in the course. All students provided some explanation.

	Survey 1	Quiz
	Pre-QBT	Post-QBT
Correct	75%	86%
Correct w/ explanation	54%	79%

TABLE V. Percentage of students answering that changing the state to the x basis does not create a new quantum state (Q3 in Fig. 5). Pretest and post-test results are shown with 17 matched students accounting for 68% of enrolled students.

	Survey 1	Survey 2
	Pre-QBT	Post-QBT
Correct	53%	82%
Corr. w/ corr. explanation	47%	59%
Corr. w/ no explanation	0%	23%

On the post-test (quiz at University B) the number of correct responses increased to 86% and the number of all students providing a correct explanation increased to 79%. There was also a noticeable improvement in the quality of students' explanations. Two example student responses after the tutorial are shown below.

> *The probabilities are still the same in the z direction. All we did was find the probability amplitudes in the n direction.*
> *A change of basis means looking at measurements from a different perspective. If we revert n to z we should get the initial probabilities from before. Going mass to moles shouldn't change our initial mass. We will still get our mass if we go from moles to mass.*

In both example responses, students argue that the probabilities would be unchanged for other measurement outcomes. On survey 1, the first student above argued that the probabilities would change because that is what happened based on the Stern-Gerlach experiment. Comparison of the percentage of correct explanations showed a medium effect size ($V = 0.27$, p value of 0.063) for our small N.

When answering the question about whether or not changing basis created a new quantum state (Q3), 53% of students answered correctly on the pretest while 47% of all students also provided a correct explanation. All the correct explanations argued that a change of basis was just a change of representation. Seventeen percent of students additionally included a statement about the reversibility of changing basis.

At the end of the semester, the percentage of students answering correctly increased to 82%, with 59% of all students providing relevant reasoning. As with survey 1, students' explanations were consistent with basis being a change in representation rather than changing the state itself. More students answered correctly with a medium effect size ($V = 0.31$, p value $= 0.067$).

Responses to both of these pre and post-test questions provide modest evidence that the tutorial successfully addresses LG1. After the tutorial, students were more likely to acknowledge that the probabilities for measurements in the original basis representation were unaffected and that changing basis did not create a new quantum state.

B. LG2: Methods used for changing basis

Assessing the changes in students' methods of changing basis is done in two parts. First, from the collected pre- and post-test data we can make comparisons between methods students used to change basis and responses to a question about which representation converts a state to a different basis (Q1 from Fig. 5). Second, since in-class assessment data were collected in previous years, we can make comparisons at equal points in the semester between students who experienced the QBT and those who did not.

TABLE VI. Student response rate on survey 1 question 1 from Fig. 5. Answer options from Fig. 5. The correct option is bolded. Note: students could select multiple choices.

Survey 1 (Pre-QBT)	
Which expression correctly converts $\lvert\psi\rangle$ into the x basis? Circle any correct options.	
Option	% Chosen
a. $\frac{1}{\sqrt{3}}\lvert+\rangle_x + \frac{\sqrt{2}}{\sqrt{3}}\lvert-\rangle_x$	23%
***b.** ${}_x\langle+\lvert\psi\rangle\lvert+\rangle_x + {}_x\langle-\lvert\psi\rangle\lvert-\rangle_x$	37%
c. $\lvert{}_x\langle+\lvert\psi\rangle\rvert^2\lvert+\rangle_x + \lvert{}_x\langle-\lvert\psi\rangle\rvert^2\lvert-\rangle_x$	20%
d. ${}_x\langle+\lvert+\rangle\lvert+\rangle_x + {}_x\langle-\lvert-\rangle\lvert-\rangle_x$	10%
e. $\frac{1}{\sqrt{3}_x}\langle+\lvert+\rangle + \frac{\sqrt{2}}{\sqrt{3}_x}\langle-\lvert-\rangle$	27%

For a pretutorial assessment, we developed a multiple-choice question related to the process of changing basis. The distractors were developed using the list of methods students used when changing basis on for-credit assessments in previous years. The question, number 1 in Fig. 5, asked students to identify which expression(s) correctly convert a state $\lvert\psi\rangle$ into the x basis. Students were allowed to select multiple answers. This question was given online after the requisite instruction related to basis and projection but before tutorial instruction. Results are presented in Table VI. Out of 30 students, 11 (37%) identified the equivalent of using projection (option b), where the coefficients were replaced with the Dirac expression for the probability amplitude. However, only 20% of all students correctly identified projection as the sole correct answer. This is consistent with preliminary findings (Table II) in which only ~30% of students used projection to change basis at this point in the semester. Each of the distractors for this question were chosen at similar rates (with the exception of option d).

As a post-test comparison, we used the change-of-basis question asked on the quiz (an isomorphic version of the question asked in years 1 and 2—see Fig. 4). The quiz was given three lectures (1.5 weeks) after the tutorial. Results from the quiz, a midterm, and the final exam at University B are presented in Table VII alongside the quiz data from previous years when the tutorial was not run.

Eighty-four percent of students used projection to change basis on the quiz post-QBT, more than double the number of students who identified the use of inner products on survey 1. Making a comparison to the quiz in previous years, the results in year 3 show a marked improvement in the use of projection: 30% (prior years) to 84% (post-QBT). The results following the QBT are a statistically significant increase in the use of projection versus both of the previous years with a large effect size ($V \geq 0.50$, p value < 0.01).

Further analysis of the quiz post-QBT shows that, of the students using projection, about half arrived at the correct answer, while another fifth made an error specifically with complex conjugation. Three weeks later on the midterm exam, students were given another change-of-basis question in an alternative context. On this midterm exam, 100% of students used the projection method with 72% arriving at the correct answer. The results from the midterm and final show that student's use of projection persisted throughout the semester. Addressing an understanding of change of basis at this early juncture could support students in learning other quantum mechanical concepts (e.g., time evolution). The above results suggest the tutorial was effective at addressing LG2.

VII. DISCUSSION

We have described the process of developing a tutorial on change of basis in quantum mechanics, and have conducted a preliminary evaluation of the effectiveness of the final product. Survey and exam performance data suggest that the tutorial helps students learn to carry out and correctly interpret a basis change in the context of spin-½ systems. Development of the tutorial involved an interplay between research into student understanding, activity design, and classroom implementation [57].

TABLE VII. Methods for changing basis from University B from various assessments over a number of years. Quiz and exam data from years prior to tutorial implementation is replicated from previous tables for comparison.

	Quiz	Quiz	Quiz	Midterm	Final
	Year 1	Year 2	Year 3	Year 3	Year 3
	$N = 29$	$N = 34$	$N = 25$	$N = 25$	$N = 25$
Attempted method	No QBT	No QBT	Post-QBT	Post-QBT	Post-QBT
Projection (correct/N)	31% (10%)	30% (18%)	84% (44%)	100% (72%)	100% (72%)
Substitution (correct/N)	48% (3%)	29% (3%)	12% (0)	0 (0)	0 (0)
System of Eqs. (correct/N)	3% (3%)	6% (0)	0 (0)	0 (0)	0 (0)
Other	7%	30%	0	0	0
Blank	7%	3%	0	0	0

Our original motivation for the QBT came from informal classroom observations in which we noticed students struggling to change basis. Analysis of interview and exam data helped us refine two primary learning goals for the activity: For students to be able to (LG1) recognize that changing basis does not change the state or the probabilities of any measurement on the state; and (LG2) use projection as a method to change basis with and without prompting. At this stage, we created an initial version of the QBT, which asks students to represent a 2D Cartesian vector in Dirac notation and its components as inner products, change basis, and plot the vector with both coordinate systems on the same graph (parts II–IV).

Observations from a classroom implementation of the alpha QBT version led us to identify the need for two auxiliary learning goals to support the primary ones: (LG3) Identify the coefficients in a basis expansion (a) physically as probability amplitudes and (b) mathematically as inner products; and (LG4) recognize that a reason for changing basis is making desired information more readily accessible. Parts I and V were added to the tutorial to address these goals, and help better achieve LG1 and LG2. Classroom observations also led to minor modifications to refine the language in the other parts of the QBT.

Following these additions and improvements, we conducted another classroom implementation (beta) of the QBT, and also administered the activity in interviews. We also administered pre- and post-tests (surveys, quizzes, and exams) with conceptual and computational questions about change of basis. This implementation provided additional opportunities for our investigation of student understanding (see Sec. II), led to some additional improvements to the activity, and allowed for preliminary evaluations of the effectiveness of the QBT. Results of the quiz and survey 2 following the beta implementation suggests that the QBT helped students achieve LG1 and LG2. A set of true-false questions showed that, following the QBT, students were better able to articulate that a change of basis is just a change of representation for that state. A comparison of quizzes to prior years showed that students more frequently and more accurately applied projection when changing basis after completing the QBT. Investigation of the QBT with respect to our auxiliary learning goals LG3 and LG4 is left to future research.

In addition to the success of the QBT in the context of spins, in our own classes we were able to use the activity as a touchstone as we progressed through more advanced topics including spin-1 systems, time evolution, and spatial wave functions. We were thus able to fall back on a familiar language and representation. Anecdotally, we observed less student discomfort [58] with the shift to continuous bases, and more spontaneous use of language and ideas about using projection as the primary tool to change basis. Overall, as instructors, we perceived that the QBT gave a meaning to changing basis beyond just being a procedure.

Student understanding of basis presents a fruitful area for research. Our continuing work explores how students connect the concepts related to change of basis for spin systems to the wave function portion of the course. The topic of basis is equally important when working with position, energy, and momentum representations where the same wave function can be written in terms of different variables or as a superposition of energy eigenfunctions which are each individually functions of position. Student feedback from the alpha version showed the activity was helpful even at the later point in the semester, but further research needs to be conducted into how ideas such as inner products or superpositions are affected by the transition from discrete spin states to continuous wave functions.

On-going work is exploring other modalities for tutorial instruction. The time intensiveness of tutorials is often a hurdle for instructors when it comes to incorporating them into the classroom. We have created an online version of the QBT as part of a larger project to adapt this and other ACE-QM [1] activities to an online environment [59]. The online version adds interactive guidance elements. Design of these guidance elements was informed by the results discussed here. In future work, we will be exploring the effectiveness of adaptive, online tutorials.

In summary, we developed the QBT through an iterative process of research and activity design. The process included multiple classroom trials, which led to modifications of the activity and contributed to our investigation of student ideas about basis and change of basis. The result is a tutorial that effectively helps students to make sense of and carry out a change of basis in quantum mechanics by drawing an analogy between quantum state vectors and 2D Cartesian vectors.

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[1] S. Pollock, G. Passante, and H. Sadaghiani, Adaptable curricular excercises for quantum mechanics, https://www.physport.org/curricula/ACEQM/.

[2] G. Corsiglia, B. P. Schermerhorn, H. Sadaghiani, A. Villasenor, S. Pollock, and G. Passante, Exploring student ideas on change of basis in quantum mechanics, Phys. Rev. Phys. Educ. Res. **18**, 010144 (2022).

[3] C. Singh and E. Marshman, Review of student difficulties in upper-level quantum mechanics, Phys. Rev. ST Phys. Educ. Res. **11**, 020117 (2015).

[4] Physics Department Oregon State University, Paradigms quantum activities, https://paradigms.oregonstate.edu/.

[5] University of Washington Physics Education Group, *Tutorials in Physics: Quantum Mechanics* (University of Washington, Seattle, WA, 2014).

[6] E. Marshman and C. Singh, Improving student understanding of Dirac notation by using analogical reasoning in the context of a three-dimensional vector space, in *Proceedings of the 2020 Physics Education Research Conference*, virtual conference (American Association of Physics Teachers, College Park, MD, 2020), pp. 309–314.

[7] R. R. Hake, Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses, Am. J. Phys. **66**, 64 (1998).

[8] S. Freeman, S. L. Eddy, M. McDonough, M. K. Smith, N. Okoroafor, H. Jordt, and M. P. Wenderoth, Active learning increases student performance in science, engineering, and mathematics, Proc. Natl. Acad. Sci. U.S.A. **111**, 8410 (2014).

[9] C. Singh and G. Zhu, Students' difficulties with quantum measurement, AIP Conf. Proc. **1413**, 77 (2011).

[10] R. Sayer, E. Marshman, and C. Singh, Case study evaluating just-in-time teaching and peer instruction using clickers in a quantum mechanics course, Phys. Rev. Phys. Educ. Res. **12**, 020133 (2016).

[11] University of St Andrews, Quvis quantum mechanics visualization project, https://www.st-andrews.ac.uk/physics/quvis/.

[12] C. Singh, Quantum interactive learning tutorials, https://www.physport.org/curricula/QuILTS/.

[13] University of Colorado Boulder, Phet interactive simulations, https://phet.colorado.edu.

[14] P. J. Emigh, G. Passante, and P. S. Shaffer, Developing and assessing tutorials for quantum mechanics: Time dependence and measurements, Phys. Rev. Phys. Educ. Res. **14**, 020128 (2018).

[15] G. Passante and A. Kohnle, Enhancing student visual understanding of the time evolution of quantum systems, Phys. Rev. Phys. Educ. Res. **15**, 010110 (2019).

[16] A. Kohnle and G. Passante, Characterizing representational learning: A combined simulation and tutorial on perturbation theory, Phys. Rev. Phys. Educ. Res. **13**, 020131 (2017).

[17] S. McKagan, K. Perkins, M. Dubson, C. Malley, S. Reid, R. LeMaster, and C. Wieman, Developing and researching PhET simulations for teaching quantum mechanics, Am. J. Phys. **76**, 406 (2008).

[18] C. Keebaugh, E. Marshman, and C. Singh, Improving student understanding of corrections to the energy spectrum of the hydrogen atom for the Zeeman effect, Phys. Rev. Phys. Educ. Res. **15**, 010113 (2019).

[19] E. Marshman and C. Singh, Investigating and improving student understanding of the expectation values of observables in quantum mechanics, Eur. J. Phys. **38**, 045701 (2017).

[20] E. Marshman and C. Singh, Investigating and improving student understanding of the expectation values of observables in quantum mechanics, Eur. J. Phys. **38**, 045701 (2017).

[21] E. Marshman and C. Singh, Investigating and improving student understanding of quantum mechanical observables and their corresponding operators in Dirac notation, Eur. J. Phys. **39**, 015707 (2018).

[22] M. Wawro, K. Watson, and W. M. Christensen, Meta-representational competence with linear algebra in quantum mechanics, in *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, edited by A. Weinberg, J. Rasmussen, J. Rabin, M. Wawro, and S. Brown (MAA, San Diego, 2017), pp. 326–337.

[23] C. Singh, Student difficulties with quantum mechanics formalism, AIP Conf. Proc. **883**, 185 (2007).

[24] E. Marshman and C. Singh, Student difficulties with quantum states while translating state vectors in Dirac notation to wave functions in position and momentum representations, in *Proceedings of the 2015 Physics Education Research Conference, College Park, MD* (American Association of Physics Teachers, College Park, MD, 2015), pp. 211–214.

[25] C. Singh and E. Marshman, Investigating student difficulties with dirac notation, in *Proceedings of the 2013 Physics Education Research Conference, Portland, OR* (American Association of Physics Teachers, College Park, MD, 2013), pp. 345–348.

[26] E. Gire and C. Manogue, Making sense of quantum operators, eigenstates and quantum measurements, AIP Conf. Proc. **1413**, 195 (2012).

[27] E. Gire and E. Price, Structural features of algebraic quantum notations, Phys. Rev. ST Phys. Educ. Res. **11**, 020109 (2015).

[28] B. P. Schermerhorn, G. Passante, H. Sadaghiani, and S. J. Pollock, Exploring student preferences when calculating expectation values using a computational features framework, Phys. Rev. Phys. Educ. Res. **15**, 020144 (2019).

[29] H. Sadaghiani and L. Bao, Student difficulties in understanding probability in quantum mechanics, AIP Conf. Proc. **818**, 61 (2006).

[30] G. Zhu and C. Singh, Improving students' understanding of quantum measurement. I. Investigation of difficulties, Phys. Rev. ST Phys. Educ. Res. **8**, 010117 (2012).

[31] C. Singh, Student understanding of quantum mechanics at the beginning of graduate instruction, Am. J. Phys. **76**, 277 (2008).

[32] G. Passante, B. Schermerhorn, S. Pollock, and H. Sadaghiani, Time evolution in quantum systems: a closer look at student understanding, Eur. J. Phys. **41**, 015705 (2020).

[33] E. Marshman and C. Singh, Framework for understanding the patterns of student difficulties in quantum mechanics, Phys. Rev. ST Phys. Educ. Res. **11**, 020119 (2015).

[34] G. Zhu and C. Singh, Students' difficulties with quantum measurement, AIP Conf. Proc. **1413**, 387 (2012).

[35] G. Zhu and C. Singh, Surveying students' understanding of quantum mechanics in one spatial dimension, Am. J. Phys. **80**, 252 (2012).

[36] T. Wan, P. J. Emigh, and P. S. Shaffer, Investigating how students relate inner products and quantum probabilities, Phys. Rev. Phys. Educ. Res. **15**, 010117 (2019).

[37] T. Wan, P. J. Emigh, and P. S. Shaffer, Investigating how students relate inner products and quantum probabilities, Phys. Rev. Phys. Educ. Res. **15**, 010117 (2019).

[38] K. S. Serbin, M. Wawro, and R. Storms, Characterizations of student, instructor, and textbook discourse related to basis and change of basis in quantum mechanics, Phys. Rev. Phys. Educ. Res. **17**, 010140 (2021).

[39] H. G. Close, C. C. Schiber, E. W. Close, and D. Donnelly, Students' dynamic geometric reasoning about quantum spin-1/2 states, in *Proceedings of the 2013 Physics Education Research Conference, Portland, OR* (American Association of Physics Teachers, College Park, MD, 2014), pp. 93–96.

[40] D. Plaxco and M. Wawro, Analyzing student understanding in linear algebra through mathematical activity, J. Math. Behav. **38**, 87 (2015).

[41] J. Hannah, S. Stewart, and M. Thomas, Emphasizing language and visualization in teaching linear algebra, Int. J. Math. Educ. Sci. Technol. **5211**, 1464 (2013).

[42] S. Stewart and M. O. J. Thomas, Student learning of basis, span and linear independence in linear algebra, Int. J. Math. Educ. Sci. Technol. **41**, 173 (2010).

[43] A. F. Heckler and T. M. Scaife, Adding and subtracting vectors: The problem with the arrow representation, Phys. Rev. ST Phys. Educ. Res. **11**, 010101 (2015).

[44] P. Barniol and G. Zavala, Force, velocity, and work: The effects of different contexts on students' understanding of vector concepts using isomorphic problems, Phys. Rev. ST Phys. Educ. Res. **10**, 020115 (2014).

[45] G. Zavala and P. Barniol, Students understanding of dot product as a projection in no-context, work and electric flux problems, AIP Conf. Proc. **1513**, 438 (2013).

[46] J. Hawkins, J. Thompson, and M. Wittmann, Students consistency of graphical vector addition method on 2D vector addition tasks, AIP Conf. Proc. **1179**, 161 (2009).

[47] B. E. Hinrichs, Writing position vectors in 3-d space: A student difficulty with spherical unit vectors in intermediate E&M, AIP Conf. Proc. **1289**, 173 (2010).

[48] B. P. Schermerhorn and J. R. Thompson, Physics students' construction of differential length vectors in an unconventional spherical coordinate system, Phys. Rev. Phys. Educ. Res. **15**, 010111 (2019).

[49] B. Farlow, M. Vega, M. E. Loverude, and W. M. Christensen, Mapping activation of resources among upper division physics students in non-Cartesian coordinate systems: A case study, Phys. Rev. Phys. Educ. Res. **15**, 020125 (2019).

[50] M. Vega, W. M. Christensen, B. Farlow, G. Passante, and M. Loverude, Student understanding of unit vectors and coordinate systems beyond cartesian coordinates in upper division physics courses, in *Proceedings of the 2016 Physics Education Research Conference, Sacramento, CA* (American Association of Physics Teachers, College Park, MD, 2016), pp. 364–367.

[51] M. Wawro, C. Rasmussen, M. Zandieh, G. F. Sweeney, and C. Larson, An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence, PRIMUS **22**, 577 (2012).

[52] G. Karaoko, Making connections among representations of eigenvector: what sort of a beast is it?, ZDM Int. J. Math. Educ. **51**, 1141 (2019).

[53] H. Sadaghiani, G. Passante, and S. Pollock, Making connections among representations of eigenvector: what sort of a beast is it?, in *Proceedings of the 2018 Physics Education Research Conference, Washington, DC* (American Association of Physics Teachers, College Park, MD, 2018).

[54] D. H. McIntyre, *Quantum Mechanics* (Pearson Education, Inc., San Francisco, CA, 2012).

[55] E. Mazur, *Peer Instruction: A User's Manual* (Prentice Hall, Englewood Cliffs, NJ, 1997).

[56] J. C. Speirs, M. R. Stetzer, B. A. Lindsey, and M. Kryjevskaia, Exploring and supporting student reasoning in physics by leveraging dual-process theories of reasoning and decision making, Phys. Rev. Phys. Educ. Res. **17**, 020137 (2021).

[57] L. C. McDermott, Millikan lecture 1990: What we teach and what is learned—Closing the gap, Am. J. Phys. **59**, 301 (1991).

[58] G. Corsiglia, T. Garcia, B. P. Schermerhorn, G. Passante, H. R. Sadaghiani, and S. J. Pollock, Characterizing and monitoring student discomfort in upper-division quantum mechanics, in *Proceedings of the 2020 Physics Education Research Conference Proceedings*, virtual conference (American Association of Physics Teachers, College Park, MD, 2020).

[59] G. Corsiglia, B. P. Schermerhorn, G. Passante, H. Sadaghiani, and S. Pollock, Interactive online activities for physics learners, https://acephysics.net.