Phase transitions to cooperation in the prisoner's dilemma

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Game theory formalizes certain interactions between physical particles or between living beings in biology, sociology, and economics and quantifies the outcomes by payoffs. The prisoner's dilemma (PD) describes situations in which it is profitable if everybody cooperates rather than defects (free rides or cheats), but as cooperation is risky and defection is tempting, the expected outcome is defection. Nevertheless, some biological and social mechanisms can support cooperation by effectively transforming the payoffs. Here, we study the related phase transitions, which can be of first order (discontinuous) or of second order (continuous), implying a variety of different routes to cooperation. After classifying the transitions into cases of equilibrium displacement, equilibrium selection, and equilibrium creation, we show that a transition to cooperation may take place even if the stationary states and the eigenvalues of the replicator equation for the PD stay unchanged. Our example is based on adaptive group pressure, which makes the payoffs dependent on the endogenous dynamics in the population. The resulting bistability can invert the expected outcome in favor of cooperation.

I. INTRODUCTION

Game theory goes back to von Neumann [1], one of the superminds of quantum mechanics. Originally intended to describe interactions in economics, sociology, and biology [1–3], it has recently become a quickly growing research area in physics, where methods from nonlinear dynamics and pattern formation [4,5], agent-based or particle-like models [5,6], network theory [7], and statistical physics [8] are applied. There are even quantum theoretical contributions [9].

When two entities characterized by the states, "strategies," or "behaviors" i and j interact with each other, game theory formalizes the result by payoffs P_{ij} and the structure of the payoff matrix (P_{ij}) determines the kind of the game. The dynamics of a system of such entities is often delineated by the so-called replicator equations

$$\frac{dp(i,t)}{dt} = p(i,t)\left[\sum_j P_{ij} p(j,t) - \sum_{j,l} p(l,t) P_{lj} p(j,t) \right] \quad (1)$$

[3]. $p(i,t)$ represents the relative frequency of behavior i in the system, which increases when the expected "success" $F_i = \sum_j P_{ij} p(j,t)$ exceeds the average one, $\sum_i F_i p(i,t)$.

Many collective phenomena in physics such as agglomeration or segregation phenomena can be studied in a game-theoretical way [5,6]. Applications also include the theory of evolution [10] and the study of ecosystems [11]. Another exciting research field is the study of mechanisms supporting the cooperation between selfish individuals [1–3] in situations such as the "prisoner's dilemma (PD)" or public goods game, where they would usually defect (free ride or cheat). Contributing to public goods and sharing them constitute ubiquitous situations, where cooperation is crucial, for example, in order to maintain a sustainable use of natural resources or a well-functioning health or social security system.

In the following, we will give an overview of the stationary solutions of the replicator Eq. (1) and their stability properties. Based on this, we will discuss several "routes to cooperation," which transform the prisoner's dilemma into

other games via different sequences of continuous or discontinuous phase transitions. These routes will then be connected to different biological or social mechanisms accomplishing such phase transitions [12]. Finally, we will introduce the concept of "equilibrium creation" and distinguish it from routes to cooperation based on "equilibrium selection" or "equilibrium displacement." A new cooperation-promoting mechanism based on adaptive group pressure will exemplify it.

II. STABILITY PROPERTIES OF DIFFERENT GAMES

Studying games with two strategies i only, the replicator Eq. (1) simplifies, and we remain with

$$\frac{dp(t)}{dt} = p(t)[1 - p(t)]\{\lambda_1[1 - p(t)] - \lambda_2 p(t)\}, \quad (2)$$

where $p(t) = p(1,t)$ represents the fraction of cooperators and $1 - p(t) = p(2,t)$ represents the fraction of defectors. $\lambda_1 = P_{12} - P_{22}$ and $\lambda_2 = P_{21} - P_{11}$ are the eigenvalues of the two stationary solutions $p = p_1 = 0$ and $p = p_2 = 1$. If $0 < \lambda_1/(\lambda_1 + \lambda_2) < 1$, there is a third stationary solution $p = p_3 = \lambda_1/(\lambda_1 + \lambda_2)$ with eigenvalue $\lambda_3 = -(1 - p_3)\lambda_1$. For the sake of our discussion, we imagine an additional fluctuation term $\xi(t)$ on the right-hand side of Eq. (2), reflecting small perturbations of the strategy distribution.

Four different cases can be classified [3]: (1) if $\lambda_1 < 0$ and $\lambda_2 > 0$, the stationary solution p_1 corresponding to defection by everybody is stable, while the stationary solution p_2 corresponding to cooperation by everyone is unstable. That is, any small perturbation will drive the system away from full cooperation towards full defection. This situation applies to the PD defined by payoffs with $P_{21} > P_{11} > P_{22} > P_{12}$. According to this, strategy $i=1$ ("cooperation") is risky, as it can yield the lowest payoff P_{12}, while strategy $i=2$ ("defection") is tempting since it can give the highest payoff P_{21}. (2) If $\lambda_1 > 0$ and $\lambda_2 < 0$, the stationary solution p_1 is unstable, while p_2 is stable. This means that the system will end up

with cooperation by everybody. Such a situation occurs for the so-called harmony game (HG) with $P_{11} > P_{21} > P_{12} > P_{22}$, as mutual cooperation gives the highest payoff P_{11}. (3) If $\lambda_1 > 0$ and $\lambda_2 > 0$, the stationary solutions p_1 and p_2 are unstable, but there exists a third stationary solution p_3, which turns out to be stable. As a consequence, the system is driven towards a situation, where a fraction p_3 of cooperators is expected to coexist with a fraction $(1-p_3)$ of defectors. Such a situation occurs for the snowdrift game (SD) (also known as hawk-dove or chicken game). This game is characterized by $P_{21} > P_{11} > P_{12} > P_{22}$ and assumes that unilateral defection is tempting as it yields the highest payoff P_{21} but also risky as mutual defection gives the lowest payoff P_{22}. (4) If $\lambda_1 < 0$ and $\lambda_2 < 0$, the stationary solutions p_1 and p_2 are both stable, while the stationary solution p_3 is unstable. As a consequence, full cooperation is possible but not guaranteed. In fact, the final state of the system depends on the initial condition $p(0)$ (the "history"): if $p(0) < p_3$, the system is expected to end up in the stationary solution p_1, i.e., with full defection. If $p(0) > p_3$, the system is expected to move towards $p_2 = 1$, corresponding to cooperation by everybody. The history dependence implies that the system is multistable (here: bistable) as it has several (locally) stable solutions. This case is found for the stag hunt game (SH) (also called assurance). This game is characterized by $P_{11} > P_{21} > P_{22} > P_{12}$, i.e., cooperation is rewarding, as it gives the highest payoff P_{11} in case of mutual cooperation but it is also risky as it yields the lowest payoff P_{12} if the interaction partner is uncooperative.

III. PHASE TRANSITIONS AND ROUTES TO COOPERATION

When facing a prisoner's dilemma, it is of vital interest to transform the payoffs in such a way that cooperation between individuals is supported. Starting with the payoffs P_{ij}^0 of a prisoner's dilemma, one can reach different payoffs P_{ij}, for example, by introducing strategy-dependent taxes $T_{ij} = P_{ij}^0 - P_{ij} > 0$. When increasing the taxes T_{ij} from 0 to T_{ij}^0, the eigenvalues will change from $\lambda_1^0 = P_{12}^0 - P_{22}^0$ and $\lambda_2^0 = P_{21}^0 - P_{11}^0$ to $\lambda_1 = \lambda_1^0 + T_{22} - T_{12}$ and $\lambda_2 = \lambda_2^0 + T_{11} - T_{21}$. In this way, one can create a variety of routes to cooperation, which are characterized by different kinds of phase transitions. We define *route 1* [PD→HG] by a direct transition from a prisoner's dilemma to a harmony game. It is characterized by a discontinuous transition from a system, in which defection by everybody is stable, to a system, in which cooperation by everybody is stable [see Fig. 1(a)]. *Route 2* [PD→SH] is defined by a direct transition from the prisoner's dilemma to a stag hunt game. After the moment t_*, where λ_2 changes from positive to negative values, the system behavior becomes history dependent: when the fluctuations $\xi(t)$ for $t > t_*$ exceed the critical threshold $p_3(t) = \lambda_1/[\lambda_1 + \lambda_2(t)]$, the system will experience a sudden transition to cooperation by everybody. Otherwise one will find defection by everyone, as in the prisoner's dilemma [see Fig. 1(b)]. In order to make sure that the perturbations $\xi(t)$ will eventually exceed $p_3(t)$ and trigger cooperation, the value of λ_2 must be reduced to sufficiently large negative values. It is also possible to have a
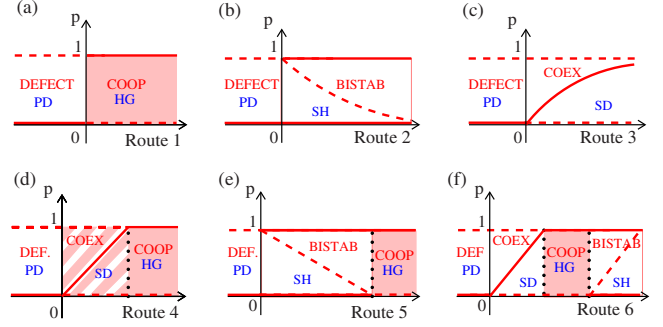

FIG. 1. (Color online) Schematic illustration of the phase transitions defining the different routes to cooperation. The order parameter is the stationary frequency of cooperators, while the control parameters are the parameters r, w, k, m, or q in Nowak's cooperation-enhancing rules [12] (see main text) or, more generally, (non)linear combination of the model parameters b and c. Solid (red) lines represent *stable* stationary proportions of cooperators, dashed lines *unstable* fix points. Diagonal lines show the additional stationary solution p_3, where $0 \le p_3 \le 1$. [p=proportion of cooperators; DEFECT=defection is stable, i.e., everybody defects; COOP =cooperation is stable, i.e., everybody cooperates; COEX =mixture of defectors with a proportion p_3 of cooperators; BISTAB=cooperation is stable if $p_3 < p(0)$, where $p(0)$ means the initial proportion of cooperators, otherwise everybody defects.]

continuous rather than sudden transition to cooperation: we define *route 3* [PD→SD] by a transition from a prisoner's dilemma to a snowdrift game. As λ_1 is changed from negative to positive values, a fraction $p_3(t) = \lambda_1(t)/[\lambda_1(t) + \lambda_2]$ of cooperators is expected to result [see Fig. 1(c)]. When increasing λ_1, this fraction rises continuously. One may also implement more complicated transitions. *Route 4*, for example, establishes the transition sequence PD→SD→HG [see Fig. 1(d)], while we define *route 5* by the transition PD→SH→HG [see Fig. 1(e)]. One may also implement the transition PD→SD→HG→SH [*route 6*, see Fig. 1(f)], establishing a path-dependence, which can guarantee cooperation by everybody in the end. (When using *route 2*, the system remains in a defective state if the perturbations do not exceed the critical value p_3.)

IV. RELATIONSHIP WITH COOPERATION-SUPPORTING MECHANISMS

We will now discuss the relationship of the above-introduced routes to cooperation with biological and social mechanisms ("rules") promoting the evolution of cooperation. Nowak performed his analysis of five such rules with the reasonable specifications $T = b > 0$, $R = b - c > 0$, $S = -c < 0$, and $P = 0$ in the limit of weak selection [12]. Cooperation is assumed to require a contribution $c > 0$ and to produce a benefit $b > c$ for the interaction partner, while defection generates no payoff ($P = 0$). As most mechanisms leave λ_1 or $\lambda = (\lambda_1 + \lambda_2)/2$ unchanged, we will now focus on the payoff-dependent parameters λ_1 and λ (rather than λ_1 and λ_2). The basic prisoner's dilemma is characterized by $\lambda_1^0 = -c$ and $\lambda^0 = 0$.

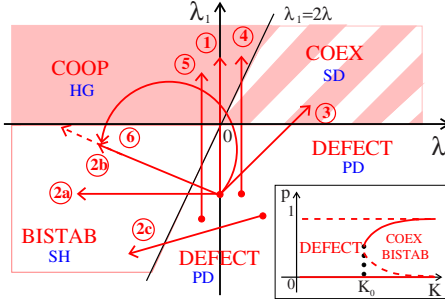According to the supporting online material of Ref. [12],

FIG. 2. (Color online) Phase diagram of expected system behaviors based on an analysis of the game-dynamical replicator Eq. (2) as a function of the parameters λ and λ_1. The different routes to cooperation are illustrated by arrows. Terms in capital letters are defined in Fig. 1. Inset: stable stationary solutions (solid lines) and unstable ones (broken lines) as functions of the parameter K when the reward depends on the proportion of cooperators. The bifurcation at the "tipping point" $K=K_0$ "inverts" the system behavior (see main text).

kin selection (genetic relatedness) transforms the payoffs into $P_{11}=P_{11}^0+r(b-c)$, $P_{12}=P_{12}^0+br$, $P_{21}=P_{21}^0-cr$, and $P_{22}=P_{22}^0$. Therefore, it leaves λ unchanged and increases λ_1 by $T_{22}-T_{12}=br$, where r represents the degree of genetic relatedness. *Direct reciprocity* (repeated interaction) does not change λ_1, but it reduces λ by $-\frac{1}{2}(b-c)[1/(1-w)-1]<0$, where w is the probability of a future interaction. *Network reciprocity* (clustering of individuals playing the same strategy) leaves λ unchanged and increases λ_1 by $H(k)$, where $H(k)$ is a function of the number k of neighbors. Finally, *group selection* (competition between different populations) increases λ_1 by $(b-c)(m-1)$, where m is the number of groups, while λ is not modified. However, λ_1 and λ may also change simultaneously. For example, *indirect reciprocity* (based on trust and reputation) increases λ_1 by cq and reduces λ by $-\frac{1}{2}(b-c)q<0$, where q quantifies social acquaintanceship.

Summarizing this, kin selection, network reciprocity, and group selection preserve $\lambda=0$ and increase the value of λ_1 (see route 1 in Fig. 2). Direct reciprocity, in contrast, preserves the value of λ_1 and reduces λ (see route 2a in Fig. 2). Indirect reciprocity promotes the same transition (see route 2b in Fig. 2). Supplementary, one can analyze *costly punishment*. Using the payoff specifications made in the Supporting Information of Ref. [13], costly punishment changes λ by $-(\beta+\gamma)/2<0$ and λ_1 by $-\gamma$ [13], i.e., when γ is increased, the values of λ and λ_1 are simultaneously reduced (see route 2c in Fig. 2). Here, $\gamma>0$ represents the punishment cost invested by a cooperator to impose a punishment fine $\beta>0$ on a defector, which decreases the payoffs of both interaction partners. Route 3 can be generated by the formation of friendship networks [14]. Route 4 may occur by kin selection, network reciprocity, or group selection when starting with a prisoner's dilemma with $\lambda^0<0$ (rather than $\lambda^0=0$ as assumed before). Route 5 may be generated by the same mechanisms if $\lambda^0>0$. Finally, route 6 can be implemented by time-dependent taxation (see Fig. 2).

V. FURTHER KINDS OF TRANSITIONS TO COOPERATION

The routes to cooperation discussed so far change the eigenvalues λ_1 and λ_2 and leave the stationary solutions p_1 and p_2 unchanged. However, transitions to cooperation can also be generated by shifting the stationary solutions or creating new ones, as we will show now. For this, we generalize the replicator Eq. (2) by replacing λ_1 with $f(p)$ and λ with $g(p)$ and by adding a term $h(p)$, which can describe effects of spontaneous transitions such as mutations. To guarantee $0\leq p(t)\leq 1$, we must have $h(p)=v(p)-pw(p)$ with functions $w(p)\geq v(p)\geq 0$. The resulting equation is $dp/dt=F(p(t))$ with $F(p)=(1-p)[f(p)-2g(p)p]p+h(p)$, and its stationary solutions p_k are given by $F(p_k)=(1-p_k)[f(p_k)-2g(p_k)p_k]p_k+h(p_k)=0$. The associated eigenvalues $\lambda_k=dF(p_k)/dp$ determining the stability of the stationary solutions p_k are

$$\lambda_k=(1-2p_k)(f_k-2p_kg_k)+p_k(1-p_k)(f_k'-2p_kg_k'-2g_k)+h_k',$$

where we have used the abbreviations $f_k=f(p_k)$, $g_k=g(p_k)$, $h_k=h(p_k)$. $f_k'=f'(p_k)$, $g_k'=g'(p_k)$, and $h_k=h'(p_k)$ are the derivatives of the functions $f(p)$, $g(p)$, and $h(p)$ in the points $p=p_k$.

VI. CLASSIFICATION

We can now distinguish different *kinds* of transitions from defection to cooperation: if the stationary solutions $p_1=0$ and $p_2=1$ of the prisoner's dilemma are modified, we talk about transitions to cooperation by equilibrium displacement. This case occurs, for example, when random mutations are not weak ($h\neq 0$). If the eigenvalues λ_1 or λ_2 of the stationary solutions $p_1=0$ and $p_2=1$ are changed, we speak of equilibrium selection. This case applies to all routes to cooperation discussed before. If a new stationary solution appears, we speak of equilibrium creation. The different cases often appear in combination with each other (see the summary below). In the following, we will discuss an interesting case, where cooperation occurs solely through equilibrium creation, i.e., the stationary solutions p_1 and p_2 of the replicator equation for the prisoner's dilemma as well as their eigenvalues λ_1 and λ_2 remain unchanged. We illustrate this by the example of an *adaptive kind of group pressure* that rewards mutual cooperation ($T_{11}<0$) or sanctions unilateral defection ($T_{21}>0$). Both, rewarding and sanctioning reduces the value of λ_2, while λ_1 remains unchanged. Assuming here that the group pressure vanishes, when everybody cooperates (as it is not needed then), while it is maximum when everybody defects (to encourage cooperation) [15], we may set $f(p)=\lambda_1^0$ and $g(p)=\lambda^0-K[1-p(t)]$, corresponding to $\lambda_2(t)=\lambda_2^0-2K[1-p(t)]$. It is obvious that we still have the two stationary solutions $p_1=0$ and $p_2=1$ with the eigenvalues $\lambda_1=\lambda_1^0<0$ and $\lambda_2=2\lambda^0-\lambda_1^0>0$ of the original prisoners dilemma with parameters λ_1^0 and λ_2^0 or λ^0. However, for large enough values of K [namely for $K>K_0=\lambda^0+|\lambda_1^0|+\sqrt{|\lambda_1^0|(2\lambda^0+|\lambda_1^0|)}$], we find two additional stationary solutions;

$$p_\pm=\frac{1}{2}-\frac{\lambda^0}{2K}\pm\sqrt{\left(\frac{1}{2}-\frac{\lambda^0}{2K}\right)^2-\frac{|\lambda_1^0|}{2K}}. \quad (3)$$

p_- is an *unstable* stationary solution with $p_1<p_-<p_+$ and

$\lambda_- = dF(p_-)/dp > 0$, while p_+ is a *stable* stationary solution with $p_- < p_+ < p_2$ and $\lambda_+ = dF(p_+)/dp < 0$ (see inset of Fig. 2). Hence, the assumed dependence of the payoffs on the proportion p of cooperators generates a *bistable* situation (BISTAB), with the possibility of a coexistence of a few defectors with a large proportion p_+ of cooperators, given $K > K_0$. If $p(0) < p_-$, where $p(0)$ denotes the initial condition, defection by everybody results, while a stationary proportion p_+ of cooperators is established for $p_- < p(0) < 1$. Surprisingly, in the limit $K \to \infty$, cooperation is established for *any* initial condition $p(0) \neq 0$ (or through fluctuations).

VII. SUMMARY

We have discussed from a physical point of view what must happen that social or biological payoff-changing interaction mechanisms can create cooperation in the prisoner's dilemma. The possible ways are (i) moving the stable stationary solution away from pure defection (routes 3, 4, and 6), (ii) stabilizing the unstable solution (routes 1, 2, 4, 5, and 6), or (iii) creating new stationary solutions, which are stable (routes 3, 4, and 6). Several of these points can be combined.

If (i) is fulfilled, we speak of equilibrium displacement if their eigenvalues change, we called this equilibrium selection, and if (iii) is the case, we talk of equilibrium creation. The first case can result from mutations, the second one applies to many social or biological cooperation-enhancing mechanisms [12]. We have discussed an interesting case of equilibrium creation, in which the outcome of the replicator equation is changed, although the stationary solutions of the PD and their eigenvalues remain unchanged. This can, for example, occur by adaptive group pressure [15], which introduces an adaptive feedback mechanism and thereby increases the order of nonlinearity of the replicator equation. Surprisingly, already a linear dependence of the payoff values P_{ij} on the endogenous dynamics $p(t)$ of the system is enough to destabilize defection and stabilize cooperation, thereby inverting the outcome of the prisoner's dilemma.

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