Mirror symmetry of 3D $\mathcal{N}=4$ gauge theories and supersymmetric indices

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We compute supersymmetric indices to test mirror symmetry of three-dimensional (3D) $\mathcal{N}=4$ gauge theories and dualities of half-Bogomol'ny Prasad Sommerfield enriched boundary conditions and interfaces in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory. We find the matching of indices as strong evidences for various dualities of the 3D interfaces conjectured by Gaiotto and Witten under the action of S-duality in Type IIB string theory.

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I. INTRODUCTION AND CONCLUSIONS

Three-dimensional (3D) $\mathcal{N}=4$ supersymmetric gauge theories have a moduli space of supersymmetric vacua consisting of a Higgs branch $\mathcal{M}_H$ and a Coulomb branch $\mathcal{M}_C$ which are hyperkähler manifolds. The theories have an intriguing duality, known as mirror symmetry, which relates theories with completely different UV descriptions where the Higgs branch and the Coulomb branch are exchanged and fayet iliopoulos parameters and mass parameters are exchanged [1]. This mapping is very nontrivial. The Higgs branch is a hyperkähler quotient realized as a zero locus of the D-term constraints divided by the gauge group. On the Higgs branch, the gauge symmetry is broken completely and the Higgs branch is not affected by the quantum correction. On the Coulomb branch, the gauge group is generically broken to its maximal torus. Unlike the Higgs branch, the Coulomb branch receives perturbative and nonperturbative quantum corrections.

The quantum Coulomb branch is studied in terms of Hilbert series in [2]. The Hilbert series is a generating function that counts chiral operators on the branches $\mathcal{M}_{H/C}$ of vacua, graded by their dimensions and quantum numbers under global symmetry. It encodes the quantum numbers of generators and relations of the chiral ring $\mathbb{C}[\mathcal{M}_{H/C}]$ of the corresponding branches $\mathcal{M}_{H/C}$ of vacua. Subsequently, the quantum Coulomb branch has been described in [3] in terms of the Abelianization map that translates the vev of a monopole operator into a linear combination of Abelian monopole operator vevs in the low-energy effective theory. Furthermore, the quantum Coulomb branch has been mathematically defined in [4–6].

Three-dimensional $\mathcal{N}=4$ gauge theories can be realized in Type IIB string theory using the brane configuration [7]. Mirror symmetry can be viewed as arising from S-duality of Type IIB string theory. Type IIB brane construction is extended to further study of half-Bogomol'ny Prasad Sommerfield boundary conditions and interfaces for four-dimensional $\mathcal{N}=4$ super Yang-Mills (SYM) theory [8–10], quarter-BPS corner configuration for four-dimensional $\mathcal{N}=4$ SYM theory [11,12], half-BPS boundary conditions for three-dimensional $\mathcal{N}=4$ gauge theory [13], and two-dimensional $\mathcal{N}=(0,4)$ supersymmetric gauge theories [14]. S-duality turns out to give a physical underpinning to geometric Langlands program [11,15–17] and symplectic duality [18,19].

In this paper, we compute supersymmetric full- and half-indices to test mirror symmetry of 3D $\mathcal{N}=4$ gauge theories and to extend the analysis in [12] for the dualities of half-BPS boundary conditions and interfaces in 4D $\mathcal{N}=4$ SYM theory.[1] These dualities were originally conjectured by Gaiotto and Witten [10]. There have been plenty of works on the subject of 3D $\mathcal{N}=4$ full superconformal indices [21–27]; however, general tests of mirror symmetry for 3D $\mathcal{N}=4$ gauge theories have not appeared in the literature except for the simplest Abelian mirror symmetry [28], in contrast to 3D $\mathcal{N}=2$ gauge theories [29–31]. Additionally, one drawback of the 3D full-indices is that they are insensitive to the boundary conditions for 4D $\mathcal{N}=4$ gauge theory, which involves singular boundary conditions specified by the Nahm poles [8]. In order to address these open issues, we evaluate supersymmetric 4D half-indices and 3D full-indices which count local operators both in four and three dimensions and present many identities of indices by

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[1]See [20] for further dualities of $\mathcal{N}=(0,4)$ half-BPS boundaries for 3D $\mathcal{N}=4$ gauge theories and quarter-BPS corners for 4D $\mathcal{N}=4$ gauge theories.

checking several terms of series expansion. As discussed in [28], after the special limits, the full-indices of 3D $\mathcal{N}=4$ gauge theories reduce to the Hilbert series. As a result, the identities of indices provide promotions of identities of Hilbert series discussed in [2,28]. Furthermore, the full-indices of 3D $\mathcal{N}=4$ gauge theories can be used to generalize the analysis in [12] of the half-BPS boundary conditions and interfaces for 4D $\mathcal{N}=4$ SYM theory. We present general half-indices for enriched half-BPS boundary conditions and interfaces in 4D $\mathcal{N}=4$ gauge theory and check precise matching of the indices for dual pairs.

The organization of this paper is straightforward. In Sec. II, we briefly review the supersymmetric indices introduced in [12] and present formulae and some properties of half-indices for 4D $\mathcal{N}=4$ gauge theories and full-indices for 3D $\mathcal{N}=4$ gauge theories. In Sec. III, we evaluate full-indices for 3D $\mathcal{N}=4$ Abelian gauge theories and check mirror symmetry. In Sec. IV, we further examine mirror symmetry by computing full-indices for 3D $\mathcal{N}=4$ non-Abelian gauge theories. We also briefly check in Sec. V Seiberg-like duality for 3D $\mathcal{N}=4$ gauge theories proposed in [10] between ugly theory and good theory in terms of full-indices. In Sec. VI, we discuss the half-BPS enriched boundary conditions for 4D $\mathcal{N}=4$ SYM theory which involve 3D $\mathcal{N}=4$ gauge theories. We present strong evidence for dualities between them conjectured from string theory by calculating half-indices, which contain nonregular Nahm pole b.c. Finally, in Sec. VII, we study the half-BPS interfaces in 4D $\mathcal{N}=4$ $U(1)$ gauge theory including 3D $\mathcal{N}=4$ Abelian gauge theories. We test dualities between the interfaces by computing half-indices. In the Appendix A, we show the q-expansions of indices generated by Mathematica as well as the confirmed orders which the indices agree up to.

II. INDICES

We begin with a definition of the quarter-index introduced in [12]. It is a generalization of superconformal index in that it can count local operators living in different dimensions, i.e., in 4D bulk, 3D boundary, and 2D junction. When the configuration has a trivial junction, it becomes the half-index that counts boundary local operators, while for the trivial interface, it becomes the full-index that counts bulk local operators. The quarter-index can be defined as the trace over the cohomology of the preserved supercharges

$$\mathbb{IV}(t,x;q) \coloneqq \mathrm{Tr}_{\mathrm{Op}}(-1)^F q^{J+\frac{H+C}{4}} t^{H-C} x^f. \tag{2.1}$$

Here F is the Fermion number, J is the generator of the $U(1)_J$ rotational symmetry in the space-time on which local operators are supported. H and C stands for the Cartan generators of the $SU(2)_H$ and $SU(2)_C$ R-symmetry groups, respectively. f is the Cartan generator of the global symmetry. The choice of fugacity in the index (2.1) is fixed in such a way that the power of q is always strictly positive for a nontrivial local operator by a unitarity bound. This ensures the convergence of the index. Consequently, the index can be a formal power series in q whose coefficients are Laurent polynomials in the other fugacities.

In this paper, we focus on the configurations of 3D $\mathcal{N}=4$ gauge theories which may couple to 4D $\mathcal{N}=4$ gauge theories so that the indices (2.1) reduce to the full-indices $\mathbb{I}$ of 3D $\mathcal{N}=4$ gauge theories and/or the half-indices $\mathbb{III}$ of 4D $\mathcal{N}=4$ gauge theories. One may compute the indices for appropriate configurations by a localization procedure. However, we will not pursue that direction in this paper, instead we will count local operators seriously from physical consideration.

In the description of indices, we use the following notation by defining q-shifted factorial:

$$\begin{aligned}
&(a;q)_0 \coloneqq 1, \qquad (a;q)_n \coloneqq \prod_{k=0}^{n-1}(1-aq^k),\\
&(q)_n \coloneqq \prod_{k=1}^{n}(1-q^k), \qquad n \geq 1,\\
&(a;q)_\infty \coloneqq \prod_{k=0}^{\infty}(1-aq^k), \qquad (q)_\infty \coloneqq \prod_{k=1}^{\infty}(1-q^k),\\
&(a^{\pm};q)_\infty \coloneqq (a;q)_\infty (a^{-1};q)_\infty,
\end{aligned} \tag{2.2}$$

where a and q are complex variables with $|q|<1$.

We compute the indices to test the 3D dualities and dualities of the half-BPS boundary conditions/interfaces which are conjectured from string theory [10]. We consider five types of branes in Type IIB string theory whose world-volumes span the following directions:

(i) D3-branes extended along $x^0x^1x^2x^6$,
(ii) NS5-branes extended along $x^0x^1x^2x^3x^4x^5$,
(iii) D5-branes extended along $x^0x^1x^2x^7x^8x^9$,
(iv) NS5′-branes extended along $x^0x^1x^6x^7x^8x^9$,
(v) D5′-branes extended along $x^0x^1x^3x^4x^5x^6$.

In other words, the brane configuration is summarized as

	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	–	–	–	○	–	–	–
NS5	○	○	○	○	○	○	–	–	–	–
D5	○	○	○	–	–	–	–	○	○	○
NS5′	○	○	–	–	–	–	○	○	○	○
D5′	○	○	–	○	○	○	○	–	–	–

(2.3)

The 3D $\mathcal{N}=4$ gauge theories are realized by considering D3-branes which are finite segments in the x^6 direction between NS5-branes and may intersect with D5-branes [7]. The half-BPS boundaries and interfaces in 4D $\mathcal{N}=4$ gauge theories are realized by considering D3-branes which are (semi-)infinite D3-branes which end on or pass through a sequence of NS5- and D5-branes [8].

Such brane setup is a nice tool for finding mirror pairs of 3D $\mathcal{N}=4$ gauge theories and dual pairs of half-BPS boundary conditions and interfaces in 4D $\mathcal{N}=4$ SYM theory by studying the action of S-duality [10].

A. Indices of 4D $\mathcal{N}=4$ Sym Theory

1. Full-indices

Four-dimensional $\mathcal{N}=4$ SYM theory has $SU(4)_R$ R-symmetry. It contains the adjoint scalar fields transforming as **6** under the $SU(4)_R$. Let X and Y be the scalar fields transforming as $(\mathbf{1},\mathbf{3})$ and $(\mathbf{3},\mathbf{1})$ under the $SU(2)_C \times SU(2)_H \subset SU(4)_R$. In the brane construction (2.3), the scalar fields X and Y describe the positions of D3-branes along the (x^7, x^8, x^9) directions and (x^3, x^4, x^5) directions, respectively. The theory also has the 4D gauginos λ transforming as $(\mathbf{2},\mathbf{2})$ under the $SU(2)_C \times SU(2)_H$.

The local operators in 4D $\mathcal{N}=4$ gauge theory of gauge group G which contribute to index have charges

	$\partial^n X$	$\partial^n Y$	$\partial^n \lambda$	$\partial^n \bar{\lambda}$
G	Adj	Adj	Adj	Adj
$U(1)_J$	n	n	$n+\frac{1}{2}$	$n+\frac{1}{2}$
$U(1)_C$	0	2	+	+
$U(1)_H$	2	0	+	+
fugacity	$q^{n+\frac{1}{2}}t^2 s^\alpha$	$q^{n+\frac{1}{2}}t^{-2}s^\alpha$	$-q^{n+1}s^\alpha$	$-q^{n+1}s^\alpha$

(2.4)

From (2.4), one can express the index for 4D $\mathcal{N}=4$ $U(N)$ gauge theory as

$$\mathbb{I}^{4\mathrm{D}\,U(N)}(t;q) = \frac{1}{N!}\frac{(q)_\infty^{2N}}{(q^{\frac{1}{2}}t^2;q)_\infty^N (q^{\frac{1}{2}}t^{-2};q)_\infty^N} \times \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \frac{\left(\frac{s_i}{s_j};q\right)_\infty \left(q\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2\frac{s_i}{s_j};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}. \tag{2.5}$$

Here the denominator comes from the scalar fields X and Y, while the numerator captures the 4D gauginos. The integration contour for gauge fugacities s_i is taken as a unit torus $\mathbb{T}^N$.

2. Half-indices

Four-dimensional $\mathcal{N}=4$ SYM theory admits half-BPS boundary conditions which preserve three-dimensional $\mathcal{N}=4$ supersymmetry with the R-symmetry group $SU(4)_R$ broken down to $SU(2)_C \times SU(2)_H$. In the brane setup (2.3), they arise when parallel D3-branes end on a single five-brane. There are two types of three-dimensional boundaries/interfaces at $x^2=0$ realized by NS5′- and D5′-branes and those at $x^6=0$ realized by NS5- and D5-branes.

Let us consider the half-BPS boundary conditions for 4D $\mathcal{N}=4$ $U(1)$ gauge theory. When a single D3-brane ends on the NS5-brane and D5-brane, one finds the Neumann b.c. $\mathcal{N}$ and Dirichlet b.c. $\mathcal{D}$ at $x^6=0$ for $U(1)$ gauge theory, respectively,

$$\begin{aligned} &\mathcal{N}: \quad F_{6\mu}|_\partial = 0, \quad \partial_\mu X|_\partial = 0, \quad \partial_6 Y|_\partial = 0 \\ &\mathcal{D}: \quad F_{\mu\nu}|_\partial = 0, \quad \partial_6 X|_\partial = 0, \quad \partial_\mu Y|_\partial = 0 \end{aligned} \qquad \mu,\nu = 0,1,2. \tag{2.6}$$

On the other hand, when the NS5′-brane and D5′-brane end on a single D3-brane, one obtains the Neumann b.c. $\mathcal{N}'$ and Dirichlet b.c. $\mathcal{D}'$ at $x^2=0$ for $U(1)$ gauge theory, respectively,

$$\begin{aligned} &\mathcal{N}': \quad F_{2\mu}|_\partial = 0, \quad \partial_2 X|_\partial = 0, \quad \partial_\mu Y|_\partial = 0 \\ &\mathcal{D}': \quad F_{\mu\nu}|_\partial = 0, \quad \partial_\mu X|_\partial = 0, \quad \partial_2 Y|_\partial = 0 \end{aligned} \qquad \mu,\nu = 0,1,6. \tag{2.7}$$

The half-indices of the Neumann b.c. $\mathcal{N}$ and Dirichlet b.c. $\mathcal{D}'$ for 4D $\mathcal{N}=4$ $U(1)$ gauge theory take the form

$$\mathbb{I}\!\!\mathbb{I}_{\mathcal{N}}^{4\mathrm{D}\,U(1)}(t;q) = \mathbb{I}\!\!\mathbb{I}_{\mathcal{D}'}^{4\mathrm{D}\,U(1)}(t;q) = \frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}. \tag{2.8}$$

The denominator is associated to the scalar fields Y charged under $U(1)_C$, while the numerator correspond to a half of the 4D gauginos. Likewise, the half-indices of Neumann b.c. $\mathcal{N}'$ and Dirichlet b.c. $\mathcal{D}$ are

$$\mathbb{I}\!\!\mathbb{I}_{\mathcal{D}}^{4\mathrm{D}\,U(1)}(t;q) = \mathbb{I}\!\!\mathbb{I}_{\mathcal{N}'}^{4\mathrm{D}\,U(1)}(t;q) = \frac{(q)_\infty}{(q^{\frac{1}{2}}t^{2};q)_\infty}. \tag{2.9}$$

The denominator captures the scalar fields X charged under $U(1)_H$, whereas the numerator is associated to a half of the 4D gauginos.

The half-BPS boundary conditions corresponding to N D3-branes ending on a single NS5-brane (or NS5') are also Neumann b.c. for the $U(N)$ gauge theory. We can denote them as $\mathcal{N}$ and $\mathcal{N}'$ as in the Abelian case. By contrast, when N multiple D3-branes end on a single D5-brane, one finds a singular boundary condition associated to a regular Nahm pole [8,32]. A single D5-brane or D5′-brane on which N D3-branes end give rise to the Nahm or Nahm′ pole boundary conditions,

$$\begin{aligned} \text{Nahm}: \quad & F_{\mu\nu}|_\partial = 0, \quad D_6\vec{X} + \vec{X}\times\vec{X}|_\partial = 0, \\ & D_\mu \vec{Y}|_\partial = 0 \quad \mu,\nu = 0,1,2 \\ \text{Nahm}': \quad & F_{\mu\nu}|_\partial = 0, \quad D_\mu \vec{X}|_\partial = 0, \\ & D_2\vec{Y} + \vec{Y}\times\vec{Y}|_\partial = 0 \quad \mu,\nu = 0,1,6, \end{aligned} \tag{2.10}$$

where we denote the scalar fields by $\vec{X}$ and $\vec{Y}$ as they are the $SU(2)_H$ triplet and the $SU(2)_C$ triplet, respectively. The Nahm equations for the scalar fields $\vec{X}$ and $\vec{Y}$ have singular solutions

$$\vec{X}(x^6) = \frac{\vec{\mathfrak{t}}}{x^6}, \qquad \vec{Y}(x^2) = \frac{\vec{\mathfrak{t}}}{x^2}, \tag{2.11}$$

where $\vec{\mathfrak{t}} = (\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3)$ is a triplet of elements of the Lie algebra $\mathfrak{g} = \mathfrak{u}(N)$ obeying the commutation relation $[\mathfrak{t}_1, \mathfrak{t}_2] = \mathfrak{t}_3$ and cyclic permutation thereof. The choice of $\vec{\mathfrak{t}}$ specifies a homomorphism of Lie algebras $\rho: \mathfrak{su}(2) \to \mathfrak{g}$ which maps the fundamental representation of $U(N)$ to the dimension N irreducible representation of $\mathfrak{su}(2)$. When N D3-branes end on multiple D5-branes, one finds other Nahm poles, including the Dirichlet b.c. as the trivial Nahm pole corresponding to the case with N D5-branes.

The half-index of Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N} = 4$ $U(N)$ gauge theory takes the form

$$\mathbb{I}^{4\mathrm{D}\,U(N)}_{\mathcal{N}}(t;q) = \frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}. \tag{2.12}$$

Again the integration contour for gauge fugacities s_i is a unit torus $\mathbb{T}^N$. The half-index of Dirichlet b.c. $\mathcal{D}$ for 4D $\mathcal{N} = 4$ $U(N)$ gauge theory is given by

$$\mathbb{I}^{4\mathrm{D}\,U(N)}_{\mathcal{D}}(t, z_i; q) = \frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^2;q)_\infty^N}\prod_{i\neq j}\frac{\left(q\frac{z_i}{z_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2\frac{z_i}{z_j};q\right)_\infty}, \tag{2.13}$$

where z_i is the fugacities associated to the boundary $U(N)$ global symmetry. The half-index for Nahm pole boundary conditions in 4D $\mathcal{N} = 4$ $U(N)$ gauge theory is

$$\mathbb{I}^{4\mathrm{D}\,U(N)}_{\mathrm{Nahm}}(t;q) = \prod_{k=1}^N\frac{(q^{\frac{k+1}{2}}t^{2(k-1)};q)_\infty}{(q^{\frac{k}{2}}t^{2k};q)_\infty}. \tag{2.14}$$

As discussed in [12], the duality between the Neumann b.c. and the regular Nahm pole b.c. implies equality between the half-indices (2.12) and (2.14). In Sec. VI, we discuss more general dual descriptions of the half-indices, including half-index (2.13) for Dirichlet b.c.

We can get similar expressions for the mirror boundary conditions, i.e., $\mathcal{N}'$, $\mathcal{D}'$, and Nahm$'$ by setting $t \to t^{-1}$.

B. Indices of 3D $\mathcal{N} = 4$ gauge theory

The 3D $\mathcal{N} = 4$ hypermultiplet consists of a pair of complex scalars $\mathbb{H}$, $\tilde{\mathbb{H}}$ forming a doublet of $SU(2)_H$ and a pair of complex fermions $\psi_+^{\mathbb{H}}$, $\psi_+^{\tilde{\mathbb{H}}}$ forming a doublet of $SU(2)_C$. The charges of 3D $\mathcal{N} = 4$ hypermultiplet is

	$\mathbb{H}$	$\tilde{\mathbb{H}}$	$\psi_+^{\mathbb{H}}$	$\psi_+^{\tilde{\mathbb{H}}}$	$\bar{\psi}_-^{\mathbb{H}}$	$\bar{\psi}_-^{\tilde{\mathbb{H}}}$
$U(1)_C$	0	0	−	−	+	+
$U(1)_H$	+	+	0	0	0	0

(2.15)

The 3D $\mathcal{N} = 4$ Abelian vector multiplet consists of a 3D $U(1)$ gauge field $A_\mu^{3\mathrm{D}}$, three scalars, which we denote by real and complex scalars σ, φ forming the $SU(2)_C$ triplet, and two complex fermions $(\lambda_\alpha^{3\mathrm{D}}, \eta_\alpha^{3\mathrm{D}})$. The charges of 3D $\mathcal{N} = 4$ vector multiplet is

	$A_\mu^{3\mathrm{D}}$	σ	φ	$\lambda_\pm^{3\mathrm{D}}$	$\bar{\lambda}_\pm^{3\mathrm{D}}$	$\eta_\pm^{3\mathrm{D}}$	$\bar{\eta}_\pm^{3\mathrm{D}}$
$U(1)_C$	0	0	2	+	−	+	−
$U(1)_H$	0	0	0	+	−	−	+

(2.16)

The 3D $\mathcal{N} = 4$ superalgebra has an outer automorphism that interchanges $SU(2)_C$ and $SU(2)_H$. This automorphism makes the ordinary supermultiplets into twisted supermultiplets. The twisted hyper and vector multiplets can be obtained by exchanging the $U(1)_H$ and $U(1)_C$ charges of the hypermultiplet and vector multiplet, respectively.

In three dimensions, a photon is electric magnetic dual to a scalar field, which we call dual photon. The dual photon is periodic when the gauge group is compact and the shift symmetry of the dual photon is a classical topological symmetry whose conserved current is $*F$ where $*$ is the Hodge star. The conservation of $*F$ follows from the Bianchi identity $dF = 0$.

1. Matter multiplets

The index of a 3D $\mathcal{N} = 2$ chiral multiplet of charge $+1$ under a $U(1)_f$ flavor symmetry with fugacity x is

$$\mathbb{I}^{3\mathrm{D\,CM}}(x;q) = \frac{(qx^{-1};q)_\infty}{(x;q)_\infty}. \tag{2.17}$$

Its denominator counts complex scalar and its ∂ derivatives, while its numerator counts fermions and its ∂ derivatives included in the 3D $\mathcal{N} = 2$ chiral multiplet.

The 3D $\mathcal{N} = 4$ hypermultiplet has the following operators which contribute to index:

	$\partial^n\mathbb{H}$	$\partial^n\tilde{\mathbb{H}}$	$\partial^n\bar{\psi}_-^{\mathbb{H}}$	$\partial^n\bar{\psi}_-^{\tilde{\mathbb{H}}}$
$U(1)_f$	+	−	+	−
$U(1)_J$	n	n	$n+\frac{1}{2}$	$n+\frac{1}{2}$
$U(1)_C$	0	0	+	+
$U(1)_H$	+	+	0	0
Fugacity	$q^{n+\frac{1}{4}}tx$	$q^{n+\frac{1}{4}}tx^{-1}$	$-q^{n+\frac{3}{4}}t^{-1}x$	$-q^{n+\frac{3}{4}}t^{-1}x^{-1}$

(2.18)

The index for 3D $\mathcal{N} = 4$ hypermultiplet is

$$\mathbb{I}^{\text{3D HM}}(t,x;q)=\frac{(q^{\frac{3}{4}}t^{-1}x;q)_\infty(q^{\frac{3}{4}}t^{-1}x^{-1};q)_\infty}{(q^{\frac{1}{4}}tx;q)_\infty(q^{\frac{1}{4}}tx^{-1};q)_\infty}=\mathbb{I}^{\text{3D CM}}\left(q^{\frac{1}{4}}tx\right)\times\mathbb{I}^{\text{3D CM}}\left(q^{\frac{1}{4}}tx^{-1}\right). \quad (2.19)$$

It can be expanded as

$$\mathbb{I}^{\text{3D HM}}(t,x;q)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(q^{\frac{1}{2}}t^{-2};q)_k(q^{\frac{1}{2}}t^{-2};q)_{n-k}}{(q)_k(q)_{n-k}}x^{n-2k}q^{\frac{n}{4}}t^n. \quad (2.20)$$

The free 3D $\mathcal{N}=4$ hypermultiplet has no Coulomb branch local operators surviving in the H-twist. Therefore, in the H-twist limit $t\to q^{\frac{1}{4}}$, the indices (2.19) and (2.20) become trivial,

$$\mathbb{I}^{\text{3D HM}}(t=q^{\frac{1}{4}},x;q)=1. \quad (2.21)$$

On the other hand, in the C-twist limit $t\to q^{-\frac{1}{4}}$, the indices reduce to

$$\mathbb{I}^{\text{3D HM}}(t=q^{-\frac{1}{4}},x;q)=\frac{1}{(1-x)(1-x^{-1})}. \quad (2.22)$$

This counts two bosonic generators in the algebra of Higgs branch local operators.

The operators in 3D $\mathcal{N}=4$ twisted hypermultiplet which contribute to index are as follows:

	$\partial^n\mathbb{T}$	$\partial^n\tilde{\mathbb{T}}$	$\partial^n\tilde{\bar{\psi}}^{\mathbb{T}}_-$	$\partial^n\tilde{\bar{\psi}}^{\tilde{\mathbb{T}}}_-$
$U(1)_f$	$+$	$-$	$+$	$-$
$U(1)_J$	n	n	$n+\frac{1}{2}$	$n+\frac{1}{2}$
$U(1)_C$	$+$	$+$	0	0
$U(1)_H$	0	0	$+$	$+$
Fugacity	$q^{n+\frac{1}{4}}t^{-1}x$	$q^{n+\frac{1}{4}}t^{-1}x^{-1}$	$-q^{n+\frac{3}{4}}tx$	$-q^{n+\frac{3}{4}}tx^{-1}$

(2.23)

The index for 3D $\mathcal{N}=4$ twisted hypermultiplet can be obtained from the index (2.19) by setting $t\to t^{-1}$,

$$\mathbb{I}^{\text{3D tHM}}(t,x;q)=\frac{(q^{\frac{3}{4}}tx;q)_\infty(q^{\frac{3}{4}}tx^{-1};q)_\infty}{(q^{\frac{1}{4}}t^{-1}x;q)_\infty(q^{\frac{1}{4}}t^{-1}x^{-1};q)_\infty}=\mathbb{I}^{\text{3D CM}}(q^{\frac{1}{4}}t^{-1}x)\cdot\mathbb{I}^{\text{3D CM}}(q^{\frac{1}{4}}t^{-1}x^{-1}). \quad (2.24)$$

Again, it has an expansion

$$\mathbb{I}^{\text{3D tHM}}(t,x;q)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(q^{\frac{1}{2}}t^{2};q)_\infty(q^{\frac{1}{2}}t^{2};q)_\infty}{(q)_k(q)_{n-k}}x^{n-2k}q^{\frac{n}{4}}t^{-n}. \quad (2.25)$$

2. Gauge multiplets

While in four-dimensional case, the index only involves integration over the gauge group [33,34], and in three-dimensional case the index would have nonperturbative contributions of monopole operators and contain the sum over the magnetic fluxes of monopole operators for all backgrounds [21,29].

Let us first consider the perturbative contributions to the index. The charges of operators in 3D $\mathcal{N}=4$ vector multiplet contributing to the index are

	$D^n(\sigma+i\rho)$	$D^n\varphi$	$D^n\bar{\lambda}^{\text{3D}}_-$	$D^n\bar{\eta}^{\text{3D}}_-$
G	Adj	Adj	Adj	Adj
$U(1)_J$	n	n	$n+\frac{1}{2}$	$n+\frac{1}{2}$
$U(1)_C$	0	2	$-$	$-$
$U(1)_H$	0	0	$-$	$+$
Fugacity	$q^n s^\alpha$	$q^{n+\frac{1}{2}}t^{-2}s^\alpha$	$-q^n s^\alpha$	$-q^{n+\frac{1}{2}}t^2 s^\alpha$

(2.26)

The perturbative index contributed from the local operators in (2.26) of 3D $\mathcal{N}=4$ $U(1)$ vector multiplet takes the form

$$\mathbb{I}^{\text{3D pert }U(1)}(t;q)=\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds}{2\pi is}, \quad (2.27)$$

where the integration contour of gauge fugacity s is a unit circle. Similarly, the perturbative index for 3D $\mathcal{N}=4$ $U(N)$ vector multiplet takes the form

$$\mathbb{I}^{\text{3D pert }U(N)}(t;q)=\frac{1}{N!}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi is_i}\prod_{i\neq j}\left(1-\frac{s_i}{s_j}\right)\frac{(q^{\frac{1}{2}}t^2\frac{s_i}{s_j})}{(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j})}, \quad (2.28)$$

where the integration contour of gauge fugacities s_i is a unit torus $\mathbb{T}^N$.

Likewise, charges of operators in 3D $\mathcal{N}=4$ twisted vector multiplet are

	$D^n(\tilde{\sigma}+i\tilde{\rho})$	$D^n\tilde{\varphi}$	$D^n\bar{\tilde{\lambda}}^{\text{3D}}_-$	$D^n\bar{\tilde{\eta}}^{\text{3D}}_-$
G	Adj	Adj	Adj	Adj
$U(1)_J$	n	n	$n+\frac{1}{2}$	$n+\frac{1}{2}$
$U(1)_C$	0	0	$+$	$+$
$U(1)_H$	0	2	$+$	$-$
Fugacity	$q^n s_\alpha$	$q^{n+\frac{1}{2}}t^2 s_\alpha$	$-q^n s_\alpha$	$-q^{n+\frac{1}{2}}t^{-2}s_\alpha$

(2.29)

We can obtain the index for 3D $\mathcal{N}=4$ twisted vector multiplet by setting $t \to t^{-1}$ for the index of 3D $\mathcal{N}=4$ vector multiplet.

As 3D $\mathcal{N}=4$ $U(1)$ gauge theory appears from 4D $\mathcal{N}=4$ $U(1)$ gauge theory on a segment with Neumann b.c. $\mathcal{N}$ at each end, we have a schematic relation

$$\mathbb{I}^{\text{3D pert } U(1)} = \frac{\mathbb{III}_{\mathcal{N}}^{\text{4D } U(1)} \times \mathbb{III}_{\mathcal{N}}^{\text{4D } U(1)}}{\mathbb{I}^{\text{4D } U(1)}}. \tag{2.30}$$

When a 4D $\mathcal{N}=4$ $U(1)$ gauge theory is put on a slab with Neumann b.c. $\mathcal{N}$ and Dirichlet b.c. $\mathcal{D}$ at each end, we have

$$1 = \frac{\mathbb{III}_{\mathcal{N}}^{\text{4D } U(1)} \times \mathbb{III}_{\mathcal{D}}^{\text{4D } U(1)}}{\mathbb{I}^{\text{4D } U(1)}}. \tag{2.31}$$

This indicates that the resulting 3D theory is a trivial theory. For a 4D $\mathcal{N}=4$ $U(1)$ gauge theory on a segment obeying Dirichlet b.c. at both ends, we find that

$$\mathbb{I}^{\text{3D pert } \widetilde{U(1)}} = \frac{\mathbb{III}_{\mathcal{N}'}^{\text{4D } U(1)} \times \mathbb{III}_{\mathcal{N}'}^{\text{4D } U(1)}}{\mathbb{I}^{\text{4D } U(1)}}. \tag{2.32}$$

This reflects the fact that the resulting 3D theory is mirror to 3D $\mathcal{N}=4$ twisted $\widetilde{U(1)}$ gauge theory appearing from a 4D theory on a segment with Neumann b.c. $\mathcal{N}'$ at both ends.

Now consider the nonperturbative contributions to indices from monopole operators. Classically, monopole operators are charged under the topological symmetry. In addition, quantum mechanically they can acquire nontrivial quantum numbers. Let us consider the canonical $U(1)$ R-charge as $\frac{1}{2}(C-H)$ so that the complex scalar $\mathbb{H}$ in the hypermultiplet carries charge $-1/2$ and the adjoint complex scalar φ in the vector multiplet have charge $+1$. Then the R-charge of a BPS bare monopole operator of magnetic charge m in the IR conformal field theory is given by

$$\Delta(m) = \frac{1}{2}\sum_{i=1}^{N_f}\sum_{\lambda_i \in \mathbf{R}_i} |\lambda_i(m)| - \sum_{\alpha \in \Delta_+} |\alpha(m)|. \tag{2.33}$$

This formula was firstly proposed in [10] and later verified in [35,36]. The first term in (2.33) is the contribution from N_f hypermultiplets labeled by $i=1,\ldots,N_f$ transforming as representations $\{\mathbf{R}_i\}_{i=1,\ldots,N_f}$ under the gauge group. The sum is taken over the weights λ_i of $\mathbf{R}_i$. The second term is the contribution from vector multiplet. The sum is taken over the positive roots $\alpha \in \Delta_+$.

The R-charge (2.33) of bare monopole can specify its energy or equivalently conformal dimension since the bare monopole is a BPS state. If all BPS monopole operators carry $\Delta(m) > \frac{1}{2}$, the theory is called *good*. If all BPS monopole operators have $\Delta(m) \geq \frac{1}{2}$ and some saturate the unitarity bound $\Delta(m) = \frac{1}{2}$, the theory is called *ugly*. Otherwise, the theory is called *bad*. In this paper, we focus on the good or ugly theories in which $\Delta(m)$ is identified with the conformal dimension and the quantum numbers $J + \frac{H+C}{4}$ of a bare monopole operator with magnetic charge m is fixed to $\frac{\Delta(m)}{2}$ in the IR superconformal field theory.

Taking into account the above, we can write the full-index of 3D $\mathcal{N}=4$ gauge theory with gauge group G and N_f hypermultiplets which takes the form

$$\begin{aligned}
&\mathbb{I}^{\text{3D } G}(t, x_i, z_i; q) \\
&= \frac{1}{|\text{Weyl}(G)|}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^{\text{rank}(G)}}{(q^{\frac{1}{2}}t^{-2};q)_\infty^{\text{rank}(G)}} \sum_{m \in \text{cochar}(G)} \oint \prod_{\alpha \in \text{roots}(G)} \\
&\quad \times \frac{ds}{2\pi i s}\frac{(1-q^{\frac{|m\cdot\alpha|}{2}}s^\alpha)(q^{\frac{1+|m\cdot\alpha|}{2}}t^2 s^\alpha;q)_\infty}{(q^{\frac{1+|m\cdot\alpha|}{2}}t^{-2}s^\alpha;q)_\infty} \\
&\quad \times \prod_{i=1}^{N_f}\prod_{\lambda_i \in \mathbf{R}_i} \frac{(q^{\frac{3}{4}+\frac{|m\cdot\lambda_i|}{2}}t^{-1}s^{\pm\lambda_i}x_i^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m\cdot\lambda_i|}{2}}ts^{\pm\lambda_i}x_i^{\pm};q)_\infty} q^{\frac{\Delta(m)}{2}} \cdot t^{-2\Delta(m)} \cdot z^m.
\end{aligned} \tag{2.34}$$

Here the second line is the contribution from vector multiplet of gauge group G. The third line is the contribution from N_f hypermultiplets transforming as representation $\{\mathbf{R}_i\}_{i=1,\ldots,N_f}$ of gauge group G. The fugacities x are associated to the flavor symmetry that rotates N_f hypermultiplets. For nonzero magnetic flux m, the expression is shifted from the index (2.19) for 3D hypermultiplet. This reflects the fact that in the presence of magnetic flux, the electrically charged states get an effective quantum numbers. The third line is the contribution from bare monopole operators. The fugacities z are associated to the topological symmetry.

In order to check mirror symmetry and dual boundary conditions, one also needs to evaluate the mirror version of the full-index for 3D $\mathcal{N}=4$ gauge theory. The mirror index of 3D $\mathcal{N}=4$ gauge theory consisting of twisted vector multiplet of gauge group $\tilde{G}$ and $\tilde{N}_f$ twisted hypermultiplets transforming as representation $\{\mathbf{R}_i\}_{i=1,\ldots,\tilde{N}_f}$ is given by

$$\tilde{\mathbb{I}}^{\text{3D } \tilde{G}}(t, x_i, z_i; q) = \mathbb{I}^{\text{3D } \tilde{G}}(t^{-1}, z_i, x_i; q), \tag{2.35}$$

where the fugacities z_i and x_i are associated to the flavor symmetry and the topological symmetry in the twisted 3D $\mathcal{N}=4$ theories as they are exchanged under mirror symmetry. In the following sections, we show various identities between indices (2.34) and (2.35) for mirror pairs.

III. ABELIAN MIRROR SYMMETRY

In this section, we consider the 3D $\mathcal{N}=4$ Abelian gauge theory and its mirror. The mirror of 3D $\mathcal{N}=4$ $U(1)$ gauge theory with N_f hypermultiplets is a twisted $U(1)^{N_f-1}$ quiver gauge theory with N_f twisted hypermultiplets [37]. The quiver diagram and the corresponding brane configuration are depicted in Fig. 1. We check that two indices for mirror pairs coincide with each other.

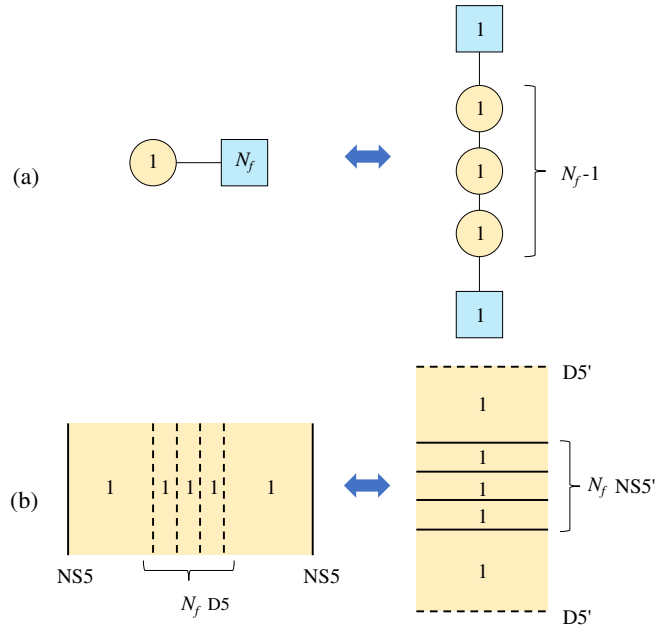

FIG. 1. (a) The quiver diagrams of SQED_{N_f} and its mirror $U(1)^{N_f-1}$ quiver gauge theory. (b) The brane configurations of SQED_{N_f} and its mirror $U(1)^{N_f-1}$ quiver gauge theory.

A. SQED_1

We begin with a 3D $\mathcal{N}=4$ Abelian gauge theory with a single charged hypermultiplet, which we call SQED_1. The 3D $\mathcal{N}=4$ SQED_1 is mirror to a 3D $\mathcal{N}=4$ twisted hypermultiplet [37]. The flavor symmetry of the twisted hypermultiplet is mapped to the topological symmetry in SQED_1. This is the simplest 3D $\mathcal{N}=4$ Abelian mirror symmetry and index computation has been already computed in [28]. We further extract a Higgsing interpretation of the indices by picking up residues at poles in the indices.

The full index of SQED_1 is given by

$$\mathbb{I}^{\text{3D SQED}_1}(t,x;q) = \frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi i s} \times \frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s;q)_\infty(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s^{-1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts;q)_\infty(q^{\frac{1}{4}+\frac{|m|}{2}}ts^{-1};q)_\infty} q^{\frac{|m|}{4}}t^{-|m|}x^m, \quad (3.1)$$

where m is the magnetic monopole charge and the fugacity x is associated to a $U(1)_t$ topological symmetry. The factors $q^{\frac{|m|}{4}}$ $t^{-|m|}$ x^m are associated to contributions from bare monopole operators with the R-charge $\Delta(m)=\frac{|m|}{2}$. The remaining factors in the integrand involve shifts of quantum numbers due to the background magnetic flux. The index for SQED_1 already appeared in [28,30], and the index (3.1) coincides with the index (2.24) of a 3D $\mathcal{N}=4$ twisted hypermultiplet.

As discussed in [12], poles in the integrand and their residues may have a physical interpretation. When fayet iliopoulos-like parameters are turned on, elementary fields with gauge charges get nontrivial vevs, which leads to a Higgsing the gauge group. Consequently, the index is written as a sum over residues which are associated to the index of the Higgsed theory. In the brane setup, this corresponds to a certain deformation of the brane configuration. In order to extract a Higgsing interpretation by expanding the index (3.1), it is enough to consider the perturbative SQED_1 index corresponding to $m=0$ that takes the form

$$\mathbb{I}^{\text{3D pert SQED}_1}(t;q) = \underbrace{\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds}{2\pi i s}}_{\mathbb{I}^{\text{3D pert }U(1)}}\underbrace{\frac{(q^{\frac{3}{4}}t^{-1}s;q)_\infty(q^{\frac{3}{4}}t^{-1}s^{-1};q)_\infty}{(q^{\frac{1}{4}}ts;q)_\infty(q^{\frac{1}{4}}ts^{-1};q)_\infty}}_{\mathbb{I}^{\text{3D HM}}(s)}. \quad (3.2)$$

This can be evaluated by considering the residues at poles of charged hypermultiplet $s=q^{\frac{1}{4}+m}t$ as

$$\mathbb{I}^{\text{3D pert SQED}_1}(t;q) = \frac{(q^{\frac{1}{2}}t^2;q)_\infty^2}{(q)_\infty^2}\sum_{m=0}^{\infty}\frac{(q^{1+m};q)_\infty^2}{(q^{\frac{1}{2}+m}t^2;q)_\infty^2}q^{\frac{m}{2}}t^{-2m}. \quad (3.3)$$

As the residue sum begins with 1, the Higgsed theory is a trivial theory. This would imply that the expansion (3.3) is associated to a Higgsing process which splits a D3-brane along the D5-brane and separates one of the NS5-branes in the $x^{7,8,9}$ directions (see Fig. 2).

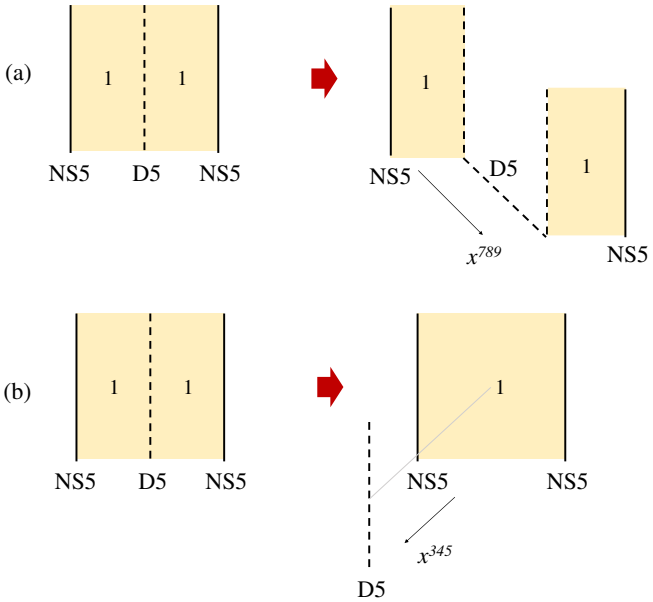

FIG. 2. (a) The Higgsing procedure of 3D $\mathcal{N}=4$ SQED_1 splitting a D3-brane along the D5-brane and separating one of the NS5-brane in the $x^{7,8,9}$ directions. (b) The Higgsing procedure of 3D $\mathcal{N}=4$ SQED_1 separating the D5-brane from the stretched D3-brane between the NS5-branes.

Correspondingly, we get the mirror index by selecting out the zero-charge sector from the index (2.24) of 3D $\mathcal{N}=4$ twisted hypermultiplet

$$\oint \frac{dx}{2\pi i x} \mathbb{I}^{\text{3D tHM}}(t,x;q) = \oint \frac{dx}{2\pi i x} \underbrace{\frac{(q^{\frac{3}{4}}tx;q)_\infty (q^{\frac{3}{4}}tx^{-1};q)_\infty}{(q^{\frac{1}{4}}t^{-1}x;q)_\infty (q^{\frac{1}{4}}t^{-1}x^{-1};q)_\infty}}_{\mathbb{I}^{\text{3D tHM}}(x)}. \tag{3.4}$$

The indices (3.3) and (3.4) can be shown to be equal. To see the equivalence, we can firstly calculate the index (3.4) by taking the sum over residues at poles of bifundamental twisted hypermultiplet $x = q^{\frac{1}{4}+m}t^{-1}$ as

$$\oint \frac{dx}{2\pi i x} \mathbb{I}^{\text{3D tHM}}(t,x;q) = \frac{(q^{\frac{1}{2}}t^2;q)_\infty (q^{\frac{1}{2}}t^{-2};q)_\infty}{(q)_\infty^2} \sum_{m=0}^{\infty} \frac{(q^{1+m};q)_\infty^2}{(q^{\frac{1}{2}+m}t^{-2};q)_\infty^2} q^{\frac{m}{2}} t^{2m}. \tag{3.5}$$

On the other hand, we can expand the index (3.2) of SQED_1 in terms of the expansion (2.20) of the hypermultiplet index as

$$\begin{aligned} \mathbb{I}^{\text{3D pert SQED}_1}(t;q) &= \frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty} \oint \frac{ds}{2\pi i s} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{\frac{1}{2}}t^{-2};q)_k (q^{\frac{1}{2}}t^{-2};q)_{n-k}}{(q)_k (q)_{n-k}} s^{n-2k} q^{\frac{n}{4}} t^n \\ &= \frac{(q^{\frac{1}{2}}t^2;q)_\infty (q^{\frac{1}{2}}t^{-2};q)_\infty}{(q)_\infty^2} \sum_{m=0}^{\infty} \frac{(q^{1+m};q)_\infty^2}{(q^{\frac{1}{2}+m}t^{-2};q)_\infty^2} q^{\frac{m}{2}} t^{2m}. \end{aligned} \tag{3.6}$$

This agrees with the expression (3.5). Also we observe that the expansion in the sum (3.6) starts from the perturbative index $\mathbb{I}^{\text{3D pert } U(1)}$ of 3D $\mathcal{N}=4$ $U(1)$ vector multiplet. This would be associated to the Higgsing process of separating the D5-brane from the stretched D3-brane between the NS5-branes (see Fig. 2).

Introducing a Wilson line operator $\mathcal{W}_n$ of charge n, we obtain the index

$$\mathbb{I}^{\text{3D SQED}_1}_{\mathcal{W}_n}(t,x;q) = \frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty} \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s;q)_\infty (q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s^{-1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts;q)_\infty (q^{\frac{1}{4}+\frac{|m|}{2}}ts^{-1};q)_\infty} s^n q^{\frac{|m|}{4}} t^{-|m|} x^m. \tag{3.7}$$

As shown in Appendix A 1 a, we have checked that this agrees with

$$\mathbb{I}^{\text{3D tHM}}_{\mathcal{V}_n}(t,x;q) = \frac{(q^{\frac{3}{4}+\frac{|n|}{2}}tx;q)_\infty (q^{\frac{3}{4}+\frac{|n|}{2}}tx^{-1};q)_\infty}{(q^{\frac{1}{4}+\frac{|n|}{2}}tx;q)_\infty (q^{\frac{1}{4}+\frac{|n|}{2}}t^{-1}x^{-1};q)_\infty} q^{\frac{|n|}{4}} t^{|n|}. \tag{3.8}$$

We see that the index (3.8) has the contributions from bare monopole operator associated with flavor symmetry and that the quantum numbers of twisted hypermultiplets are affected by the flux.

B. $T[SU(2)]$

The next example is a 3D $\mathcal{N}=4$ Abelian gauge theory with two charged hypermultiplets, which we call $T[SU(2)]$. The dimension of the Higgs branch is $\dim_{\mathbb{C}} \mathcal{M}_H^{T[SU(2)]} = 2\cdot(1\times 2 - 1^2) = 2$, and the dimension of the Coulomb branch is $\dim_{\mathbb{C}} \mathcal{M}_C^{T[SU(2)]} = 2\cdot 1 = 2$. This is self-mirror with two global symmetries, a $SU(2)_f$ flavor symmetry on $\mathcal{M}_H$, and enhanced $SU(2)_t$ topological symmetry on $\mathcal{M}_C$ which are exchanged under mirror symmetry.

The index of $T[SU(2)]$ reads

$$\mathbb{I}^{T[SU(2)]}(t,x_\alpha,z_\alpha;q) = \frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty} \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s^{\pm}x_1^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts^{\pm}x_1^{\pm};q)_\infty} \frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s^{\pm}x_2^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts^{\pm}x_2^{\pm};q)_\infty} q^{\frac{|m|}{2}} t^{-2|m|} z_1^m z_2^{-m}, \tag{3.9}$$

where the fugacities x_α are associated to the $SU(2)$ flavor symmetry with $x_1x_2=1$ and z_α are the fugacities for the topological symmetry.

The index (3.9) of $T[SU(2)]$ coincides with the index of $T[\widetilde{SU(2)}]$ which is obtained from $T[SU(2)]$ by replacing their supermultiplets with the twisted counterparts,

$$\mathbb{I}^{\widetilde{T[SU(2)]}}(t,x_\alpha,z_\alpha;q)=\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{2};q)_\infty}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}ts^{\pm}z_1^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}t^{-1}s^{\pm}z_1^{\pm};q)_\infty}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}ts^{\pm}z_2^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}t^{-1}s^{\pm}z_2^{\pm};q)_\infty}q^{\frac{|m|}{2}}t^{2|m|}x_1^m x_2^{-m}. \tag{3.10}$$

Here the role of fugacities x_α and z_α is exchanged.

We can draw a lesson from the expansion of the index. To get a Higgsing procedure, we simply look at the sector with zero flavor charge of the perturbative index of $T[SU(2)]$,

$$\oint\frac{dx_1}{2\pi ix_1}\frac{dx_2}{2\pi ix_2}\mathbb{I}^{\text{3D pert }T[SU(2)]}(t;q)=\underbrace{\frac{(q^{\frac{1}{2}}t^{2};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds_1}{2\pi is_1}}_{\mathbb{I}^{\text{3D pert }U(1)}}\oint\frac{dx_1}{2\pi ix_1}\oint\frac{dx_2}{2\pi ix_2}\underbrace{\frac{(q^{\frac{3}{4}}t^{-1}s^{\pm}x_1^{\pm};q)_\infty}{(q^{\frac{1}{4}}ts^{\pm}x_1^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D HM}}(sx_1)}\cdot\underbrace{\frac{(q^{\frac{3}{4}}t^{-1}s^{\pm}x_2^{\pm};q)_\infty}{(q^{\frac{1}{4}}ts^{\pm}x_2^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D HM}}(sx_2)}. \tag{3.11}$$

This can be evaluated as

$$\oint\frac{dx_1}{2\pi ix_1}\frac{dx_2}{2\pi ix_2}\mathbb{I}^{\text{pert }T[SU(2)]}(t;q)=\frac{(q^{\frac{1}{2}}t^{2};q)_\infty(q^{\frac{1}{2}}t^{-2};q)_\infty^3}{(q)_\infty^4}\sum_{n,m=0}^{\infty}\frac{(q^{1+n};q)_\infty^2(q^{1+m};q)_\infty^2}{(q^{\frac{1}{2}+n}t^{-2};q)_\infty^2(q^{\frac{1}{2}+m}t^{-2};q)_\infty^2}q^{\frac{n+m}{2}}t^{2(n+m)}. \tag{3.12}$$

Again the expansion (3.12) has a Higgsing interpretation. As its first term is identified with the index $\mathbb{I}^{\text{3D pert }U(1)}$ of 3D $\mathcal{N}=4$ $U(1)$ vector multiplet. It is associated to a Higgsing manipulation of separation of two D5-branes from a suspended D3-brane between two NS5-branes.

C. SQED$_{N_f}$

Now we would like to discuss the generalization of full-indices for 3D $\mathcal{N}=4$ Abelian gauge theories. Consider a 3D $\mathcal{N}=4$ Abelian gauge theory with N_f charged hypermultiplets, which we call SQED$_{N_f}$.

For SQED$_{N_f}$, the R-charge of a bare monopole is $\Delta(m)=\frac{N_f|m|}{2}$, and the index takes the form

$$\mathbb{I}^{\text{3D SQED}_{N_f}}(t,x_\alpha,z_\alpha;q)=\frac{(q^{\frac{1}{2}}t^{2};q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}\prod_{\alpha=1}^{N_f}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s^{\pm}x_\alpha^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts^{\pm}x_\alpha^{\pm};q)_\infty}q^{\frac{N_f|m|}{4}}t^{-N_f|m|}z_1^m z_2^{-m}, \tag{3.13}$$

where x_α are the fugacities for the $SU(N_f)$ flavor symmetry with $\prod_{\alpha=1}^{N_f}x_\alpha=1$, and z_α are the fugacities for the topological symmetry.

As discussed in [28], one can obtain the Hilbert series for branches of vacua in the special fugacity limit of the indices. We briefly check this in our notation as follows. The Coulomb branch of SQED$_{N_f}$ is the A_{N_f-1} singularity $\mathbb{C}^2/\mathbb{Z}_{N_f}$ [1]. Taking the Coulomb limit $q\to 0$ while keeping $\tilde{\mathfrak{t}}^2=q^{\frac{1}{4}}t^{-1}$ constant and setting $z=z_1z_2^{-1}$, the index (3.13) reduces to

$$\begin{aligned}\mathbb{I}^{\text{3D SQED}_{N_f}}_{\text{Coulomb}}(\tilde{\mathfrak{t}},z)&=\frac{1}{(1-\tilde{\mathfrak{t}})}\sum_{m\in\mathbb{Z}}\tilde{\mathfrak{t}}^{\frac{N_f|m|}{2}}z^m\\&=\frac{1-\tilde{\mathfrak{t}}^{N_f}}{(1-\tilde{\mathfrak{t}})(1-\tilde{\mathfrak{t}}^{\frac{N_f}{2}}z)(1-\tilde{\mathfrak{t}}^{\frac{N_f}{2}}z^{-1})}.\end{aligned} \tag{3.14}$$

This is identified with the refined Hilbert series of the Coulomb branch of SQED$_{N_f}$ [2]. The factors $1/(1-\tilde{\mathfrak{t}})$, $1/(1-\tilde{\mathfrak{t}}^{\frac{N_f}{2}}z)$, and $1/(1-\tilde{\mathfrak{t}}^{\frac{N_f}{2}}z^{-1})$ would correspond to the scalar field φ, the monopole operator V_+ of magnetic flux $+1$, and the monopole operator V_- of magnetic flux -1, respectively. They obey the relation $V_+V_-=\varphi^{N_f}$ at dimension N_f and topological charge 0 [38] which is encoded by the numerator.

In fact, by setting $z=1$ in (3.14), we get the Hilbert series of the A_{N_f-1} singularity $\mathbb{C}^2/\mathbb{Z}_{N_f}$ [39],

$$\mathbb{I}^{\text{3D SQED}_{N_f}}_{\text{Coulomb}}(\tilde{\mathfrak{t}},z=1)=\frac{1-\tilde{\mathfrak{t}}^{N_f}}{(1-\tilde{\mathfrak{t}})(1-\tilde{\mathfrak{t}}^{\frac{N_f}{2}})^2}=\text{Hilb}[\mathbb{C}^2/\mathbb{Z}_{N_f}]. \tag{3.15}$$

The mirror of SQED$_{N_f}$, which we denote by $[1]-\widetilde{(1)^{N_f-1}}-[1]$, has a gauge group $U(1)^{N_f}/U(1)$ and the twisted hypermultiplets that are associated to the links of the extended Dynkin diagram of A_{N_f-1}. The twisted hypers carry charges $(+,-,0,\ldots,0),(0,+,-,\ldots,0),\cdots,(-,0,\ldots,0,+)$ under the $U(1)^{N_f}$. For $[1]-\widetilde{(1)^{N_f-1}}-[1]$ theory, the magnetic fluxes are labeled by N_f-1 integers $(m_1,\ldots,m_{N_f-1})$ corresponding to (N_f-1) $U(1)$ topological symmetries.

The conformal dimension of bare monopole is $\Delta(m)=\frac{1}{2}(|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|)$. Then the index of the mirror quiver gauge theory $[1]-\widetilde{(1)^{N_f-1}}-[1]$ reads

$$\begin{aligned}
&\mathbb{I}^{3\mathrm{D}\ [1]-\widetilde{(1)^{N_f-1}}-[1]}(t,x_\alpha,z_\alpha;q)\\
&=\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^{N_f-1}}{(q^{\frac{1}{2}}t^{2};q)_\infty^{N_f-1}}\sum_{m_1,\ldots,m_{N_f-1}\in\mathbb{Z}}\oint\prod_{i=1}^{N_f-1}\frac{ds_i}{2\pi i s_i}\\
&\times\frac{(q^{\frac{3}{4}+\frac{|m_1|}{2}}ts_1^{\pm}z_1^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1|}{2}}t^{-1}s_1^{\pm}z_1^{\mp};q)_\infty}\cdot\prod_{i=1}^{N_f-2}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_{i+1}|}{2}}ts_i^{\pm}s_{i+1}^{\mp};q)_\infty}{(q^{\frac{3}{4}+\frac{|m_i-m_{i+1}|}{2}}t^{-1}s_i^{\pm}s_{i+1}^{\mp};q)_\infty}\cdot\frac{(q^{\frac{3}{4}+\frac{|m_{N_f-1}|}{2}}ts_{N_f-1}^{\pm}z_2^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{N_f-1}|}{2}}t^{-1}s_{N_f-1}^{\pm}z_2^{\mp};q)_\infty}\\
&\times q^{\frac{|m_1|}{4}+\frac{|m_{N_f-1}|}{4}+\sum_{i=1}^{N_f-2}\frac{|m_i-m_{i+1}|}{4}}t^{|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|}\prod_{\alpha=1}^{N_f-1}\left(\frac{x_\alpha}{x_{\alpha+1}}\right)^{m_\alpha}\left(\frac{x_{N_f}}{x_1}\right)^{m_1+m_{N_f-1}}.
\end{aligned}\tag{3.16}$$

Here the fugacities z_α are used for the flavor symmetry, while the fugacities x_α are associated to the topological symmetry. The index (3.16) would be equal to the index (3.13) of SQED_{N_f}. As shown in Appendix A 1 b, we have confirmed that they agree with each other for $N=1,2,3,4$ up to order q^5.

The Coulomb branch of $[1]-(1)^{N_f-1}-[1]$ is the reduced moduli space of one instanton of $SU(N_f)$. Taking the Higgs limit $q\to 0$ while keeping $\mathfrak{t}^{\frac{1}{2}}=q^{\frac{1}{4}}t$ constant, setting $z_\alpha=x_\alpha x_{\alpha+1}^{-1}$ and $m_0=m_{N_f}\equiv 0$, the index (3.16) becomes

$$\begin{aligned}
&\mathbb{I}^{3\mathrm{D}\ [1]-\widetilde{(1)^{N_f-1}}-[1]}_{\mathrm{Higgs}}(\mathfrak{t},z_i)\\
&=\frac{1}{(1-\mathfrak{t})^{N_f-1}}\sum_{m_1,\ldots,m_{N_f-1}}\mathfrak{t}^{\frac{1}{2}\sum_{i=0}^{N_f-1}|m_i-m_{i+1}|}\prod_{i=1}^{N_f-1}z_i^{m_i}\\
&=\mathrm{Hilb}^{3\mathrm{D}\ [1]-\widetilde{(1)^{N_f-1}}-[1]}(\mathfrak{t},z_i),
\end{aligned}\tag{3.17}$$

where we have removed decoupled $U(1)$ by gauge fixing to set $m_1+m_{N_f-1}=0$. This is the refined Hilbert series of the Coulomb branch of $[1]-(1)^{N_f-1}-[1]$.

IV. NON-ABELIAN MIRROR SYMMETRY

In this section, we test 3D $\mathcal{N}=4$ mirror symmetry for non-Abelian gauge theories by computing 3D full-indices. We confirm that two indices nicely agree to each other.

A. $(N)-[2N]$

Let us argue for the balanced $U(N)$ gauge theory with $2N$ hypers which we denote by $(N)-[2N]$. The magnetic charges for $U(N)$ gauge theory are given by N-tuples of integers $(m_1,\ldots,m_N)$. The dimension of Coulomb branch is

$$\dim_{\mathbb{C}}\mathcal{M}_C^{(N)-[2N]}=2N,\tag{4.1}$$

while the dimension of Higgs branch is

$$\dim_{\mathbb{C}}\mathcal{M}_H^{(N)-[2N]}=2\cdot(N\times 2N-N^2)=2N^2.\tag{4.2}$$

The R-charge of bare monopole is

$$\Delta(m)=N\sum_{i=1}^{N}|m_i|-\sum_{i<j}|m_i-m_j|.\tag{4.3}$$

Here the first terms are contributed from the $2N$ fundamental hypers, while the second terms are the contributions from the $U(N)$ vector multiplet.

The index of $(N)-[2N]$ takes the form

$$\begin{aligned}
&\mathbb{I}^{(N)-[2N]}(t,x_\alpha,z_\alpha;q)\\
&=\frac{1}{N!}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\sum_{m_1,\ldots,m_N\in\mathbb{Z}}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{(1-q^{\frac{|m_i-m_j|}{2}}s_i^{\pm}s_j^{\mp})(q^{\frac{1+|m_i-m_j|}{2}}t^2s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1+|m_i-m_j|}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times\prod_{i=1}^{N}\prod_{\alpha=1}^{2N}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}ts_i^{\pm}x_\alpha^{\pm};q)_\infty}q^{\frac{N}{2}\sum_{i=1}^{N}|m_i|-\sum_{i<j}\frac{|m_i-m_j|}{2}}\cdot t^{-2N\sum_{i=1}^{N}|m_i|+2\sum_{i<j}|m_i-m_j|}\cdot\left(\frac{z_1}{z_2}\right)^{\sum_{i=1}^{N}m_i},
\end{aligned}\tag{4.4}$$

where x_α are the fugacities for the $SU(2N)$ flavor symmetry with $\prod_{i=1}^{2N} x_\alpha = 1$, and z_α are the fugacities for the topological symmetry.

The mirror of $(N)-[2N]$ is the quiver gauge theory $(1)-(2)-\cdots-\widetilde{(N)}-\cdots-(2)-(1)$ with $[2]$ attached to $\widetilde{(N)}$. The dimension of the Coulomb branch is

$$\dim_{\mathbb{C}}\mathcal{M}_C = 2\cdot\left(2\sum_{k=1}^{N-1}k+N\right)=2N^2, \tag{4.5}$$

and the dimension of Higgs branch is

$$\dim_{\mathbb{C}}\mathcal{M}_H = 2\cdot\left(2\times\sum_{k=1}^{N-1}k(k+1)+2\times N-2\sum_{k=1}^{N-1}k^2-N^2\right)=2N. \tag{4.6}$$

As predicted by mirror symmetry, the dimensions (4.5) and (4.6) agree with the dimensions (4.2) and (4.1), respectively.

Classically, there is a $U(1)^{N^2}$ topological symmetry and we label the magnetic fluxes by N sets of k-tuple of integers $\{m_i^{(k)}\}_{i=1,\ldots,k}$ with $k=1,\ldots,N$ and $(N-1)$ sets of $(2N-k)$-tuple of integers $\{m_i^{(k)}\}_{i=1,\ldots,2N-k}$ with $k=N+1,\ldots,2N-1$, which are in total N^2 integers. The conformal dimension of bare monopole is

$$\Delta(m)=\frac{1}{2}\sum_{k=1}^{2N-2}\sum_i\sum_j|m_i^{(k)}-m_j^{(k+1)}| +\sum_{i=1}^{N}|m_i^{(N)}|-\sum_{k=1}^{2N-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|. \tag{4.7}$$

The first terms are the contributions from the bifundamental twisted hypermultiplets. The sum over i runs from 1 to k for $k\le N$ and from 1 to $2N-k$ for $N<k$, while the sum over j runs from 1 to $k+1$ for $k\le N-1$ and from 1 to $2N-k-1$ for $N-1<k$. The second terms are the contributions from the two twisted hypermultiplets transforming as fundamental representation under the $U(N)$ gauge symmetry, and the third terms are the contributions from the twisted vector multiplets.

Then the index of the quiver gauge theory $(1)-(2)-\cdots-\widetilde{(N)}-\cdots-(2)-(1)$ with $[2]$ attached to $\widetilde{(N)}$ reads

$$\begin{aligned}
&\mathbb{I}^{\text{3D}}_{(1)-(2)-\cdots-\widetilde{(N)}-\cdots-(2)-(1),\,[2]}(t,x_\alpha,z_\alpha;q)\\
&=\prod_{k=1}^{N}\frac{1}{k!}\frac{(q^{\frac12}t^{-2};q)_\infty^k}{(q^{\frac12}t^2;q)_\infty^k}\sum_{m_1^{(k)},\ldots,m_k^{(k)}\in\mathbb{Z}}\oint\prod_{i=1}^{k}\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}\\
&\times\prod_{i<j}\frac{\left(1-q^{\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}s_i^{(k)\pm}s_j^{(k)\mp}\right)\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{-2}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}{\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{2}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}\\
&\times\prod_{k=1}^{N-1}\frac{1}{(N-k)!}\frac{(q^{\frac12}t^{-2};q)_\infty^{N-k}}{(q^{\frac12}t^2;q)_\infty^{N-k}}\sum_{m_1^{(N+k)},\ldots,m_{N-k}^{(N+k)}\in\mathbb{Z}}\oint\prod_{i=1}^{N-k}\frac{ds_i^{(N+k)}}{2\pi i s_i^{(N+k)}}\\
&\times\prod_{i<j}\frac{\left(1-q^{\frac{|m_i^{(N+k)}-m_j^{(N+k)}|}{2}}s_i^{(N+k)\pm}s_j^{(N+k)\mp}\right)\left(q^{\frac{1+|m_i^{(N+k)}-m_j^{(N+k)}|}{2}}t^{-2}s_i^{(N+k)\pm}s_j^{(N+k)\mp};q\right)_\infty}{\left(q^{\frac{1+|m_i^{(N+k)}-m_j^{(N+k)}|}{2}}t^{2}s_i^{(N+k)\pm}s_j^{(N+k)\mp};q\right)_\infty}\\
&\times\prod_{k=1}^{2N-2}\prod_i\prod_j\frac{\left(q^{\frac34+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}ts_i^{(k)\pm}s_j^{(k+1)\mp};q\right)_\infty}{\left(q^{\frac14+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}t^{-1}s_i^{(k)\pm}s_j^{(k+1)\mp};q\right)_\infty}\cdot\prod_{i=1}^{N}\prod_{\alpha=1}^{2}\frac{\left(q^{\frac34+\frac{|m_i^{(N)}|}{2}}ts_i^{(N)\pm}z_\alpha^{\pm};q\right)_\infty}{\left(q^{\frac14+\frac{|m_i^{(N)}|}{2}}t^{-1}s_i^{(N)\pm}z_\alpha^{\pm};q\right)_\infty}
\end{aligned}$$

$$\times q^{\sum_{k=1}^{2N-2}\sum_i\sum_j\frac{|m_i^{(k)}-m_j^{(k+1)}|}{4}+\sum_{i=1}^{N}\frac{|m_i^{(N)}|}{2}-\sum_{k=1}^{2N-1}\sum_{i<j}\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}$$
$$\times t^{\sum_{k=1}^{2N-2}\sum_i\sum_j|m_i^{(k)}-m_j^{(k+1)}|+2\sum_{i=1}^{N}|m_i^{(N)}|-2\sum_{k=1}^{2N-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|}$$
$$\times\prod_{k=1}^{2N-1}\left(\frac{x_k}{x_{k+1}}\right)^{\sum_{i=1}^{k}m_i^{(k)}}\cdot\left(\frac{x_{2N}}{x_1}\right)^{m_1^{(1)}+\sum_{i=1}^{2N-1}m_i^{(2N-1)}}, \tag{4.8}$$

where z_α are the fugacities for the flavor symmetry and x_α are the fugacities for the topological symmetry. We expect that the index (4.4) coincides with the index (4.8).

For example, consider 3D $\mathcal{N}=4$ $U(2)$ gauge theory with four fundamental hypermultiplets, which we denote by $(2)-[4]$. This is the simplest balanced non-Abelian gauge theory. The quiver diagram and the brane construction are shown in Fig. 3.

The index of $(2)-[4]$ is

$$\mathbb{I}^{\text{3D }(2)-[4]}(t,x_\alpha,z_\alpha;q)=\frac{1}{2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^2}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}\sum_{m_1,m_2\in\mathbb{Z}}\oint\frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}$$
$$\times\frac{(1-q^{\frac{|m_1-m_2|}{2}}\frac{s_1}{s_2})(1-q^{\frac{|m_1-m_2|}{2}}\frac{s_2}{s_1})(q^{\frac{1+|m_1-m_2|}{2}}t^2\frac{s_1}{s_2};q)_\infty(q^{\frac{1+|m_1-m_2|}{2}}t^2\frac{s_2}{s_1};q)_\infty}{(q^{\frac{1+|m_1-m_2|}{2}}t^{-2}\frac{s_1}{s_2};q)_\infty(q^{\frac{1+|m_1-m_2|}{2}}t^{-2}\frac{s_2}{s_1};q)_\infty}$$
$$\times\prod_{i=1}^{2}\prod_{\alpha=1}^{4}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}ts_i^{\pm}x_\alpha^{\pm};q)_\infty}q^{|m_1|+|m_2|-\frac{|m_1-m_2|}{2}}t^{-4|m_1|-4|m_2|+2|m_1-m_2|}z_1^{m_1+m_2}z_2^{-m_1-m_2}, \tag{4.9}$$

where x_α are the fugacities for the $SU(4)$ flavor symmetry with $\prod_{\alpha=1}^{4}x_\alpha=1$ and z_α are the fugacities for the topological symmetry.

In the Coulomb limit, where one keeps $\tilde{\mathfrak{t}}^{\frac{1}{2}}=q^{\frac{1}{4}}t^{-1}$ constant and sends $q\to 0$ and $t\to\infty$, the index (4.9) reduces to

$$\mathbb{I}^{\text{3D }(2)-[4]}_{\text{Coulomb}}(\tilde{\mathfrak{t}},z)\frac{1}{2}\frac{1}{(1-\tilde{\mathfrak{t}})^2}\sum_{m_1,m_2\in\mathbb{Z}}\oint\frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}\delta_{m_1,m_2}\frac{(1-s_1^{\pm}s_2^{\mp})}{(1-\tilde{\mathfrak{t}}s_1^{\pm}s_2^{\mp})}\tilde{\mathfrak{t}}^{\Delta(m)}z^{m_1+m_2}, \tag{4.10}$$

where we have defined $z:=z_1z_2^{-1}$. Let $\lambda(m_i)$ be a partition of 2 which is associated to the magnetic flux m_i obeying $\sum_{i=1}^{2}\lambda_j(m_i)=2$ and $\lambda_j(m_i)\geq\lambda_{j+1}(m_i)$. Making use of the formula

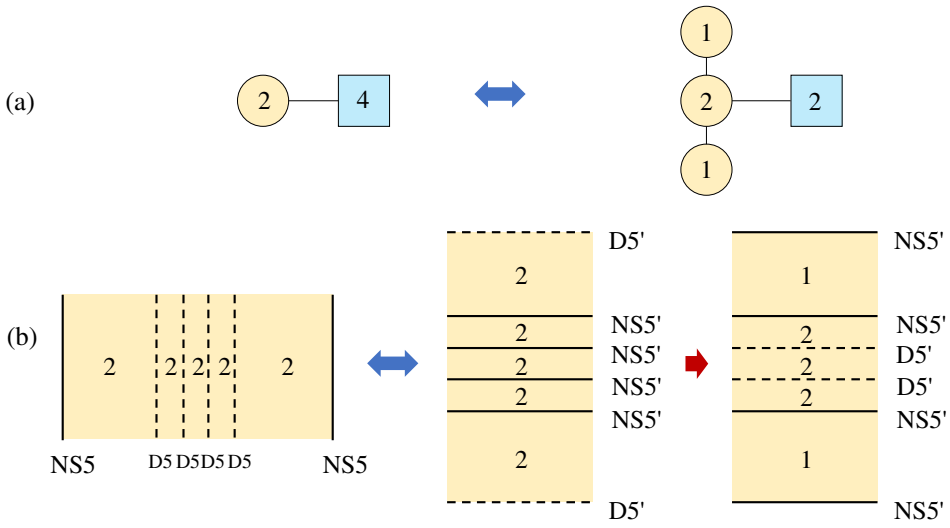

FIG. 3. (a) The quiver diagrams of $(2)-[4]$ and its mirror quiver gauge theory. (b) The brane configurations of $(2)-[4]$ and its mirror quiver gauge theory.

$$\frac{1}{N!}\frac{1}{(1-x)^N}\oint\prod_{i=}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(1-\frac{s_i}{s_j})}{(1-x\frac{s_i}{s_j})}=\prod_{k=1}^{N}\frac{1}{1-x^k}, \tag{4.11}$$

we get

$$\mathbb{I}^{\text{3D }(2)-[4]}_{\text{Coulomb}}(\tilde{\mathfrak{t}},z)=\sum_{m_1,m_2\in\mathbb{Z}}\tilde{\mathfrak{t}}^{2|m_1|+2|m_2|-|m_1-m_2|}\cdot z^{m_1+m_2}\cdot P_{U(2)}(\tilde{\mathfrak{t}},m_i). \tag{4.12}$$

Here

$$P_{U(2)}(\tilde{\mathfrak{t}},m_i)=\prod_{k=1}^{2}\frac{1}{(1-\tilde{\mathfrak{t}}^k)^{\lambda_k^T(m_i)}} \tag{4.13}$$

is the factor which counts the number of Casimir where $\lambda_k^T(m_i)$ is the length of the kth row of the transposed Young tableau $\lambda^T(m_i)$. The expression (4.12) is identified with the refined Hilbert series for the Coulomb branch of $(2)-[4]$ which can be further written as [2]

$$\mathbb{I}^{\text{3D }(2)-[4]}_{\text{Coulomb}}(\tilde{\mathfrak{t}},z)=\text{Hilb}^{\text{3D }(2)-[4]}(\tilde{\mathfrak{t}},z)=\prod_{i=1}^{2}\frac{1-\tilde{\mathfrak{t}}^{5-i}}{(1-\tilde{\mathfrak{t}}^i)(1-z\tilde{\mathfrak{t}}^{3-i})(1-z^{-1}\tilde{\mathfrak{t}}^{3-i})}. \tag{4.14}$$

The factors $1/(1-\tilde{\mathfrak{t}}^i)$ with $i=1,2$ correspond to the two generators $\text{Tr}\varphi^i$ where φ is the adjoint scalar field. The factors $1/(1-z\tilde{\mathfrak{t}}^{3-i})$ and $1/(1-z^{-1}\tilde{\mathfrak{t}}^{3-i})$ with $i=1,2$ describe the monopole operators V_+ with magnetic flux $(+,0)$ and V_- with magnetic flux $(-,0)$ dressed by the adjoint complex scalar field.

The index of the mirror quiver gauge theory $\begin{matrix}(1)-\widetilde{(2)}-(1)\\ \quad\quad|\\ \quad\quad[2]\end{matrix}$ reads

$$\begin{aligned}
&\mathbb{I}^{\text{3D }\begin{matrix}(1)-\widetilde{(2)}-(1)\\ \quad\quad|\\ \quad\quad[2]\end{matrix}}(t,x_\alpha,z_\alpha;q)\\
&=\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{2};q)_\infty}\sum_{m_1\in\mathbb{Z}}\oint\frac{ds_1}{2\pi i s_1}\cdot\frac{1}{2}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}{(q^{\frac{1}{2}}t^{2};q)_\infty^2}\sum_{m_2,m_3\in\mathbb{Z}}\oint\frac{ds_2}{2\pi i s_2}\frac{ds_3}{2\pi i s_3}\\
&\times\frac{(1-q^{\frac{|m_2-m_3|}{2}}\frac{s_2}{s_3})(1-q^{\frac{|m_2-m_3|}{2}}\frac{s_3}{s_2})(q^{\frac{1+|m_2-m_3|}{2}}t^{-2}\frac{s_2}{s_3};q)_\infty(q^{\frac{1+|m_2-m_3|}{2}}t^{-2}\frac{s_3}{s_2};q)_\infty}{(q^{\frac{1+|m_2-m_3|}{2}}t^{2}\frac{s_2}{s_3};q)_\infty(q^{\frac{1+|m_2-m_3|}{2}}t^{2}\frac{s_3}{s_2};q)_\infty}\\
&\times\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{2};q)_\infty}\sum_{m_4\in\mathbb{Z}}\oint\frac{ds_4}{2\pi i s_4}\\
&\times\prod_{i=2}^{3}\frac{(q^{\frac{3}{4}+\frac{|m_1-m_i|}{2}}ts_1^{\pm}s_i^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1-m_i|}{2}}t^{-1}s_1^{\pm}s_i^{\mp};q)_\infty}\cdot\prod_{i=2}^{3}\prod_{\alpha=1}^{2}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}ts_i^{\pm}z_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}z_\alpha^{\pm};q)_\infty}\cdot\prod_{i=2}^{3}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_4|}{2}}ts_i^{\pm}s_4^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_4|}{2}}t^{-1}s_i^{\pm}s_4^{\mp};q)_\infty}\\
&\times q^{\frac{|m_1-m_2|+|m_1-m_3|}{4}+\frac{|m_2|}{2}+\frac{|m_3|}{2}+\frac{|m_2-m_4|+|m_3-m_4|}{4}-\frac{|m_2-m_3|}{2}}\\
&\times t^{|m_1-m_2|+|m_1-m_3|+2|m_2|+2|m_3|+|m_2-m_4|+|m_3-m_4|-2|m_2-m_3|}\\
&\times\left(\frac{x_1}{x_2}\right)^{m_1}\left(\frac{x_2}{x_3}\right)^{m_2+m_3}\left(\frac{x_3}{x_4}\right)^{m_4}\left(\frac{x_4}{x_1}\right)^{m_1+m_4},
\end{aligned} \tag{4.15}$$

where x_α are the fugacities for the topological symmetry, while z_α are the fugacities for the flavor symmetry. As expected, we find that the indices (4.9) and (4.15) coincide with each other up to order q^3 (see Appendix A 1 c).

B. $T[SU(N)]$

Consider the quiver gauge theory $(1)-(2)-\cdots-(N-1)-[N]$, which we call $T[SU(N)]$. For example, for $T[SU(3)]$ theory, the quiver diagram and the brane construction are illustrated in Fig. 4.

The dimension of the Coulomb branch of $T[SU(N)]$ is

$$\dim_{\mathbb{C}}\mathcal{M}_C^{T[SU(N)]}=2\sum_{k=1}^{N-1}k=N(N-1), \quad (4.16)$$

and the dimension of the Higgs branch of $T[SU(N)]$ is

$$\dim_{\mathbb{C}}\mathcal{M}_H^{T[SU(N)]}=2\left[\sum_{k=1}^{N-1}k(k+1)-\sum_{k=1}^{N-1}k^2\right]=N(N-1). \quad (4.17)$$

The quiver gauge theory $T[SU(N)]$ is a self-mirror theory whose Coulomb branch and Higgs branch are identical as the dimensions (4.16) and (4.17) are equal.

From brane construction, a D3-brane ending on the NS5-brane can be viewed as a magnetic monopole. Hence, the Coulomb branch or Higgs branch of $T[SU(N)]$ is interpreted as the moduli space of $SU(N)$ monopoles which are formed by one with magnetic charge $(+,-,0,\ldots,0)$, two with magnetic charge $(0,+,-,\ldots,0),\ldots,(N-1)$ with magnetic charge $(0,0,\ldots,+,-)$ together with N fixed Dirac monopoles with magnetic charge $(0,0,\ldots,0,+)$.

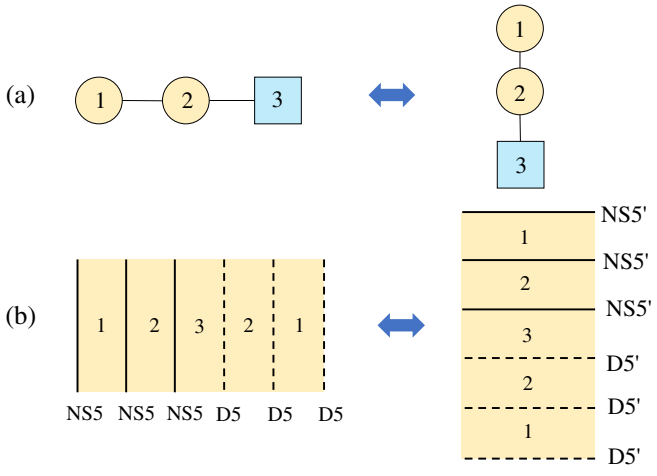

FIG. 4. (a) The quiver diagrams of $T[SU(3)]$. (b) The brane configurations of $T[SU(3)]$.

The magnetic fluxes for $T[SU(N)]$ is labeled by $N-1$ sets of k-tuple of integers, that is $\frac{N(N-1)}{2}$ integers $\{m_i^{(k)}\}_{i=1,\ldots,k}$ with $k=1,\ldots,N-1$. The canonical R-charge of bare monopole is

$$\Delta(m)=\frac{1}{2}\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|+\frac{N}{2}\sum_{i=1}^{N-1}|m_i^{(N-1)}|-\sum_{k=1}^{N-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|, \quad (4.18)$$

where the first terms are the contributions from the bifundamental hypers, the second terms are those from the N fundamental hypers, and the third terms are those from the vector multiplets.

The index of $T[SU(N)]$ takes the form

$$\begin{aligned}
&\mathbb{I}^{T[SU(N)]}(t,x_\alpha,z_\alpha;q)\\
&=\prod_{k=1}^{N-1}\left[\frac{1}{k!}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^k}{(q^{\frac{1}{2}}t^{-2};q)_\infty^k}\sum_{m_1^{(k)},\ldots,m_k^{(k)}\in\mathbb{Z}}\oint\prod_{i=1}^{k}\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}\right.\\
&\quad\left.\times\prod_{i<j}\frac{\left(1-q^{\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}s_i^{(k)\pm}s_j^{(k)\mp}\right)\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^2s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}{\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{-2}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}\right]\\
&\quad\times\prod_{k=1}^{N-2}\prod_{i=1}^{k}\prod_{j=1}^{k+1}\frac{\left(q^{\frac{3}{4}+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}t^{-1}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}ts_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}\prod_{i=1}^{N-1}\prod_{\alpha=1}^{N}\frac{\left(q^{\frac{3}{4}+\frac{|m_i^{(N-1)}|}{2}}t^{-1}s_i^{(N-1)\pm}x_\alpha^{\pm};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(N-1)}|}{2}}ts_i^{(N-1)\pm}x_\alpha^{\pm};q\right)_\infty}\\
&\quad\times q^{\frac{1}{4}\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|+\frac{N}{4}\sum_{i=1}^{N-1}|m_i^{(N-1)}|-\frac{1}{2}\sum_{k=1}^{N-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|}\\
&\quad\times t^{-\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|-N\sum_{i=1}^{N-1}|m_i^{(N-1)}|+2\sum_{k=1}^{N-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|}\\
&\quad\times\prod_{k=1}^{N-1}\left(\frac{z_k}{z_{k+1}}\right)^{\sum_{i=1}^{k}m_i^{(k)}}\cdot\left(\frac{z_N}{z_1}\right)^{m_1^{(1)}+\sum_{i=1}^{N-1}m_i^{(N-1)}},
\end{aligned} \quad (4.19)$$

where x_α are the fugacities for the $SU(N)$ flavor symmetry with $\prod_{\alpha=1}^{N} x_\alpha = 1$, and z_α are the fugacities for the topological symmetry.

We expect that the index (4.19) is equal to the index of $T[\widetilde{SU(N)}]$,

$$
\begin{aligned}
&\mathbb{I}^{T[\widetilde{SU(N)}]}(t, x_\alpha, z_\alpha; q) \\
&= \prod_{k=1}^{N-1} \Bigg[\frac{1}{k!} \frac{(q^{\frac{1}{2}} t^{-2}; q)_\infty^k}{(q^{\frac{1}{2}} t^{2}; q)_\infty^k} \sum_{m_1^{(k)},\ldots,m_k^{(k)} \in \mathbb{Z}} \oint \prod_{i=1}^{k} \frac{ds_i^{(k)}}{2\pi i s_i^{(k)}} \\
&\times \prod_{i<j} \frac{\left(1 - q^{\frac{|m_i^{(k)} - m_j^{(k)}|}{2}} s_i^{(k)\pm} s_j^{(k)\mp}\right) \left(q^{\frac{1+|m_i^{(k)} - m_j^{(k)}|}{2}} t^{-2} s_i^{(k)\pm} s_j^{(k)\mp}; q\right)_\infty}{\left(q^{\frac{1+|m_i^{(k)} - m_j^{(k)}|}{2}} t^{2} s_i^{(k)\pm} s_j^{(k)\mp}; q\right)_\infty} \Bigg] \\
&\times \prod_{k=1}^{N-2} \prod_{i=1}^{k} \prod_{j=1}^{k+1} \frac{\left(q^{\frac{3}{4}+\frac{|m_i^{(k)} - m_j^{(k+1)}|}{2}} t s_i^{(k)\pm} s_j^{(k)\mp}; q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(k)} - m_j^{(k+1)}|}{2}} t^{-1} s_i^{(k)\pm} s_j^{(k)\mp}; q\right)_\infty} \prod_{i=1}^{N-1} \prod_{\alpha=1}^{N} \frac{(q^{\frac{3}{4}+\frac{|m_i^{(N-1)}|}{2}} t s_i^{(N-1)\pm} z_\alpha^{\pm}; q)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(N-1)}|}{2}} t^{-1} s_i^{(N-1)\pm} z_\alpha^{\pm}; q\right)_\infty} \\
&\times q^{\frac{1}{4}\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1} |m_i^{(k)} - m_j^{(k+1)}| + \frac{N}{4}\sum_{i=1}^{N-1} |m_i^{(N-1)}| - \frac{1}{2}\sum_{k=1}^{N-1}\sum_{i<j} |m_i^{(k)} - m_j^{(k)}|} \\
&\times t^{\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1} |m_i^{(k)} - m_j^{(k+1)}| + N\sum_{i=1}^{N-1} |m_i^{(N-1)}| - 2\sum_{k=1}^{N-1}\sum_{i<j} |m_i^{(k)} - m_j^{(k)}|} \\
&\times \prod_{k=1}^{N-1} \left(\frac{x_k}{x_{k+1}}\right)^{\sum_{i=1}^{k} m_i^{(k)}} \cdot \left(\frac{x_N}{x_1}\right)^{m_1^{(1)} + \sum_{i=1}^{N-1} m_i^{(N-1)}},
\end{aligned}
\tag{4.20}
$$

with z_α being the fugacities for the flavor symmetry satisfying $\prod_{\alpha=1}^{N} z_\alpha = 1$, and x_α being the fugacities for the topological symmetry. In fact, we have checked that the indices (4.19) and (4.20) agree up to order q^3 (see Appendix A 1 c).

C. (1) – (2) – (1)

Consider a quiver gauge theory $\begin{matrix}(1) & - & (2) & - & (1)\\ | & & | & & |\\ [1] & & [2] & & [1]\end{matrix}$ whose quiver diagram and the brane construction are illustrated in Fig. 5. This is a self-mirror quiver theory.

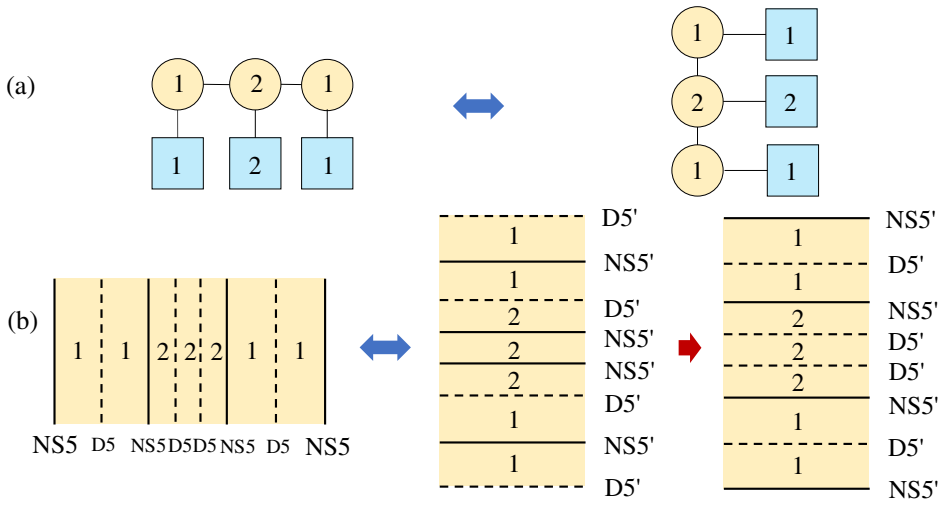

FIG. 5. (a) The quiver diagrams of the self-mirror $U(1) \times U(2) \times U(1)$ quiver gauge theory. (b) The brane configurations of the self-mirror $U(1) \times U(2) \times U(1)$ quiver gauge theory.

The dimension of the Coulomb branch is

$$\dim_{\mathbb{C}}\mathcal{M}_C\begin{matrix}(1)&-&(2)&-&(1)\\|&&|&&|\\[1]&&[2]&&[1]\end{matrix}=2\cdot(1+2+1)=8, \tag{4.21}$$

and the dimension of the Higgs branch is

$$\dim_{\mathbb{C}}\mathcal{M}_H\begin{matrix}(1)&-&(2)&-&(1)\\|&&|&&|\\[1]&&[2]&&[1]\end{matrix}=2\cdot(1\times1+1\times2+2\times2+2\times1+1\times1-1^2-2^2-1^2)=8. \tag{4.22}$$

Let us label the magnetic fluxes for $\begin{matrix}(1)&-&(2)&-&(1)\\|&&|&&|\\[1]&&[2]&&[1]\end{matrix}$ by four integers m_1, m_2, m_3, m_4. The R-charge of bare monopole operator is given by

$$\Delta(m)=\frac{|m_1|}{2}+\frac{|m_1-m_2|+|m_1-m_3|}{2}+|m_2|+|m_3|+\frac{|m_2-m_4|+|m_3-m_4|}{2}+\frac{|m_4|}{2}-|m_2-m_3|. \tag{4.23}$$

The terms $\frac{|m_1|}{2}+\frac{|m_4|}{2}$ are contributed from the charged hypers under the $U(1)$ gauge factors, while the terms $|m_2|+|m_3|$ are the contributions from the fundamental hypers of $U(2)$ gauge symmetry. The terms $\prod_{i<j}|m_i-m_j|$ are the contributions from the bifundamental hypers. The terms $-|m_2-m_3|$ are the contributions from the $U(2)$ vector multiplet.

The index of $\begin{matrix}(1)&-&(2)&-&(1)\\|&&|&&|\\[1]&&[2]&&[1]\end{matrix}$ is evaluated as

$$\begin{aligned}
&\mathbb{I}^{\text{3D}}\begin{matrix}(1)&-&(2)&-&(1)\\|&&|&&|\\[1]&&[2]&&[1]\end{matrix}(t,x_\alpha,z_\alpha;q)\\
&=\frac{(q^{\frac12}t^2;q)_\infty}{(q^{\frac12}t^{-2};q)_\infty}\sum_{m_1\in\mathbb{Z}}\oint\frac{ds_1}{2\pi is_1}\\
&\times\frac12\frac{(q^{\frac12}t^2;q)_\infty^2}{(q^{\frac12}t^{-2};q)_\infty^2}\sum_{m_2,m_3\in\mathbb{Z}}\oint\frac{ds_2}{2\pi is_2}\frac{ds_3}{2\pi is_3}\frac{(1-q^{\frac{|m_2-m_3|}{2}}s_2^{\pm}s_3^{\mp})(q^{\frac{1+|m_2-m_3|}{2}}t^2s_2^{\pm}s_3^{\mp};q)_\infty}{(q^{\frac{1+|m_2-m_3|}{2}}t^{-2}s_2^{\pm}s_3^{\mp};q)_\infty}\\
&\times\frac{(q^{\frac12}t^2;q)_\infty}{(q^{\frac12}t^{-2};q)_\infty}\sum_{m_4\in\mathbb{Z}}\oint\frac{ds_4}{2\pi is_4}\\
&\times\frac{(q^{\frac34+\frac{|m_1|}{2}}t^{-1}s_1^{\pm}x_1^{\pm};q)_\infty}{(q^{\frac14+\frac{|m_1|}{2}}ts_1^{\pm}x_1^{\pm};q)_\infty}\cdot\prod_{i=2}^{3}\frac{(q^{\frac34+\frac{|m_1-m_i|}{2}}t^{-1}s_1^{\pm}s_i^{\mp};q)_\infty}{(q^{\frac14+\frac{|m_1-m_i|}{2}}ts_1^{\pm}s_i^{\mp};q)_\infty}\prod_{i=2}^{3}\prod_{\alpha=2}^{3}\frac{(q^{\frac34+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac14+\frac{|m_i|}{2}}ts_i^{\pm}x_\alpha^{\pm};q)_\infty}
\end{aligned}$$

$$\times \prod_{i=2}^{3} \frac{(q^{\frac{3}{4}+\frac{|m_i-m_4|}{2}} t^{-1} s_i^{\pm} s_4^{\mp}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_4|}{2}} t s_i^{\pm} s_4^{\mp}; q)_\infty} \cdot \frac{(q^{\frac{3}{4}+\frac{|m_4|}{2}} t^{-1} s_4^{\pm} x_4^{\pm}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_4|}{2}} t s_4^{\pm} x_4^{\pm}; q)_\infty}$$
$$\times q^{\frac{|m_1|}{4}+\frac{|m_1-m_2|+|m_1-m_3|}{4}+\frac{|m_2|+|m_3|}{2}+\frac{|m_2-m_4|+|m_3-m_4|}{4}+\frac{|m_4|}{4}-\frac{|m_2-m_3|}{2}}$$
$$\times t^{-|m_1|-|m_1-m_2|-|m_1-m_3|-2|m_2|-2|m_3|-|m_2-m_4|-|m_3-m_4|-|m_4|+2|m_2-m_3|}$$
$$\times \left(\frac{z_1}{z_2}\right)^{m_1} \left(\frac{z_2}{z_3}\right)^{m_2+m_3} \left(\frac{z_3}{z_4}\right)^{m_4} \left(\frac{z_4}{z_1}\right)^{m_1+m_4}, \tag{4.24}$$

where x_α are the fugacities for the flavor symmetry, and z_α are the fugacities for the topological symmetry.

As expected, we have confirmed that the index (4.24) coincides with the index of $\overset{(1)}{[1]} - \widetilde{\overset{(2)}{[2]}} - \overset{(1)}{[1]}$ which consists of twisted supermultiplets,

$$\mathbb{I}^{\text{3D}}_{\overset{(1)}{[1]} - \widetilde{\overset{(2)}{[2]}} - \overset{(1)}{[1]}}(t, x_\alpha, z_\alpha; q)$$
$$= \frac{(q^{\frac{1}{2}} t^{-2}; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi i s_1}$$
$$\times \frac{1}{2} \frac{(q^{\frac{1}{2}} t^{-2}; q)_\infty^2}{(q^{\frac{1}{2}} t^2; q)_\infty^2} \sum_{m_2, m_3 \in \mathbb{Z}} \oint \frac{ds_2}{2\pi i s_2} \frac{ds_3}{2\pi i s_3} \frac{(1 - q^{\frac{|m_2-m_3|}{2}} s_2^{\pm} s_3^{\mp})(q^{\frac{1+|m_2-m_3|}{2}} t^{-2} s_2^{\pm} s_3^{\mp}; q)_\infty}{(q^{\frac{1+|m_2-m_3|}{2}} t^2 s_2^{\pm} s_3^{\mp}; q)_\infty}$$
$$\times \frac{(q^{\frac{1}{2}} t^{-2}; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty} \sum_{m_4 \in \mathbb{Z}} \oint \frac{ds_4}{2\pi i s_4}$$
$$\times \frac{(q^{\frac{3}{4}+\frac{|m_1|}{2}} t s_1^{\pm} z_1^{\pm}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1|}{2}} t^{-1} s_1^{\pm} z_1^{\pm}; q)_\infty} \cdot \prod_{i=2}^{3} \frac{(q^{\frac{3}{4}+\frac{|m_1-m_i|}{2}} t s_1^{\pm} s_i^{\mp}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1-m_i|}{2}} t^{-1} s_1^{\pm} s_i^{\mp}; q)_\infty} \prod_{i=2}^{3} \prod_{\alpha=2}^{3} \frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}} t s_i^{\pm} z_\alpha^{\pm}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}} t^{-1} s_i^{\pm} z_\alpha^{\pm}; q)_\infty}$$
$$\times \prod_{i=2}^{3} \frac{(q^{\frac{3}{4}+\frac{|m_i-m_4|}{2}} t s_i^{\pm} s_4^{\mp}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_4|}{2}} t^{-1} s_i^{\pm} s_4^{\mp}; q)_\infty} \cdot \frac{(q^{\frac{3}{4}+\frac{|m_4|}{2}} t s_4^{\pm} z_4^{\pm}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_4|}{2}} t^{-1} s_4^{\pm} z_4^{\pm}; q)_\infty}$$
$$\times q^{\frac{|m_1|}{4}+\frac{|m_1-m_2|+|m_1-m_3|}{4}+\frac{|m_2|+|m_3|}{2}+\frac{|m_2-m_4|+|m_3-m_4|}{4}+\frac{|m_4|}{4}-\frac{|m_2-m_3|}{2}}$$
$$\times t^{|m_1|+|m_1-m_2|+|m_1-m_3|+2|m_2|+2|m_3|+|m_2-m_4|+|m_3-m_4|+|m_4|-2|m_2-m_3|}$$
$$\times \left(\frac{x_1}{x_2}\right)^{m_1} \left(\frac{x_2}{x_3}\right)^{m_2+m_3} \left(\frac{x_3}{x_4}\right)^{m_4} \left(\frac{x_4}{x_1}\right)^{m_1+m_4}, \tag{4.25}$$

where z_α are the fugacities for the flavor symmetry, and x_α are the fugacities for the topological symmetry (see Appendix A 1 c).

As a next example, let us consider a quiver gauge theory $\overset{(1)}{[1]} - \overset{(2)}{[2]} - \overset{(1)}{[1]}$. This is not self-mirror and the quiver diagram and the brane construction are drawn in Fig. 6. The dimension of the Coulomb branch is

$$\dim_{\mathbb{C}} \mathcal{M}_C \left[\overset{(1)}{[1]} - \overset{(2)}{[2]} - \overset{(1)}{[1]}\right] = 2 \cdot (1 + 2 + 1) = 8, \tag{4.26}$$

while the dimension of the Higgs branch is

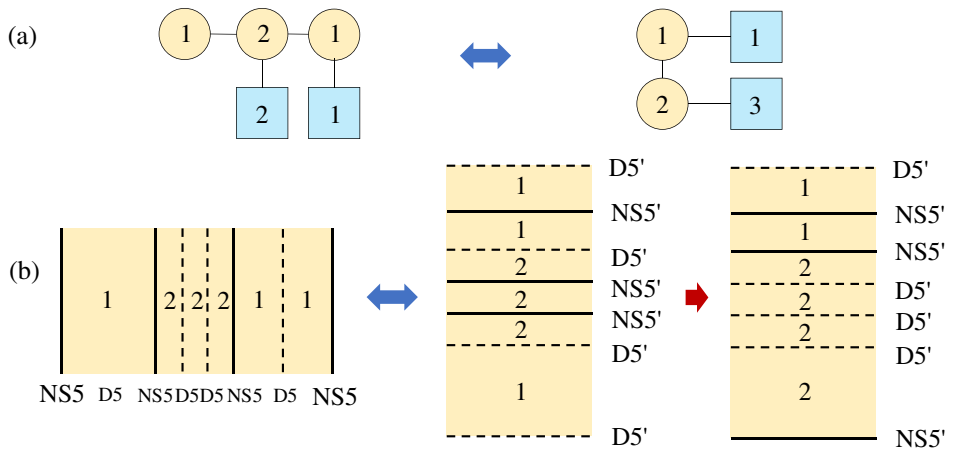

FIG. 6. (a) The quiver diagrams of the $U(1) \times U(2) \times U(1)$ quiver gauge theory with two fundamental hypers for the centered $U(2)$ gauge node and one charged hyper for the end of the $U(1)$ node and its mirror theory. (b) The corresponding brane configurations.

$$\dim_{\mathbb{C}} \mathcal{M}_H \quad \begin{matrix} (1) & - & (2) & - & (1) \\ & & | & & | \\ & & [2] & & [1] \end{matrix} \quad = 2 \cdot (1 \times 2 + 2 \times 2 + 2 \times 1 + 1 \times 1 - 1^2 - 2^2 - 1^2) = 6. \tag{4.27}$$

Again the magnetic fluxes are labeled by integers m_1, m_2, m_3, m_4. The canonical R-charge of bare monopole is given by

$$\Delta(m) = \frac{|m_1 - m_2| + |m_1 - m_3|}{2} + |m_2| + |m_3| + \frac{|m_2 - m_4| + |m_3 - m_4|}{2} + \frac{|m_4|}{2} - |m_2 - m_3|, \tag{4.28}$$

which is obtained by eliminating the contribution $\frac{|m_1|}{2}$ of the charged hyper from the R-charge (4.23) for self-mirror quiver gauge theory $\begin{matrix} (1) & - & (2) & - & (1) \\ | & & | & & | \\ [1] & & [2] & & [1] \end{matrix}$.

We can evaluate the index of $\begin{matrix} (1) & - & (2) & - & (1) \\ & & | & & | \\ & & [2] & & [1] \end{matrix}$ as

$$\begin{aligned} &{}^{3D}\mathbb{I} \quad \begin{matrix} (1) & - & (2) & - & (1) \\ & & | & & | \\ & & [2] & & [1] \end{matrix} \quad (t, x_\alpha, z_\alpha; q) \\ &= \frac{(q^{\frac{1}{2}}t^2; q)_\infty}{(q^{\frac{1}{2}}t^{-2}; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi i s_1} \\ &\times \frac{1}{2} \frac{(q^{\frac{1}{2}}t^2; q)_\infty^2}{(q^{\frac{1}{2}}t^{-2}; q)_\infty^2} \sum_{m_2, m_3 \in \mathbb{Z}} \oint \frac{ds_2}{2\pi i s_2} \frac{ds_3}{2\pi i s_3} \frac{(1 - q^{\frac{|m_2 - m_3|}{2}} s_2^{\pm} s_3^{\mp})(q^{\frac{1+|m_2-m_3|}{2}} t^2 s_2^{\pm} s_3^{\mp}; q)_\infty}{(q^{\frac{1+|m_2-m_3|}{2}} t^{-2} s_2^{\pm} s_3^{\mp}; q)_\infty} \\ &\times \frac{(q^{\frac{1}{2}}t^2; q)_\infty}{(q^{\frac{1}{2}}t^{-2}; q)_\infty} \sum_{m_4 \in \mathbb{Z}} \oint \frac{ds_4}{2\pi i s_4} \end{aligned}$$

$$\times \prod_{i=2}^{3} \frac{(q^{\frac{3}{4}+\frac{|m_1-m_i|}{2}} t^{-1} s_1^{\pm} s_i^{\mp}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1-m_i|}{2}} t s_1^{\pm} s_i^{\mp}; q)_\infty} \cdot \prod_{i=2}^{3} \prod_{\alpha=1}^{2} \frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}} t^{-1} s_i^{\pm} x_\alpha^{\pm}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}} t s_i^{\pm} x_\alpha^{\pm}; q)_\infty}$$
$$\times \prod_{i=2}^{3} \frac{(q^{\frac{3}{4}+\frac{|m_i-m_4|}{2}} t^{-1} s_i^{\pm} s_4^{\mp}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_4|}{2}} t s_i^{\pm} s_4^{\mp}; q)_\infty} \cdot \frac{(q^{\frac{3}{4}+\frac{|m_4|}{2}} t^{-1} s_4^{\pm} x_3^{\pm}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_4|}{2}} t s_4^{\pm} x_3^{\pm}; q)_\infty}$$
$$\times q^{\frac{|m_1-m_2|}{4}+\frac{|m_1-m_3|}{4}+\frac{|m_2|+|m_3|}{2}+\frac{|m_2-m_4|}{4}+\frac{|m_3-m_4|}{4}+\frac{|m_4|}{4}-\frac{|m_2-m_3|}{2}}$$
$$\times t^{-|m_1-m_2|-|m_1-m_3|-2|m_2|-2|m_3|-|m_2-m_4|-|m_3-m_4|-|m_4|+2|m_2-m_3|}$$
$$\times \left(\frac{z_4}{z_2}\right)^{m_1} \left(\frac{z_2}{z_3}\right)^{m_2+m_3} \left(\frac{z_3}{z_1}\right)^{m_4} \left(\frac{z_1}{z_4}\right)^{m_1+m_4}, \tag{4.29}$$

where the fugacities x_α are associated to the flavor symmetry, while the fugacities z_α are associated to the topological symmetry.

The mirror of $\overset{(1)}{\underset{[2]}{|}} - \overset{(2)}{\underset{[1]}{|}} - \overset{(1)}{\underset{}{|}}$ is the quiver gauge theory $\overset{(2)}{\underset{[3]}{|}} \widetilde{-} \overset{(1)}{\underset{[1]}{|}}$ whose quiver diagram and brane construction are illustrated in Fig. 6.

The dimension of the Coulomb branch is

$$\dim_{\mathbb{C}} \mathcal{M}_C^{\overset{(2)}{\underset{[3]}{|}} \widetilde{-} \overset{(1)}{\underset{[1]}{|}}} = 2 \cdot (2+1) = 6, \tag{4.30}$$

and the dimension of the Higgs branch is

$$\dim_{\mathbb{C}} \mathcal{M}_H^{\overset{(2)}{\underset{[3]}{|}} \widetilde{-} \overset{(1)}{\underset{[1]}{|}}} = 2 \cdot (2\times 3 + 2\times 1 + 1\times 1 - 2^2 - 1^2) = 8. \tag{4.31}$$

The magnetic fluxes are labeled by three integers m_1, m_2, m_3. We have the bare monopole operator with dimension

$$\Delta(m) = \frac{3}{2}|m_1| + \frac{3}{2}|m_2| + \frac{|m_1-m_3|}{2} + \frac{|m_2-m_3|}{2} + \frac{|m_3|}{2} - |m_1-m_2|. \tag{4.32}$$

Here the first two terms are contributed from three fundamental twisted hypers for $U(2)$ gauge symmetry, the next two terms are the contributions from bifundamental twisted hypers, the second from the last is contributed from charged twisted hyper for $U(1)$ gauge symmetry, and the last comes from the $U(2)$ twisted vector multiplet.

The index of $\begin{matrix}(2)\\|\\[3]\end{matrix}\widetilde{\ -\ }\begin{matrix}(1)\\|\\[1]\end{matrix}$ is given by

$$\mathbb{I}^{\text{3D}}_{\begin{matrix}(2)\\|\\[3]\end{matrix}\widetilde{\ -\ }\begin{matrix}(1)\\|\\[1]\end{matrix}}(t,x_\alpha,z_\alpha;q)$$
$$=\frac{1}{2}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}{(q^{\frac{1}{2}}t^2;q)_\infty^2}\sum_{m_1,m_2\in\mathbb{Z}}\oint\frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}\frac{(1-q^{\frac{|m_1-m_2|}{2}}s_1^{\pm}s_2^{\mp})(q^{\frac{1+|m_1-m_2|}{2}}t^{-2}s_1^{\mp}s_2^{\mp};q)_\infty}{(q^{\frac{1+|m_1-m_2|}{2}}t^2s_1^{\mp}s_2^{\mp};q)_\infty}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\sum_{m_3\in\mathbb{Z}}\oint\frac{ds_3}{2\pi i s_3}$$
$$\times\prod_{i=1}^{2}\prod_{\alpha=1}^{3}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}ts_i^{\pm}z_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}z_\alpha^{\pm};q)_\infty}\prod_{i=1}^{2}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_3|}{2}}ts_i^{\pm}s_3^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_3|}{2}}t^{-1}s_i^{\pm}s_3^{\mp};q)_\infty}\cdot\frac{(q^{\frac{3}{4}+\frac{|m_3|}{2}}ts_3^{\pm}z_4^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_3|}{2}}t^{-1}s_3^{\pm}z_4^{\pm};q)_\infty}$$
$$\times q^{\frac{3}{4}|m_1|+\frac{3}{4}|m_2|+\frac{|m_1-m_3|}{4}+\frac{|m_2-m_3|}{4}+\frac{|m_3|}{4}-\frac{|m_1-m_2|}{2}}$$
$$\times t^{3|m_1|+3|m_2|+|m_1-m_3|+|m_2-m_3|+|m_3|-2|m_1-m_2|}$$
$$\times\left(\frac{x_1}{x_2}\right)^{m_1+m_2}\left(\frac{x_2}{x_3}\right)^{m_3}\left(\frac{x_3}{x_1}\right)^{m_1+m_2+m_3}. \tag{4.33}$$

As predicted from mirror symmetry, the associated global symmetry of fugacities x_α and z_α are swapped and the indices (4.29) and (4.33) coincide with each other (see Appendix A 1 c).

D. (1) – (2) – [4]

Consider the 3D $\mathcal{N}=4$ quiver gauge theory $(1)-(2)-[4]$. The quiver diagram and the brane construction are illustrated in Fig. 7.

The dimension of the Coulomb branch is

$$\dim_{\mathbb{C}}\mathcal{M}_C^{(1)-(2)-[4]}=2\cdot(1+2)=6, \tag{4.34}$$

and the dimension of the Higgs branch is

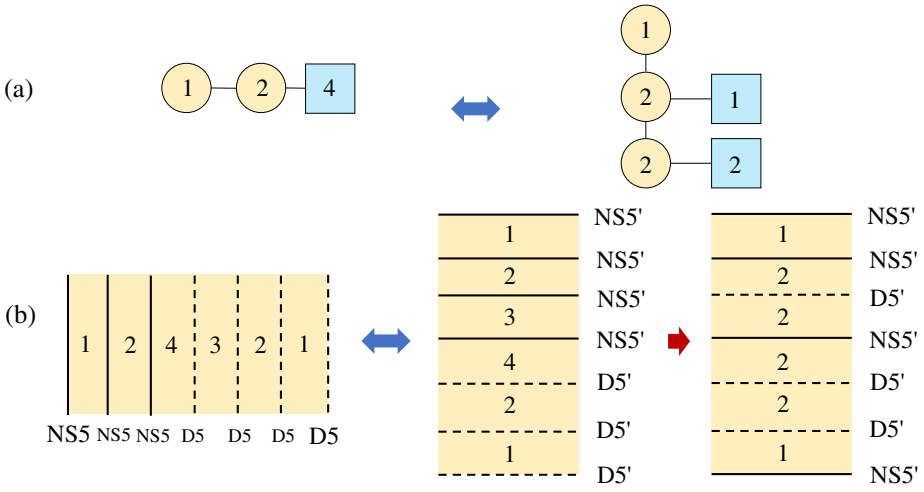

FIG. 7. (a) The quiver diagrams of the $(1)-(2)-[4]$ quiver gauge theory and its mirror theory. (b) The corresponding brane configurations.

$$\dim_{\mathbb{C}}\mathcal{M}_H^{(1)-(2)-[4]} = 2\cdot(1\times 2+2\times 4-1^2-2^2)=10. \tag{4.35}$$

For $(1)-(2)-[4]$, we can label the magnetic fluxes in terms of three integers m_1, m_2, m_3. The R-charge of bare monopole operator is expressed as

$$\Delta(m)=\frac{|m_1-m_2|}{2}+\frac{|m_1-m_3|}{2}+2|m_2|+2|m_3|-|m_2-m_3|, \tag{4.36}$$

where the first two terms are the contributions from bifundamental hypers, the next two terms are those from four fundamental hypers for the $U(2)$ gauge node, and the last is contributed from the $U(2)$ vector multiplet.

The index of $(1)-(2)-[4]$ is

$$\begin{aligned}
&\mathbb{I}^{3\mathrm{D}(1)-(2)-[4]}(t,x_\alpha,z_\alpha;q)\\
&=\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\sum_{m_1\in\mathbb{Z}}\oint\frac{ds_1}{2\pi i s_1}\\
&\times\frac{1}{2}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^2}{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}\sum_{m_2,m_3\in\mathbb{Z}}\oint\frac{ds_2}{2\pi i s_2}\frac{ds_3}{2\pi i s_3}\frac{(1-q^{\frac{|m_2-m_3|}{2}}s_2^{\pm}s_3^{\mp})(q^{\frac{1+|m_2-m_3|}{2}}t^2s_2^{\pm}s_3^{\mp};q)_\infty}{(q^{\frac{1+|m_2-m_3|}{2}}t^{-2}s_2^{\pm}s_3^{\mp};q)_\infty}\\
&\times\prod_{i=2}^{3}\frac{(q^{\frac{3}{4}+\frac{|m_1-m_i|}{2}}t^{-1}s_1^{\pm}s_i^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1-m_i|}{2}}ts_1^{\pm}s_i^{\mp};q)_\infty}\cdot\prod_{i=2}^{3}\prod_{\alpha=1}^{4}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}ts_i^{\pm}x_\alpha^{\pm};q)_\infty}\\
&\times q^{\frac{|m_1-m_2|}{4}+\frac{|m_1-m_3|}{4}+|m_2|+|m_3|-\frac{|m_2-m_3|}{2}}\\
&\times t^{-|m_1-m_2|-|m_1-m_3|-4|m_2|-4|m_3|+2|m_2-m_3|}\\
&\times\left(\frac{z_3}{z_1}\right)^{m_1}\left(\frac{z_1}{z_2}\right)^{m_2+m_3}\left(\frac{z_2}{z_3}\right)^{m_1+m_2+m_3},
\end{aligned} \tag{4.37}$$

where x_α and z_α are the fugacities for flavor symmetry and topological symmetry, respectively.

The mirror of $(1)-(2)-[4]$ is the quiver gauge theory $\begin{matrix}(2)&-&\widetilde{(2)}&-&(1)\\ |&&|&&\\ [2]&&[1]&&\end{matrix}$. The corresponding quiver diagram and brane configuration are shown in Fig. 7.

The dimension of the Coulomb branch is

$$\dim_{\mathbb{C}}\mathcal{M}_C^{\begin{matrix}(2)&-&\widetilde{(2)}&-&(1)\\ |&&|&&\\ [2]&&[1]&&\end{matrix}} = 2\cdot(2+2+1)=10, \tag{4.38}$$

and the dimension of the Higgs branch is

$$\dim_{\mathbb{C}}\mathcal{M}_H^{\begin{matrix}(2)&-&\widetilde{(2)}&-&(1)\\ |&&|&&\\ [2]&&[1]&&\end{matrix}} = 2\cdot(2\times 2+2\times 2+2\times 1+2\times 1-2^2-2^2-1^2)=6. \tag{4.39}$$

The dimension of monopole operator is

$$\Delta(m)=|m_1|+|m_2|+\sum_{i=1}^{2}\sum_{j=3}^{4}\frac{|m_i-m_j|}{2}+\frac{|m_3|+|m_4|}{2}+\sum_{i=3}^{4}\frac{|m_i-m_5|}{2}-\frac{|m_1-m_2|}{2}-\frac{|m_3-m_4|}{2}, \tag{4.40}$$

where the first line includes the contributions from the twisted hyper multiplets, while the second line describes the contributions from two $U(2)$ twisted vector multiplets.

As shown in Appendix A 1 c, we have checked that the index (4.37) beautifully coincides with the index of the mirror quiver gauge theory,

$$
\begin{array}{ccccccc}
 & (2) & - & \widetilde{(2)} & - & (1) & \\
\text{3D} & | & & | & & & \\
\mathbb{I} & [2] & & [1] & & &
\end{array}
(t, x_\alpha, z_\alpha; q)
$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}{(q^{\frac{1}{2}}t^{2};q)_\infty^2}\sum_{m_1,m_2\in\mathbb{Z}}\oint\frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}\frac{(1-q^{\frac{|m_1-m_2|}{2}}s_1^{\pm}s_2^{\mp})(q^{\frac{1+|m_1-m_2|}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q)_\infty}{(q^{\frac{1+|m_1-m_2|}{2}}t^{2}s_2^{\pm}s_3^{\mp};q)_\infty}\\
&\times\frac{1}{2}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^2}{(q^{\frac{1}{2}}t^{2};q)_\infty^2}\sum_{m_3,m_4\in\mathbb{Z}}\oint\frac{ds_3}{2\pi i s_3}\frac{ds_4}{2\pi i s_4}\frac{(1-q^{\frac{|m_1-m_2|}{2}}s_3^{\pm}s_4^{\mp})(q^{\frac{1+|m_1-m_2|}{2}}t^{-2}s_3^{\pm}s_4^{\mp};q)_\infty}{(q^{\frac{1+|m_1-m_2|}{2}}t^{2}s_3^{\pm}s_4^{\mp};q)_\infty}\\
&\times\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty}{(q^{\frac{1}{2}}t^{2};q)_\infty}\sum_{m_5\in\mathbb{Z}}\oint\frac{ds_5}{2\pi i s_5}\\
&\times\prod_{i=1}^{2}\prod_{\alpha=1}^{2}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}ts_i^{\pm}z_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}z_\alpha^{\pm};q)_\infty}\cdot\prod_{i=1}^{2}\prod_{j=3}^{4}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}}ts_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}}t^{-1}s_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times\prod_{i=3}^{4}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}ts_i^{\pm}z_3^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}z_3^{\pm};q)_\infty}\cdot\prod_{i=3}^{4}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_5|}{2}}ts_i^{\pm}s_5^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_5|}{2}}t^{-1}s_i^{\pm}s_5^{\mp};q)_\infty}\\
&\times q^{\frac{|m_1|+|m_2|}{2}+\sum_{i=1}^{2}\sum_{j=3}^{4}\frac{|m_i-m_j|}{4}+\frac{|m_3|+|m_4|}{4}+\sum_{i=3}^{4}\frac{|m_i-m_5|}{2}-\frac{|m_1-m_2|}{2}-\frac{|m_3-m_4|}{2}}\\
&\times t^{2|m_1|+2|m_2|+\sum_{i=1}^{2}\sum_{j=3}^{4}|m_i-m_j|+|m_3|+|m_4|+\sum_{i=3}^{4}|m_i-m_5|-2|m_1-m_2|-2|m_3-m_4|}\\
&\times\left(\frac{x_1}{x_2}\right)^{m_1+m_2}\left(\frac{x_2}{x_3}\right)^{m_3+m_4}\left(\frac{x_3}{x_4}\right)^{m_5}\left(\frac{x_4}{x_1}\right)^{m_1+m_2+m_5},
\end{aligned}
\tag{4.41}
$$

where the fugacities z_α and x_α are associated to the flavor and topological symmetries of the mirror theory, respectively.

V. SEIBERG-LIKE DUALITY

The 3D $\mathcal{N}=4$ $U(N)$ gauge theory with $(2N-1)$ hypermultiplets is expected to be equivalent in the IR to a $U(N-1)$ gauge theory with $(2N-1)$ hypermultiplets times a free twisted hypermultiplet [10]. This is interpreted as Seiberg-like duality between the ugly and good theories. The equality of the Hilbert series for the Coulomb branch is shown in [2] and that for the Hilbert series for the Higgs branch is shown in [28]. Here we would like to check the equalities of 3D full-indices.

We conjecture the identity of indices,

$$
\begin{aligned}
&\mathbb{I}^{\text{3D}(N)-[2N-1]}(t, x_\alpha, z_\alpha; q)\\
&\quad= \mathbb{I}^{\text{3D }(N-1)-[2N-1]}(t, x_\alpha, z_\alpha; q)\times\mathbb{I}^{\text{3D tHM}}\left(t, \frac{z_1}{z_2}; q\right),
\end{aligned}
\tag{5.1}
$$

where

$$
\begin{aligned}
&\mathbb{I}^{\text{3D }(N)-[N_f]}(t, x_\alpha, z_\alpha; q)\\
&= \frac{1}{N!}\frac{(q^{\frac{1}{2}}t^{2};q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\sum_{m_1,\ldots,m_N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{(1-q^{\frac{|m_i-m_j|}{2}}s_i^{\pm}s_j^{\mp})(q^{\frac{1+|m_i-m_j|}{2}}t^{2}s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1+|m_i-m_j|}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times\prod_{i=1}^{N}\prod_{\alpha=1}^{N_f}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}ts_i^{\pm}x_\alpha^{\pm};q)_\infty}\\
&\times q^{\frac{N_f}{4}\sum_{i=1}^{N}|m_i|-\sum_{i<j}\frac{|m_i-m_j|}{2}}\cdot t^{-N_f\sum_{i=1}^{N}|m_i|+2\sum_{i<j}|m_i-m_j|}\cdot\left(\frac{z_1}{z_2}\right)^{\sum_{i=1}^{N}m_i}
\end{aligned}
\tag{5.2}
$$

is the index for 3D $\mathcal{N}=4$ $U(N)$ gauge theory with N_f fundamental hypermultiplets. In fact, we have checked that (5.1) holds for $N=2,3$ up to order q^3 (see Appendix A 1 d for the q-expansions).

VI. DUALITIES OF BOUNDARY CONDITIONS

Making use of the results in Secs. III and IV, we construct the dual half-BPS boundary conditions for 4D $\mathcal{N}=4$ gauge theories by including 3D $\mathcal{N}=4$ gauge theories. In order to check the dualities conjectured from the action of S-duality in string theory, we compute the half-indices which encode the half-BPS boundary conditions for 4D $\mathcal{N}=4$ gauge theories, as well as the full-indices for 3D $\mathcal{N}=4$ gauge theories. The indices for some half-BPS interfaces in 4D $\mathcal{N}=4$ gauge theory were studied in [12,40]. The analysis in this section and in Sec. VII provides more general examples which realize 4D-3D dualities.

A. 4D $U(N)$|3D $U(M)$

Now let us consider the enriched Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ gauge theory which is coupled to 3D $\mathcal{N}=4$ $U(M)$ vector multiplet by the 3D bifundamental hypermultiplet. We denote this boundary condition by 4D $U(N)$|3D $U(M)$. It can be realized in the brane construction as in Fig. 8. There are finite M D3-branes stretched between the NS5-branes and semi-infinite N D3-branes extending along the one side of the NS5-brane in the x^6 direction. The computation of the half-indices

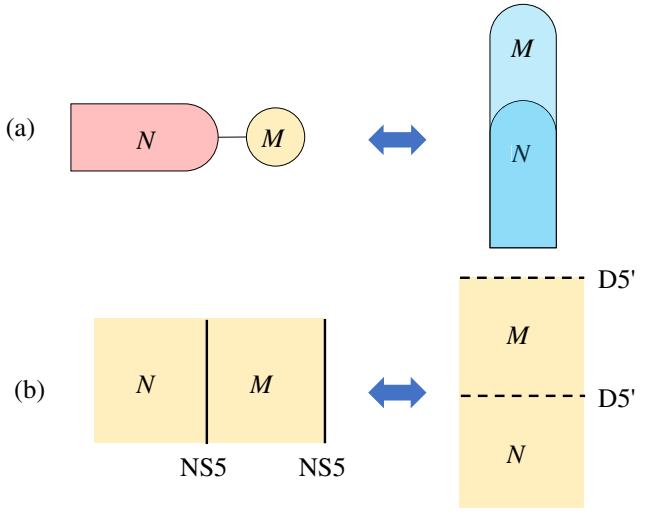

FIG. 8. (a) The quiver diagrams of the enriched Neumann b.c. 4D $U(N)$|3D $U(M)$ and its mirror. (b) The corresponding brane constructions. While the enriched Neumann b.c. 4D $U(N)$|3D $U(M)$ involves 4D gauge and 3D gauge symmetries, its mirror b.c. breaks 4D gauge symmetry.

for the enriched b.c. 4D $U(N)$|3D $U(M)$ involves two sets of contour integral corresponding to the 4D gauge and 3D gauge symmetries. On the other hand, the mirror boundary condition is rather simple as it has no gauge symmetry.

We can compute the half-index for the boundary condition 4D $U(N)$|3D $U(M)$ as

$$
\begin{aligned}
&\mathbb{I}_{\mathcal{N}}^{\text{4D } U(N)|(M)|}(t,z_\alpha;q)\\
&=\underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(\frac{s_i}{s_j};q)_\infty}{(q^{\frac12}t^{-2}\frac{s_i}{s_j};q)_\infty}}_{\mathbb{I}_{\mathcal{N}}^{\text{4D } U(N)}}\\
&\times\frac{1}{M!}\frac{(q^{\frac12}t^2;q)_\infty^M}{(q^{\frac12}t^{-2};q)_\infty^M}\sum_{m_1,\ldots,m_M\in\mathbb{Z}}\oint\prod_{i=N+1}^{N+M}\prod_{i<j}\frac{(1-q^{\frac{|m_{i-N}-m_{j-N}|}{2}}s_i^{\pm}s_j^{\mp})(q^{\frac{1+|m_{i-N}-m_{j-N}|}{2}}t^2s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1+|m_{i-N}-m_{j-N}|}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times\prod_{i=1}^N\prod_{j=N+1}^{N+M}\frac{(q^{\frac34+\frac{|m_{j-N}|}{2}}t^{-1}s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac14+\frac{|m_{j-N}|}{2}}ts_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times q^{\sum_{i=1}^M\frac{N|m_i|}{4}-\sum_{i<j}\frac{|m_i-m_j|}{2}}\cdot t^{-\sum_{i=1}^M N|m_i|+2\sum_{i<j}|m_i-m_j|}\cdot\left(\frac{z_1}{z_2}\right)^{\sum_{i=1}^M m_i}.
\end{aligned}
\tag{6.1}
$$

The contributions in the second line and third line are the half-index for 4D $U(N)$ gauge theory satisfying Neumann b.c. $\mathcal{N}$ and full-index for 3D $U(M)$ gauge theory, respectively. The contributions in the fourth line count the 3D bifundamental hypermultiplet. The last line involves the contribution from the monopole operator whose canonical R-charge is

$$
\Delta(m)=\sum_{i=1}^M\frac{N|m_i|}{2}-\sum_{i<j}|m_i-m_j|. \tag{6.2}
$$

We assume the condition $N\geq 2M-1$, which guarantees that all monopole operators are above the unitarity bound so that the half-indices are convergent.

The S-dual boundary condition 4D $U(N)|[M]|$ is associated with two D5′-branes. The D5′-brane defect would break the 4D gauge group $U(N)$ down to $U(M)$ block-diagonal subgroup. This may lead to the singular boundary condition specified by the Nahm pole of rank $(N-M)$. The another D5′-brane on which M D3-branes terminate requires the singular boundary specified by the Nahm pole of rank M. There would be M different types of contributions to half-index from broken gauge theory characterized by the Nahm pole of rank $(N-M)$, as discussed in [12].

The half-index for the boundary condition 4D $U(N)|[M]|$ takes the form

$$\mathbb{III}^{4\mathrm{D}\ U(N)|[M]|}_{\mathcal{D}'}(t,z_\alpha;q)=\underbrace{\prod_{k=1}^{N-M}\frac{(q^{\frac{k+1}{2}}t^{-2(k-1)};q)_\infty}{(q^{\frac{k}{2}}t^{-2k};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(N-M)}_{\mathrm{Nahm}'}}\cdot\underbrace{\prod_{l=1}^{M}\frac{(q^{\frac{l+1}{2}}t^{-2(l-1)};q)_\infty}{(q^{\frac{l}{2}}t^{-2l};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(M)}_{\mathrm{Nahm}'}}\prod_{l=1}^{M}\frac{(q^{\frac{N-2M}{4}+\frac{l+1}{2}}t^{-(N-2M)-2(l-1)}z_1^{\pm}z_2^{\mp};q)_\infty}{(q^{\frac{N-2M}{4}+\frac{l}{2}}t^{-(N-2M)-2l}z_1^{\pm}z_2^{\mp};q)_\infty}. \tag{6.3}$$

We expect that the half-index (6.1) for the enriched Neumann boundary condition 4D $U(N)|3\mathrm{D}\ U(M)$ is equal to the half-index (6.3) for its dual boundary condition 4D $U(N)|[M]|$. As shown in Appendix A 2 a, we have checked that they coincide for $(N,M)=(1,1)$, (2,1), (3,1), (3,2), (4,2), and (5,2) up to certain orders of q.

B. 4D $U(N)|3\mathrm{D}\ U(N-1)\times\cdots\times U(1)$

Now we present the half-index of the enriched Neumann boundary condition 4D $U(N)|3\mathrm{D}\ U(N-1)\times\cdots\times U(1)$, which is the Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ gauge theory coupled to 3D $\mathcal{N}=4$ $U(N-1)\times U(N-2)\times\cdots\times U(1)$ quiver gauge theory through the 3D hypermultiplet transforming as $(\mathbf{N},\overline{\mathbf{N-1}})\oplus(\overline{\mathbf{N}},\mathbf{N-1})$ under the $U(N)\times U(N-1)$ gauge group. This is expected to be dual to the Dirichlet boundary condition $\mathcal{D}'$ for 4D $\mathcal{N}=4$ $U(N)$ SYM theory [8] (see Fig. 9).

One can compute the half-index of the enriched Neumann boundary condition 4D $U(N)|3\mathrm{D}\ U(N-1)\times\cdots\times U(1)$ as

$$\begin{aligned}
&\mathbb{III}^{4\mathrm{D}\ U(N)|(N-1)-(N-2)-\cdots-(1)}_{\mathcal{N}}(t,z_\alpha;q)\\
&=\underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(\frac{s_i}{s_j};q)_\infty}{(q^{\frac12}t^{-2}\frac{s_i}{s_j};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(N)}_{\mathcal{N}}}\\
&\times\prod_{k=1}^{N-1}\left[\frac{1}{k!}\frac{(q^{\frac12}t^2;q)_\infty^k}{(q^{\frac12}t^{-2};q)_\infty^k}\sum_{m_1^{(k)},\ldots,m_k^{(k)}}\oint\prod_{i=1}^{k}\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}\prod_{i\neq j}\frac{\left(\frac{s_i^{(k)}}{s_j^{(k)}};q\right)_\infty}{\left(q^{\frac12}t^{-2}\frac{s_i^{(k)}}{s_j^{(k)}};q\right)_\infty}\right]\\
&\times\prod_{i=1}^{N}\prod_{j=1}^{N-1}\frac{(q^{\frac34+\frac{|m_j^{(N-1)}|}{2}}t^{-1}s_i^{\pm}s_j^{(N-1)\mp};q)_\infty}{(q^{\frac14+\frac{|m_j^{(N-1)}|}{2}}ts_i^{\pm}s_j^{(N-1)\mp};q)_\infty}\prod_{k=1}^{N-2}\prod_{i=1}^{k}\prod_{j=1}^{k+1}\frac{\left(q^{\frac34+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}t^{-1}s_i^{(k)\pm}s_j^{(k+1)\mp};q\right)_\infty}{\left(q^{\frac14+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}ts_i^{(k)\pm}s_j^{(k+1)\mp};q\right)_\infty}\\
&\times q^{\frac{N}{4}\sum_{i=1}^{N-1}|m_i^{(N-1)}|+\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1}\frac{|m_i^{(k)}-m_j^{(k+1)}|}{4}-\sum_{k=1}^{N-1}\sum_{i<j}\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}\\
&\times t^{-N\sum_{i=1}^{N-1}|m_i^{(N-1)}|-\sum_{k=1}^{N-2}\sum_{i=1}^{k}\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|+2\sum_{k=1}^{N-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|}\\
&\times\prod_{k=1}^{N-1}\left(\frac{z_k}{z_{k+1}}\right)^{\sum_{i=1}^{k}m_i^{(k)}}\cdot\left(\frac{z_N}{z_1}\right)^{m_1^{(1)}+\sum_{i=1}^{N-1}m_i^{(N-1)}}.
\end{aligned}\tag{6.4}$$

The half-index (6.4) would agree with the half-index of Dirichlet boundary $\mathcal{D}'$ for 4D $\mathcal{N}=4$ $U(N)$ gauge theory

$$\mathbb{III}^{4\mathrm{D}\ U(N)}_{\mathcal{D}'}(t,z_\alpha;q)=\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\prod_{i\neq j}\frac{(q\frac{z_i}{z_j};q)_\infty}{(q^{\frac12}t^{-2}\frac{z_i}{z_j};q)_\infty}. \tag{6.5}$$

In fact, we have checked that they coincide with each other for $N=1,2$ up to certain orders of q by using Mathematica (see Appendix A 2 a).

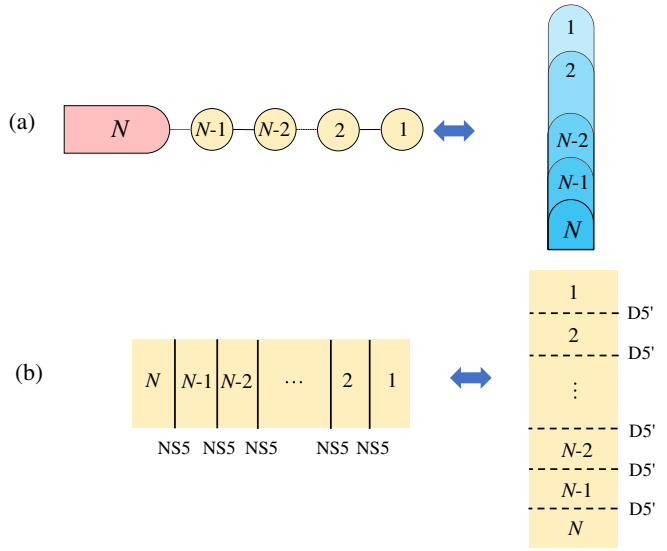

FIG. 9. (a) The quiver diagrams of the enriched Neumann b.c. 4D $U(N)|$3D $U(N-1)\times U(N-2)\times\cdots\times U(1)$ and its mirror, that is Dirichlet b.c. $\mathcal{D}'$ for 4D $U(N)$ gauge theory. (b) The corresponding brane constructions.

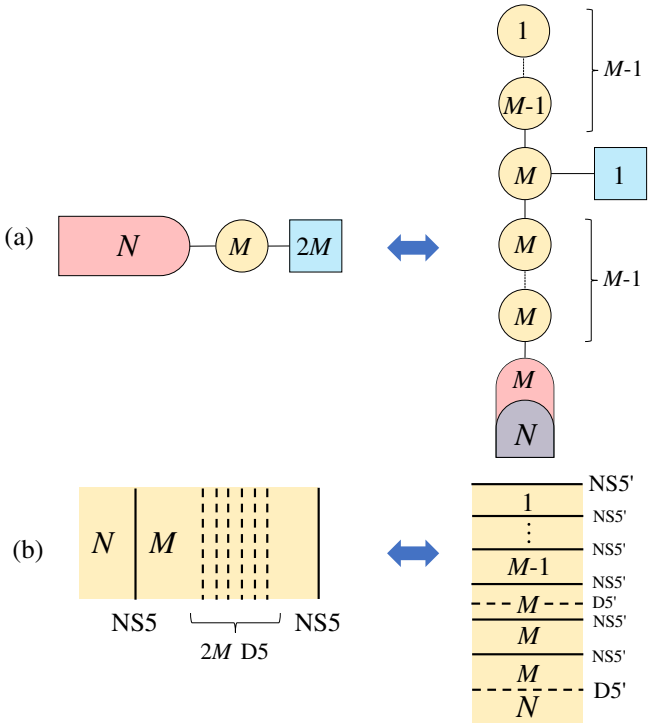

FIG. 10. (a) The quiver diagrams of the enriched Neumann b.c. 4D $U(N)|$3D $(M)-[2M]$ and its mirror. (b) The corresponding brane constructions.

C. 4D $U(N)|$3D $(M)-[2M]$

Let us consider the enriched Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ gauge theory coupled to balanced 3D $\mathcal{N}=4$ $U(M)$ gauge theory with $2M$ hypermultiplets. We represent this boundary condition by 4D $U(N)|$3D $(M)-[2M]$. The corresponding quiver diagram and brane construction are shown in Fig. 10. In contrast to the enriched boundary conditions in Sec. VI A, the dual boundary conditions can admit both 4D and 3D gauge symmetries. The dualities of boundary conditions produce a generalization of mirror symmetry for $(N)-[2N]$ discussed in Sec. IV A.

The enriched Neumann b.c. 4D $U(N)|$3D $(M)-[2M]$ consists of the Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ gauge theory coupled to 3D $\mathcal{N}=4$ $U(M)$ balanced gauge theory with $2M$ fundamental hypermultiplets via 3D boundary hypermultiplets transforming as $(\mathbf{N},\overline{\mathbf{M}})\oplus(\overline{\mathbf{N}},\mathbf{M})$ under the $U(N)\times U(M)$ gauge symmetry. Here we assume that $N\geq M$.

The half-index for the enriched Neumann b.c. 4D $U(N)|$3D $(M)-[2M]$ reads

$$
\begin{aligned}
&\mathbb{III}_{\mathcal{N}}^{\text{4D } U(N)|(M)-[2M]}(t,x_\alpha,z_\alpha;q)\\
&=\underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\prod_{i=1}^{N}\oint\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac12}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{III}_{\mathcal{N}}^{\text{4D } U(N)}}\\
&\times\frac{1}{M!}\frac{(q^{\frac12}t^2;q)_\infty^M}{(q^{\frac12}t^{-2};q)_\infty^M}\sum_{m_1,\ldots,m_M\in\mathbb{Z}}\oint\prod_{i=N+1}^{N+M}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(1-q^{\frac{|m_i-m_j|}{2}}s_i^{\pm}s_j^{\mp})(q^{\frac{1+|m_i-m_j|}{2}}t^2s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1+|m_i-m_j|}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q)_\infty}\\
&\times\prod_{i=1}^{N}\prod_{j=N+1}^{N+M}\frac{(q^{\frac34+\frac{|m_j-N|}{2}}t^{-1}s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac14+\frac{|m_j-N|}{2}}ts_i^{\pm}s_j^{\mp};q)_\infty}\cdot\prod_{i=N+1}^{N+M}\prod_{\alpha=1}^{2M}\frac{(q^{\frac34+\frac{|m_i-N|}{2}}t^{-1}s_i^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac14+\frac{|m_i-N|}{2}}ts_i^{\pm}x_\alpha^{\pm};q)_\infty}\\
&\times q^{\frac{N+2M}{4}\sum_{i=1}^{M}|m_i|-\sum_{i<j}\frac{|m_i-m_j|}{2}}\cdot t^{-(N+2M)\sum_{i=1}^{M}|m_i|+2\sum_{i<j}|m_i-m_j|}\cdot\left(\frac{z_1}{z_2}\right)^{\sum_{i=1}^{M}m_i}.
\end{aligned}
\tag{6.6}
$$

The second and third lines describe the half-index of Neumann b.c. $\mathcal{N}$ for 4D $U(N)$ gauge theory and the full-index of 3D $U(M)$ vector multiplet, respectively. The fourth line counts the boundary bifundamental hypermultiplet and the fundamental hypermultiplet in $(M)-[2M]$. The last line counts the monopole operator with the canonical R-charge

$$\Delta(m)=\frac{N+2M}{2}\sum_{i=1}^{M}|m_i|-\sum_{i<j}|m_i-m_j|. \quad (6.7)$$

This is generalized from the formula (4.3) by including the additional contributions from the boundary bifundamental hypermultiplet coupled to 4D $U(N)$ SYM theory.

The S-dual boundary condition can be read off from the brane configuration in Fig. 10. The 4D gauge symmetry must break down to $U(M)$ due to the presence of the D5′-brane interface. When $N=M$, there is a defect 3D twisted hypermultiplet at the D5′-brane interface transforming under fundamental representation under the $U(N)$ gauge group. For $N>M$, there is no fundamental twisted hypermultiplet at the defect; however, there are contributions to the half-index from the broken $U(N)$ gauge theory associated with the Nahm pole of rank $(N-M)$ [12].

The surviving 4D $U(M)$ gauge theory should obey the Neumann b.c. $\mathcal{N}'$ due to the NS5′-brane. It further couples to 3D twisted quiver gauge theory through the 3D bifundamental twisted hypermultiplet. The 3D twisted quiver gauge theory has $U(M)^M\times U(M-1)\times U(M-2)\times U(1)$ gauge symmetry and a single $U(1)$ flavor node for the Mth $U(M)$ gauge node, which we denote by

$$\begin{array}{ccccccccccc}(M)^{M-1} & - & (M) & - & (\widetilde{M-1}) & - & (M-2) & - & \cdots & - & (1)\\ & & | & & & & & & & & \\ & & [1] & & & & & & & & \end{array}.$$

We expect that the half-index (6.6) agrees with

$$\begin{array}{ccccccccccc}(M)^{M-1} & - & (M) & - & (\widetilde{M-1}) & - & (M-2) & - & \cdots & - & (1)\\ \text{4D } U(N)\to U(M)| & & | & & & & & & & & \\ \mathbb{III}_{\mathcal{N}'} & & [1] & & & & & & & & (t,x_\alpha,z_\alpha;q)\end{array}$$

$$=\underbrace{\frac{1}{M!}\frac{(q)_\infty^M}{(q^{\frac12}t^2;q)_\infty^M}\oint\prod_{i=1}^{M}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(\frac{s_i}{s_j};q)_\infty}{(q^{\frac12}t^2\frac{s_i}{s_j};q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{\text{4D }U(M)}}\cdot\underbrace{\prod_{k=1}^{N-M}\frac{(q^{\frac{k+1}{2}}t^{-2(k-1)};q)_\infty}{(q^{\frac{k}{2}}t^{-2k};q)_\infty}}_{\mathbb{III}_{\text{Nahm}'}^{\text{4D }U(N-M)}}$$

$$\times\left(\frac{1}{M!}\frac{(q^{\frac12}t^{-2};q)_\infty^M}{(q^{\frac12}t^2;q)_\infty^M}\right)^M\prod_{k=1}^{M}\Bigg[\sum_{m_1^{(k)},\ldots,m_M^{(k)}\in\mathbb{Z}}\oint\prod_{i=1}^{M}\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}$$

$$\times\prod_{i<j}\frac{(1-q^{\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}s_i^{(k)\pm}s_j^{(k)\mp})(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{-2}s_i^{(k)\pm}s_j^{(k)\mp};q)_\infty}{(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}ts_i^{(k)\pm}s_j^{(k)\mp};q)_\infty}\Bigg]$$

$$\times\prod_{k=M+1}^{2M-1}\Bigg[\frac{1}{(2M-k)!}\frac{(q^{\frac12}t^{-2};q)_\infty^{2M-k}}{(q^{\frac12}t^2;q)_\infty^{2M-k}}\sum_{m_1^{(k)},\ldots,m_{2M-k}^{(k)}\in\mathbb{Z}}\oint\prod_{i=1}^{2M-k}\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}$$

$$\times\prod_{i<j}\frac{(1-q^{\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}s_i^{(k)\pm}s_j^{(k)\mp})(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{-2}s_i^{(k)\pm}s_j^{(k)\mp};q)_\infty}{(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}ts_i^{(k)\pm}s_j^{(k)\mp};q)_\infty}\Bigg]$$

$$\times\prod_{i=1}^{M}\frac{(q^{\frac34+\frac{N-M}{4}}t^{1-(N-M)}s_i^{\pm}z_1^{\pm};q)_\infty}{(q^{\frac14+\frac{N-N_c}{4}}t^{-1-(N-M)}s_i^{\pm}z_1^{\pm};q)_\infty}\cdot\prod_{i=1}^{M}\prod_{j=1}^{M}\frac{(q^{\frac34+\frac{|m_j^{(1)}|}{2}}ts_i^{\pm}s_j^{(1)\mp};q)_\infty}{(q^{\frac14+\frac{|m_j^{(1)}|}{2}}t^{-1}s_i^{\pm}s_j^{(1)\mp};q)_\infty}$$

$$\times\prod_{k=1}^{2M-2}\prod_i\prod_j\frac{(q^{\frac34+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}ts_i^{(k)\pm}s_j^{(k+1)\mp};q)_\infty}{(q^{\frac14+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}t^{-1}s_i^{(k)\pm}s_j^{(k+1)\mp};q)_\infty}\cdot\prod_{i=1}^{M}\frac{(q^{\frac34+\frac{|m_i^{(M)}|}{2}}ts_i^{(M)\pm}z_2^{\pm};q)_\infty}{(q^{\frac14+\frac{|m_i^{(M)}|}{2}}t^{-1}s_i^{(M)\pm}z_2^{\pm};q)_\infty}$$

$$\times q^{\frac{M}{4}\sum_{i=1}^{M}|m_i^{(1)}|+\sum_{k=1}^{2M-2}\sum_i\sum_j\frac{|m_i^{(k)}-m_j^{(k+1)}|}{4}+\frac14\sum_{i=1}^{M}|m_i^{(M)}|-\sum_{k=1}^{2M-1}\sum_{i<j}\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}$$

$$\times t^{M\sum_{i=1}^{M}|m_i^{(1)}|+\sum_{k=1}^{2M-2}\sum_i\sum_j|m_i^{(k)}-m_j^{(k+1)}|+\sum_{i=1}^{M}|m_i^{(M)}|-2\sum_{k=1}^{2M-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|}$$
$$\times\prod_{k=1}^{N_c}\left(\frac{x_k}{x_{k+1}}\right)^{\sum_{i=1}^{M}m_i^{(k)}}\cdot\sum_{k=M+1}^{2M-1}\left(\frac{x_k}{x_{k+1}}\right)^{\sum_{i=1}^{2M-k}m_i^{(k)}}\cdot\left(\frac{x_{2M}}{x_1}\right)^{\sum_{i=1}^{M}m_i^{(1)}+m_1^{(2M-1)}}. \tag{6.8}$$

The second line includes the half-index of Nahm′ boundary condition for 4D $U(N-M)$ gauge theory corresponding to the defect of D5′-brane and the half-index of Neumann b.c. $\mathcal{N}'$ for 4D $U(N)$ gauge theory corresponding to the NS5′-brane on which semi-infinite N D3-branes terminate. The contributions from third to sixth line are the full-index of 3D twisted vector multiplets. The first factors in the seventh line describe the defect twisted hypermultiplet for $N=M$ and the local operators appearing from the broken $U(N)$ gauge theory for $N>M$. The remaining contributions are the 3D twisted hypers and monopole operator with conformal dimension

$$\Delta(m)=\frac{M}{2}\sum_{i=1}^{M}|m_i^{(1)}|+\sum_{k=1}^{2M-2}\sum_i\sum_j\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}+\frac{1}{2}\sum_{i=1}^{M}|m_i^{(M)}|-\sum_{k=1}^{2M-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|, \tag{6.9}$$

where the first terms are the contributions from the bifundamental twisted hyper coupled to 4D $U(M)$ and 3D $U((M)$ gauge symmetries, the second and third are the contributions from the bifundamental twisted hyper in the 3D quiver gauge theory, and the last terms are the contributions from the 3D twisted vector multiplets.

As shown in Appendix A 2 a, we have checked that the indices (6.6) and (6.8) agree with each others for $(N,M)=(1,1)$, (2,1), (2,2), and (3,2) up to certain orders of q.

D. 4D $U(N)|T[SU(M)]$

Let us study the enriched Neumann b.c. 4D $U(N)|T[SU(M)]$ for 4D $U(N)$ SYM theory. The corresponding quiver diagram and brane configuration are depicted in Fig. 11. We have already examined the case with $M=2$ in Sec. VI C. Although $T[SU(N)]$ is self-mirror, the enriched boundary condition 4D $U(N)|T[SU(M)]$ is not self-mirror. We further check the dualities for $M=3$ and propose the generalization.

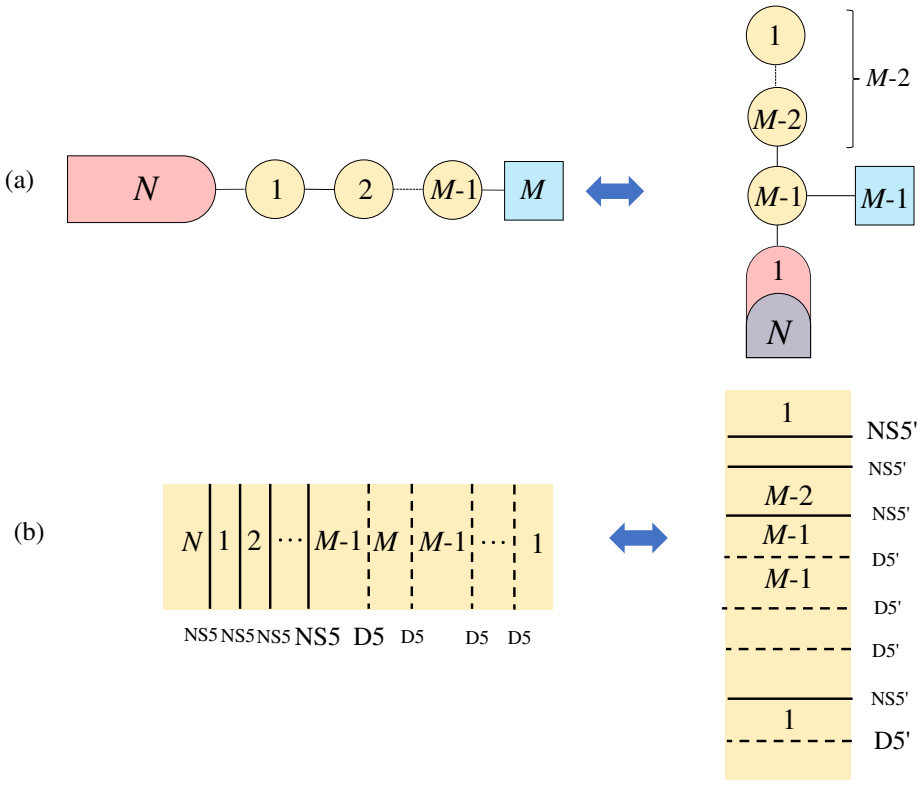

FIG. 11. (a) The quiver diagrams of the enriched Neumann b.c. 4D $U(N)|T[SU(M)]$ and its mirror. (b) The corresponding brane constructions.

The enriched Neumann boundary condition 4D $U(N)|T[SU(M)]$ is constructed from the brane setup in Fig. 11, which is the Neumann b.c. $\mathcal{N}$ for 4D $U(N)$ SYM theory coupled to $T[SU(M)]$ via the boundary 3D hypermultiplet transforming as $(\mathbf{N},-)\oplus(\overline{\mathbf{N}},+)$ under the 4D $U(N)$ gauge and 3D $U(1)$ gauge symmetries.

We can write the half-index as

$$
\begin{aligned}
&\mathbb{I}_{\mathcal{N}}^{\text{4D } U(N)|T[SU(M)]}(t,x_\alpha,z_\alpha;q)\\
&=\underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac12}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{I}_{\mathcal{N}}^{\text{4D } U(N)}}\\
&\times\prod_{k=1}^{M-1}\left[\frac{1}{k!}\frac{(q^{\frac12}t^2;q)_\infty^k}{(q^{\frac12}t^{-2};q)_\infty^k}\sum_{m_1^{(k)},\ldots,m_k^{(k)}\in\mathbb{Z}}\oint\prod_{i=1}^k\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}\right.\\
&\left.\times\prod_{i<j}\frac{\left(1-q^{\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}s_i^{(k)\pm}s_j^{(k)\mp}\right)\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^2s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}{\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{-2}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}\right]\\
&\times\prod_{i=1}^N\frac{(q^{\frac34+\frac{|m_1^{(1)}|}{2}}t^{-1}s_i^\pm s_j^\mp;q)_\infty}{\left(q^{\frac14+\frac{|m_1^{(1)}|}{2}}ts_i^\pm s_j^\mp;q\right)_\infty}\cdot\prod_{k=1}^{M-2}\prod_{i=1}^k\prod_{j=1}^{k+1}\frac{\left(q^{\frac34+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}t^{-1}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}{\left(q^{\frac14+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}ts_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}\\
&\times\prod_{i=1}^{M-1}\prod_{\alpha=1}^M\frac{\left(q^{\frac34+\frac{|m_i^{(M-1)}|}{2}}t^{-1}s_i^{(M-1)\pm}x_\alpha^\pm;q\right)_\infty}{\left(q^{\frac14+\frac{|m_i^{(M-1)}|}{2}}ts_i^{(M-1)\pm}x_\alpha^\pm;q\right)_\infty}\\
&\times q^{\frac{N}{4}|m_1^{(1)}|+\frac14\sum_{k=1}^{M-2}\sum_{i=1}^k\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|+\frac{M}{4}\sum_{i=1}^{M-1}|m_i^{(M-1)}|-\sum_{k=1}^{M-1}\sum_{i<j}\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}\\
&\times t^{-N|m_1^{(1)}|-\sum_{k=1}^{M-2}\sum_{i=1}^k\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|-M\sum_{i=1}^{M-1}|m_i^{(M-1)}|+2\sum_{k=1}^{M-1}|m_i^{(k)}-m_j^{(k)}|}\\
&\times\prod_{k=1}^{M-1}\left(\frac{z_k}{z_{k+1}}\right)^{\sum_{i=1}^k m_i^{(k)}}\cdot\left(\frac{z_M}{z_1}\right)^{m_1^{(1)}+\sum_{i=1}^{M-1}m_i^{(M-1)}}.
\end{aligned}
\tag{6.10}
$$

The terms appearing from the second to fourth line describe the half-index of Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ SYM theory and the full-index for 3D $U(1)\times U(2)\times\cdots\times U(M-1)$ vector multiplet. The associated magnetic fluxes are labeled by $\frac{M(M-1)}{2}$ integers $\{m_i^{(k)}\}_{i=1,\ldots,k}$ where $k=1,\ldots,M-1$. The contributions in the fifth and sixth lines describe 3D $\mathcal{N}=4$ hypermultiplets. The other terms count the bare monopole with the R-charge

$$
\begin{aligned}
\Delta(m)=&\frac{N}{2}|m_1^{(1)}|+\frac12\sum_{k=1}^{M-2}\sum_{i=1}^k\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|\\
&+\frac{M}{2}\sum_{i=1}^{M-1}|m_i^{(M-1)}|-\sum_{k=1}^{M-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|.
\end{aligned}
\tag{6.11}
$$

This formula is generalized from (4.18) by the additional contributions appearing in the first terms. They come from the boundary hypermultiplet transforming as $(\mathbf{N},-)\oplus(\bar{\mathbf{N}},+)$ under the 4D $U(N)$ gauge and 3D $U(1)$ gauge symmetries.

The dual quiver diagram and the corresponding brane setup are illustrated in Fig. 11. It is identified with the boundary condition for 4D $U(N)$ gauge theory including the Nahm$'$ pole of rank $(N-1)$ that breaks the 4D gauge symmetry down to $U(1)$. While for $N=1$, one finds a defect twisted hypermultiplet arising from D3-D5$'$ string; for $N>1$, the half-index receives contributions from broken gauge theory as discussed in [12].

In addition, the surviving $U(1)$ gauge theory should satisfy the enriched Neumann b.c. $\mathcal{N}'$ corresponding to NS5$'$-brane with a coupling to 3D twisted quiver gauge theory $\begin{array}{c}(M-1)-\widetilde{(M-2)}-\cdots-(1)\\ |\\ [M-1]\end{array}$ through the 3D boundary twisted hypermultiplet.

We find the half-index

$$
\begin{array}{ccccccccc}
 & & \widetilde{(M-1)} & - & (M-2) & - & \cdots & - & (1) \\
{\scriptstyle 4\mathrm{D}\ U(N)\to U(1)|} & & | & & & & & & \\
\mathbb{III}_{\mathcal{N}'} & & [M-1] & & & & & & (t,x_\alpha,z_\alpha;q)
\end{array}
$$

$$
\begin{aligned}
&= \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\oint\frac{ds_1}{2\pi i s_1}}_{\mathbb{III}_{\mathcal{N}'}^{4\mathrm{D}\ U(1)}}\cdot\underbrace{\prod_{l=1}^{N-1}\frac{(q^{\frac{l+1}{2}}t^{-2(l-1)};q)_\infty}{(q^{\frac{l}{2}}t^{-2l};q)_\infty}}_{\mathbb{III}_{\mathrm{Nahm}'}^{4\mathrm{D}\ U(N-1)}}\\
&\times\prod_{k=1}^{M-1}\left[\frac{1}{k!}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^k}{(q^{\frac{1}{2}}t^2;q)_\infty^k}\sum_{m_1^{(k)},\ldots,m_k^{(k)}\in\mathbb{Z}}\oint\prod_{i=1}^k\frac{ds_i^{(k)}}{2\pi i s_i^{(k)}}\right.\\
&\left.\times\prod_{i<j}\frac{\left(1-q^{\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}s_i^{(k)\pm}s_j^{(k)\mp}\right)\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^{-2}s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}{\left(q^{\frac{1+|m_i^{(k)}-m_j^{(k)}|}{2}}t^2s_i^{(k)\pm}s_j^{(k)\mp};q\right)_\infty}\right]\\
&\times\frac{(q^{\frac{3}{4}+\frac{N-1}{4}}t^{1-(N-1)}s_1^\pm z_M^\pm;q)_\infty}{(q^{\frac{1}{4}+\frac{N-1}{4}}t^{-1-(N-1)}s_1^\pm z_M^\pm;q)_\infty}\cdot\prod_{i=1}^{M-1}\frac{\left(q^{\frac{3}{4}+\frac{|m_i^{(M-1)}|}{2}}ts_1^\pm s_i^{(M-1)\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(M-1)}|}{2}}t^{-1}s_1^\pm s_i^{(M-1)\mp};q\right)_\infty}\\
&\times\prod_{i=1}^{M-1}\prod_{\alpha=1}^{M-1}\frac{\left(q^{\frac{3}{4}+\frac{|m_i^{(M-1)}|}{2}}ts_i^{(M-1)\pm}z_\alpha^\pm;q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(M-1)}|}{2}}t^{-1}s_i^{(M-1)\pm}z_\alpha^\pm;q\right)_\infty}\cdot\prod_{k=1}^{M-2}\prod_{i=1}^{k}\prod_{j=1}^{k+1}\frac{\left(q^{\frac{3}{4}+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}ts_i^{(k)\pm}s_j^{(k+1)\mp};q\right)_\infty}{\left(q^{\frac{1}{4}+\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}}t^{-1}s_i^{(k)\pm}s_j^{(k+1)\mp};q\right)_\infty}\\
&\times q^{\frac{M}{4}\sum_{i=1}^M|m_i^{(M-1)}|+\sum_{k=1}^{M-2}\sum_{i=1}^k\sum_{j=1}^{k+1}\frac{|m_i^{(k)}-m_j^{(k+1)}|}{4}-\sum_{k=1}^{M-1}\sum_{i<j}\frac{|m_i^{(k)}-m_j^{(k)}|}{2}}\\
&\times t^{M\sum_{i=1}^{M-1}|m_i^{(M-1)}|+\sum_{k=1}^{M-2}\sum_{i=1}^k\sum_{j=1}^{k+1}|m_i^{(k)}-m_j^{(k+1)}|-2\sum_{k=1}^{M-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|}\\
&\times\prod_{k=1}^{M-1}\left(\frac{x_k}{x_{k+1}}\right)^{\sum_{i=1}^k m_i^{(k)}}\cdot\left(\frac{x_M}{x_1}\right)^{m_1^{(1)}+\sum_{i=1}^{M-1}m_i^{(M-1)}}. \qquad (6.12)
\end{aligned}
$$

The contributions in the second line are the half-index of Nahm′ b.c. for 4D $U(N-1)$ gauge theory and the half-index of Neumann b.c. for 4D $U(1)$ gauge theory. The next two lines are the full-index of 3D $U(M-1)\times U(M-2)\times\cdots\times U(1)$ twisted vector multiplet. The first terms in the fifth line correspond to the defect hypermultiplet for $N=1$ or the extra fields appearing from the broken gauge theory [12]. The remaining terms count 3D $\mathcal{N}=4$ twisted hypers and monopole operator of dimension

$$
\begin{aligned}
\Delta(m)=&\frac{M}{2}\sum_{i=1}^M|m_i^{(M-1)}|+\sum_{k=1}^{M-2}\sum_{i=1}^k\sum_{j=1}^{k+1}\frac{|m_i^{(k)}-m_j^{(k+1)}|}{2}\\
&-\sum_{k=1}^{M-1}\sum_{i<j}|m_i^{(k)}-m_j^{(k)}|. \qquad (6.13)
\end{aligned}
$$

We conjecture that the half-indices (6.10) and (6.12) give the same result. In fact, we have found that for $(N,M)=(1,3)$ and (2,3), they agree with each other up to certain orders of q (see Appendix A 2 a).

VII. DUALITIES OF INTERFACES

In this section, we would like to study the dualities of interfaces for a pair of 4D $\mathcal{N}=4$ gauge theories including additional 3D $\mathcal{N}=4$ gauge theories.

A. 4D $U(N)|$3D $U(N)^{k-1}|$4D $U(N)$

We study the interface for a pair of two 4D $\mathcal{N}=4$ $U(N)$ gauge theories which involves 3D $\mathcal{N}=4$ $U(N)^{k-1}$ quiver gauge theory. We denote this by 4D $U(N)|$3D $U(N)^{k-1}|$4D $U(N)$. The corresponding quiver diagram and brane configuration are given in Fig. 12. It is realized by N D3-branes intersecting with k NS5-branes. The dual configuration is realized by N D3-branes intersecting with k D5′-branes. According to the presence of D5′

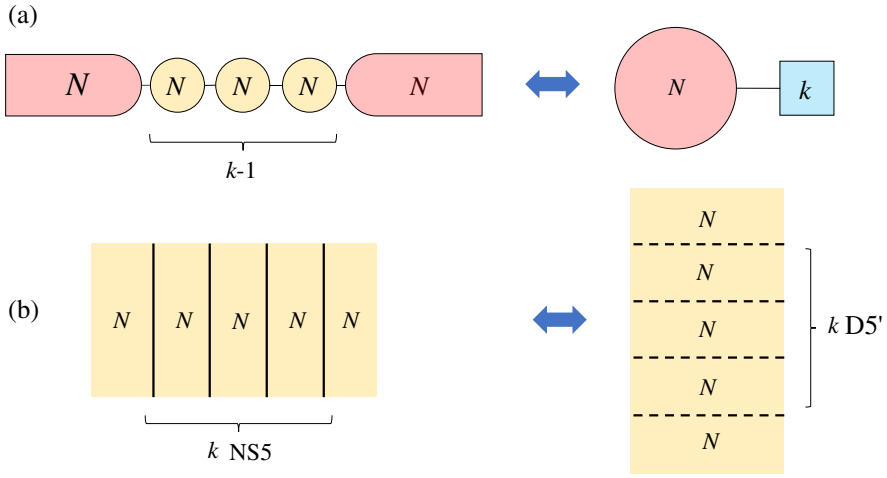

FIG. 12. (a) The quiver diagrams of the 4D $U(N)|$3D $U(N)^{k-1}|$4D $U(N)$ interface and its mirror. (b) The corresponding brane configurations.

interface, the 4D $U(N)^{k-1}$ gauge symmetry is broken down to a diagonal $U(N)$ so that four-dimensional $U(N)$ gauge fields couple to 3D $\mathcal{N}=4$ twisted hypermultiplets. The analysis for $k=1$ where the interface has no 3D gauge theory is already given in [12]. We present more general results and check that the half-indices for the dual interfaces agree with each other.

The interface 4D $U(N)|$3D $U(N)^{k-1}|$4D $U(N)$ has a pair of 4D $U(N)$ SYM theories obeying Neumann b.c. $\mathcal{N}$ corresponding to the NS5-branes. Both of them are coupled to 3D $U(N)^{k-1}$ vector multiplet at the interface through the 3D bifundamental hypermultiplets. Let us label the magnetic fluxes by $k-1$ sets of N-tuple of integers, i.e., $N(k-1)$ integers $\{m_i^{(l)}\}_{i=1,\ldots,N}$ with $l=1,\ldots,k-1$.

The half-index takes the form

$$
\begin{aligned}
&\mathbb{III}^{\text{4D } U(N)|(N)^k|\text{4D } U(N)}_{\mathcal{N}\mathcal{N}}(t,z_\alpha;q)\\
&=\underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(\frac{s_i}{s_j};q)_\infty}{(q^{\frac12}t^{-2}\frac{s_i}{s_j};q)_\infty}}_{\mathbb{III}^{\text{4D } U(1)}_{\mathcal{N}}}\\
&\times\left[\frac{1}{N!}\frac{(q^{\frac12}t^2;q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\right]^{k-1}\prod_{l=1}^{k-1}\sum_{m_1^{(l)},\ldots,m_N^{(l)}}\oint\prod_{i=1}^N\frac{ds_i^{(l)}}{2\pi i s_i^{(l)}}\\
&\times\frac{(1-q^{\frac{|m_i^{(l)}-m_j^{(l)}|}{2}}s_i^{(l)\pm}s_j^{(l)\mp})(q^{\frac{1+|m_i^{(l)}-m_j^{(l)}|}{2}}t^2s_i^{(l)\pm}s_j^{(l)\mp};q)_\infty}{(q^{\frac{1+|m_i^{(l)}-m_j^{(l)}|}{2}}t^{-2}s_i^{(l)\pm}s_j^{(l)\mp};q)_\infty}\\
&\times\underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac12}t^{-2};q)_\infty^N}\oint\prod_{i=N+1}^{2N}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(\frac{s_i}{s_j};q)_\infty}{(q^{\frac12}t^{-2}\frac{s_i}{s_j};q)_\infty}}_{\mathbb{III}^{\text{4D } U(1)}_{\mathcal{N}}}\\
&\times\prod_{i=1}^N\prod_{j=1}^N\frac{(q^{\frac34+\frac{|m_i^{(1)}|}{2}}t^{-1}s_i^{\pm}s_j^{(1)\mp};q)_\infty}{(q^{\frac14+\frac{|m_i^{(1)}|}{2}}ts_i^{\pm}s_j^{(1)\mp};q)_\infty}\cdot\prod_{l=1}^{k-2}\prod_{i=1}^N\prod_{j=1}^N\frac{(q^{\frac34+\frac{|m_i^{(l)}-m_j^{(l+1)}|}{2}}t^{-1}s_i^{(l)\pm}s_j^{(l+1)\mp};q)_\infty}{(q^{\frac14+\frac{|m_i^{(l)}-m_j^{(l+1)}|}{2}}ts_i^{(l)\pm}s_j^{(l+1)\mp};q)_\infty}
\end{aligned}
$$

$$\times \prod_{i=1}^{N}\prod_{j=N+1}^{2N}\frac{(q^{\frac{3}{4}+\frac{|m_i^{(k-1)}|}{2}}t^{-1}s_i^{(k-1)\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i^{(k-1)}|}{2}}ts_i^{(k-1)\pm}s_j^{\mp};q)_\infty}$$
$$\times q^{\sum_{i=1}^{N}\frac{N|m_i^{(1)}|}{4}+\sum_{l=1}^{k-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{|m_i^{(l)}-m_j^{(l+1)}|}{4}+\sum_{i=1}^{N}\frac{N|m_i^{(k-1)}|}{4}-\sum_{l=1}^{k-1}\sum_{i<j}\frac{|m_i^{(l)}-m_j^{(l)}|}{2}}$$
$$\times t^{-\sum_{i=1}^{N}N|m_i^{(1)}|-\sum_{l=1}^{k-2}\sum_{i=1}^{N}\sum_{j=1}^{N}|m_i^{(l)}-m_j^{(l+1)}|-\sum_{i=1}^{N}N|m_i^{(k-1)}|+2\sum_{l=1}^{k-1}\sum_{i<j}|m_i^{(l)}-m_j^{(l)}|}$$
$$\times \prod_{\alpha=1}^{k-1}\left(\frac{z_\alpha}{z_{\alpha+1}}\right)^{\sum_{i=1}^{N}m_i^{(\alpha)}}\cdot\left(\frac{z_M}{z_1}\right)^{\sum_{i=1}^{N}m_i^{(1)}+\sum_{j=1}^{N}m_j^{(k-1)}}. \tag{7.1}$$

The terms appearing from the second to fifth line are the two half-indices of Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ SYM theory and the full-index of 3D $\mathcal{N}=4$ $U(N)^{k-1}$ vector multiplet. The terms in the sixth and seventh lines are the contributions from 3D $\mathcal{N}=4$ hypermultiplets. The remaining terms describe the monopole contributions whose canonical R-charge is

$$\Delta(m)=\sum_{i=1}^{N}\frac{N|m_i^{(1)}|}{2}+\sum_{l=1}^{k-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{|m_i^{(l)}-m_j^{(l+1)}|}{2}$$
$$+\sum_{i=1}^{N}\frac{N|m_i^{(k-1)}|}{2}-\sum_{l=1}^{k-1}\sum_{i<j}|m_i^{(l)}-m_j^{(l)}|, \tag{7.2}$$

where the first and third terms are contributed from bifundamental hypers coupled to 4D and 3D gauge theories, the second terms are the contributions from 3D bifundamental hypers between gauge nodes in 3D quiver gauge theory, and the last terms are contributed from 3D $U(N)^{k-1}$ vector multiplet.

Under S-duality, we find the dual interface which is identified with a 4D $\mathcal{N}=4$ $U(N)$ SYM theory with k defects corresponding to k D5′-branes, which couple the 4D $U(N)$ gauge theory to k 3D $\mathcal{N}=4$ fundamental twisted hypermultiplets.

The half-index for the dual interface should be computed as

$$\mathbb{III}^{\text{4D }U(N)+k\text{ hypers}}_{\mathcal{D}',\ldots,\mathcal{D}'}(t,z_\alpha;q)=\underbrace{\frac{1}{N!}\frac{(q)_\infty^{2N}}{(q^{\frac{1}{2}}t^2;q)_\infty^N(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{(\frac{s_i}{s_j};q)_\infty(q\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{2}}t^2\frac{s_i}{s_j};q)_\infty(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q)_\infty}}_{\mathbb{I}^{\text{4D }U(N)}}\prod_{i=1}^{N}\prod_{\alpha=1}^{k}\underbrace{\frac{(q^{\frac{3}{4}}ts_i^{\pm}z_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}}t^{-1}s_i^{\pm}z_\alpha^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D tHM}}(s_iz_\alpha)}. \tag{7.3}$$

We expect that the half-index (7.1) is equal to (7.3). In fact, we have confirmed that they agree up to certain orders of q for $(N,k)=(1,2)$, (1,3), (2,2), and (2,3) (see Appendix A 2 b).

B. 4D $U(L)$|3D $U(M)$|4D $U(N)$

We consider the interface 4D $U(L)$|3D $U(M)$|4D $U(N)$, which involves a pair of 4D $\mathcal{N}=4$ $U(L)$ and $U(N)$ gauge theories with Neumann b.c. $\mathcal{N}$ and 3D $U(M)$ vector multiplet where L, M, and N are not all equal. The corresponding quiver diagram and brane setup are depicted in Fig. 13. According to unequal numbers of D3-branes, the dual interface can involve singular boundary conditions specified by Nahm poles corresponding to the D5′-branes. We compute the half-indices for a pair of dual interfaces and check the matching. We find that the half-indices have additional contributions when one of 4D gauge symmetries is smaller than 3D gauge symmetry.

The interface has a pair of 4D $\mathcal{N}=4$ $U(L)$ and $U(N)$ gauge theories which obey Neumann b.c. $\mathcal{N}$ at the 3D boundary. Each of them couples to 3D $\mathcal{N}=4$ $U(M)$ vector multiplet through the 3D boundary bifundamental hypermultiplets. We label magnetic fluxes for 3D $U(M)$ gauge

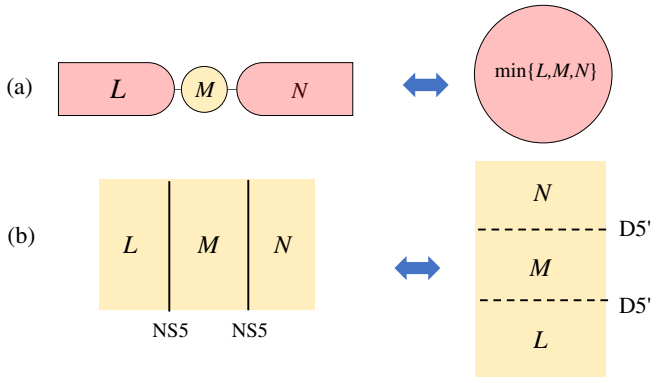

FIG. 13. (a) The quiver diagrams of the 4D $U(L)$|3D $U(M)$|4D $U(N)$ interface and its mirror. (b) The corresponding brane configurations.

theory by M integers $m_1, \ldots, m_M$. The interface includes bare monopole whose canonical R-charge is given by

$$\Delta(m) = \sum_{i=1}^{M} \frac{(L+N)|m_i|}{2} - \sum_{i<j} |m_i - m_j|. \quad (7.4)$$

To ensure that all monopole operators are above the unitarity bound, we assume that $L+N \geq 2M-1$. In fact, for $L+N < 2M-1$, the half-indices may not be convergent. Without loss of generality, we further assume that $L \leq N$.

The half-index takes the form

$$\begin{aligned}
&\mathbb{III}^{4\mathrm{D}\ U(L)|(M)|4\mathrm{D}\ U(N)}_{\mathcal{N}\mathcal{N}}(t, z_\alpha; q) \\
&= \underbrace{\frac{1}{L!}\frac{(q)_\infty^L}{(q^{\frac{1}{2}}t^{-2};q)_\infty^L}\oint \prod_{i=1}^{L}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(L)}_{\mathcal{N}}} \\
&\times \frac{1}{M!}\frac{(q^{\frac{1}{2}}t^2;q)_\infty^M}{(q^{\frac{1}{2}}t^{-2};q)_\infty^M}\sum_{m_1,\ldots,m_M\in\mathbb{Z}}\oint \prod_{i=L+1}^{L+M}\frac{ds_i}{2\pi i s_i}\prod_{i<j}\frac{(1-q^{\frac{|m_{i-L}-m_{j-L}|}{2}}s_i^{\pm}s_j^{\mp})(q^{\frac{1+|m_{i-L}-m_{j-L}|}{2}}t^2 s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1+|m_{i-L}-m_{j-L}|}{2}}t^2 s_i^{\pm}s_j^{\mp};q)_\infty} \\
&\times \underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint \prod_{i=L+M+1}^{L+M+N}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(N)}_{\mathcal{N}}} \\
&\times \prod_{i=1}^{L}\prod_{j=L+1}^{L+M}\frac{(q^{\frac{3}{4}+\frac{|m_{j-L}|}{2}}t^{-1}s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{j-L}|}{2}}ts_i^{\pm}s_j^{\mp};q)_\infty}\cdot\prod_{i=L+1}^{L+M}\prod_{j=L+M+1}^{L+M+N}\frac{(q^{\frac{3}{4}+\frac{|m_{i-L}|}{2}}t^{-1}s_i^{\pm}s_j^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{i-L}|}{2}}ts_i^{\pm}s_j^{\mp};q)_\infty} \\
&\times q^{\sum_{i=1}^{M}\frac{(L+N)|m_i|}{4}-\sum_{i<j}\frac{|m_i-m_j|}{2}}\cdot t^{-\sum_{i=1}^{M}(L+N)|m_i|+2\sum_{i<j}|m_i-m_j|}\cdot\left(\frac{z_1}{z_2}\right)^{\sum_{i=1}^{M}m_i}. \quad (7.5)
\end{aligned}$$

The terms from the second to fourth line describe the half-indices of Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(L)$ and $U(N)$ gauge theories and full-index for 3D $\mathcal{N}=4$ $U(M)$ vector multiplet. The first and second terms in the fifth line count 3D $\mathcal{N}=4$ bifundamental hypermultiplets which couple to 4D $U(L)$ and 3D $U(M)$ gauge theories and those which couple to 4D $U(N)$ and 3D $U(M)$ gauge theories, respectively. The remaining terms count bare monopole operator of dimension (7.4).

Now consider the dual interface. The corresponding quiver and brane configuration are illustrated in Fig. 13. When $M<L$, the 4D $U(L)\times U(M)\times U(N)$ gauge symmetry is broken down to $U(M)$ and the whole 4D $U(M)$ gauge theory remains. For the two defects corresponding to two D5′-branes, we would have two Nahm′ boundary conditions of rank $L-M$ and rank $N-M$. In addition, there are extra contributions to the half-index which come from the broken $U(L)$ and $U(N)$ gauge theories.

The half-index (7.5) for the dual interface with $M<L$ will be computed as

$$\begin{aligned}
\mathbb{III}^{4\mathrm{D}\ U(L)|[M]|4\mathrm{D}\ U(N)}_{\mathcal{D}'}(t, z_\alpha; q) &= \underbrace{\frac{1}{M!}\frac{(q)_\infty^{2M}}{(q^{\frac{1}{2}}t^2;q)_\infty^M(q^{\frac{1}{2}}t^{-2};q)_\infty^M}\oint\prod_{i=1}^{M}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty\left(q\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2\frac{s_i}{s_j};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{I}^{4\mathrm{D}\ U(M)}} \\
&\times \underbrace{\prod_{k=1}^{L-M}\frac{(q^{\frac{k+1}{2}}t^{-2(k-1)};q)_\infty}{(q^{\frac{k}{2}}t^{-2k};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(L-M)}_{\mathrm{Nahm}'}}\cdot\underbrace{\prod_{l=1}^{N-M}\frac{(q^{\frac{l+1}{2}}t^{-2(l-1)};q)_\infty}{(q^{\frac{l}{2}}t^{-2l};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(N-M)}_{\mathrm{Nahm}'}} \\
&\times \prod_{i=1}^{M}\frac{(q^{\frac{3}{4}+\frac{L-M}{4}}t^{1-(L-M)}s_i^{\pm}z_1^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{L-M}{4}}t^{-1-(L-M)}s_i^{\pm}z_1^{\pm};q)_\infty}\cdot\prod_{i=1}^{M}\frac{(q^{\frac{3}{4}+\frac{N-M}{4}}t^{1-(N-M)}s_i^{\pm}z_1^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{N-M}{4}}t^{-1-(N-M)}s_i^{\pm}z_1^{\pm};q)_\infty}. \quad (7.6)
\end{aligned}$$

The terms in the second and third lines are the full-index for 4D $U(M)$ gauge theory and half-indices for Nahm′ boundary conditions of rank $(L-M)$ and rank $(N-M)$. The terms in the last line describe the extra contributions from broken gauge theories.

We conjecture that the half-indices (7.5) and (7.6) give the same answer for $M<L$. In fact, we have found that they match up to certain order of q for $(L,M,N)=(2,1,1)$ and (2,1,3) (see Appendix A 2 b).

For $L\le M$, the gauge symmetry $U(L)\times U(M)\times U(N)$ is broken to $U(L)$ and the $U(L)$ gauge symmetry is kept in the whole 4D theory. For $L<M$, the defect of D5′-brane which has L and M D3-branes in their sides may give rise to the Nahm′ b.c. of rank $L-M$ and the associated extra degrees of freedom (d.o.f.) at the interface which couple to the surviving 4D $U(L)$ gauge theory. For $L=M$, it couples 3D fundamental twisted hypermultiplet to the 4D $U(L)$ gauge theory. On the other defect of D5′-brane which has M and N D3-branes in their its sides would lead to the Nahm′ b.c. of rank $N-M$ and the associated extra d.o.f. at the interface. These extra operators would couple to the surviving $U(L)$ gauge theory; however, further d.o.f. which do not couple to the surviving $U(L)$ gauge theory would appear in contrast to the case with $M<L$.

Thus, the half-index will be evaluated as

$$\begin{aligned}\mathbb{III}^{4\mathrm{D}\ U(L)||[M]|4\mathrm{D}\ U(N)}_{\mathcal{D}'}(t,z_a;q)=&\underbrace{\frac{1}{L!}\frac{(q)_\infty^{2L}}{(q^{\frac12}t^2;q)_\infty^L(q^{\frac12}t^{-2};q)_\infty^L}\oint\prod_{i=1}^L\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty\left(q\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac12}t^2\frac{s_i}{s_j};q\right)_\infty\left(q^{\frac12}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{I}^{4\mathrm{D}\ U(L)}}\\
&\times\underbrace{\prod_{k=1}^{M-L}\frac{(q^{\frac{k+1}{2}}t^{-2(k-1)};q)_\infty}{(q^{\frac{k}{2}}t^{-2k};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(M-L)}_{\mathrm{Nahm}'}}\cdot\underbrace{\prod_{l=1}^{N-M}\frac{(q^{\frac{l+1}{2}}t^{-2(l-1)};q)_\infty}{(q^{\frac{l}{2}}t^{-2l};q)_\infty}}_{\mathbb{III}^{4\mathrm{D}\ U(N-M)}_{\mathrm{Nahm}'}}\\
&\times\prod_{i=1}^L\frac{(q^{\frac34+\frac{M-L}{4}}t^{1-(M-L)}s_i^\pm z_1^\pm;q)_\infty}{(q^{\frac14+\frac{M-L}{4}}t^{-1-(M-L)}s_i^\pm z_1^\pm;q)_\infty}\cdot\prod_{i=1}^L\frac{(q^{\frac34+\frac{N-M}{4}}t^{1-(N-M)}s_i^\pm z_1^\pm;q)_\infty}{(q^{\frac14+\frac{N-M}{4}}t^{-1-(N-M)}s_i^\pm z_1^\pm;q)_\infty}\\
&\times\frac{(q^{\frac34+\frac{N-M}{4}}t^{1-(N-M)}z_1^\pm z_2^\mp;q)_\infty}{(q^{\frac14+\frac{N-M}{4}}t^{-1-(N-M)}z_1^\pm z_2^\mp;q)_\infty}.\end{aligned}\tag{7.7}$$

The terms in the second and third lines are the full-index for 4D $U(L)$ gauge theory and half-indices for Nahm′ boundary conditions of rank $(M-L)$ and rank $(N-M)$. The terms in the fourth line are the extra contributions from broken gauge theories which couple to the surviving gauge theory. The terms in the last line are those which are neutral under the surviving gauge symmetry.

For $L\le M$, the half-index (7.5) will be equal to the half-indices (7.7). As shown in Appendix A 2 b, we have checked that they match up to certain orders of q for $(L,M,N)=(1,1,2)$, (1,2,2), (2,2,3), (1,2,3), (1,2,4), (1,2,5), and (2,3,3).

C. 4D $U(1)|$3D SQED$_{N_f}|$4D $U(1)$

Let us turn to the interfaces which include 3D $\mathcal{N}=4$ gauge theories with flavors. We consider the interface which has a pair of 4D $\mathcal{N}=4$ $U(1)$ gauge theories with Neumann b.c. $\mathcal{N}$ coupled to 3D $\mathcal{N}=4$ SQED$_{N_f}$ through the 3D bifundamental hypermultiplets. We denote this interface by 4D $U(1)|$3D SQED$_{N_f}|$4D $U(1)$. The corresponding quiver diagram and brane configuration are illustrated in Fig. 14. It is realized by a single D3-brane intersecting with two NS5-branes and N_f D5-branes. On the other hand, the S-dual configuration contains a single D3-brane interacting with

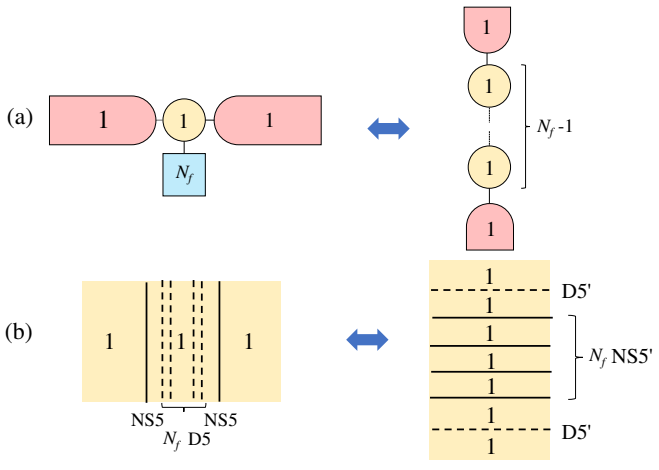

FIG. 14. (a) The quiver diagrams of the interface 4D $U(1)|$3D SQED$_{N_f}|$4D $U(1)$ and its mirror. (b) The corresponding brane configurations.

N_f NS5′-branes and two D5′-branes. For the corresponding dual interface involves 3D Abelian quiver gauge theories as for the Abelian mirror symmetry, however, the topological symmetry in the original interface is mapped to the flavor symmetry for the twisted hypers living at defects in the bulk 4D theory which are decoupled from the 3D quiver gauge theories. We compute the half-indices to check the dualities of the interfaces.

For the interface 4D $U(1)|$3D SQED$_{N_f}|$4D $U(1)$, we have a pair of 4D $\mathcal{N}=4$ $U(1)$ gauge theories living in half-spaces obeying Neumann b.c. $\mathcal{N}$. The interface has 3D $\mathcal{N}=4$ SQED$_{N_f}$ coupled to the 4D theories in either sides.

The half-index of the 4D $U(1)|$3D SQED$_{N_f}|$4D $U(1)$ interface is

$$\begin{aligned}\mathbb{III}^{\text{4D } U(1)|\text{SQED}_{N_f}|\text{4D } U(1)}_{\mathcal{N}\mathcal{N}}(t,x_\alpha,z_\alpha;q) &= \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds_1}{2\pi i s_1}}_{\mathbb{III}^{\text{4D}U(1)}_{\mathcal{N}}}\cdot\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\sum_{m\in\mathbb{Z}}\oint\frac{ds_2}{2\pi i s_2}\underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds_3}{2\pi i s_3}}_{\mathbb{III}^{\text{4D } U(1)}_{\mathcal{N}}}\\ &\times\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_1^{\pm}s_2^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_1^{\pm}s_2^{\mp};q)_\infty}\cdot\prod_{\alpha=1}^{N_f}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_2^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_2^{\pm}x_\alpha^{\pm};q)_\infty}\cdot\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_2^{\pm}s_3^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_2^{\pm}s_3^{\mp};q)_\infty}\\ &\times q^{\frac{(N_f+2)|m|}{4}}t^{-(N_f+2)|m|}z_1^m z_2^{-m},\end{aligned}\tag{7.8}$$

where x_α are the fugacities for the $SU(N_f)$ flavor symmetry of 3D SQED$_{N_f}$ with $\prod_{\alpha=1}^{N_f}x_\alpha=1$ and z_α are the fugacities for the topological symmetries of the 3D SQED$_{N_f}$. The terms in the second line are the two half-indices of Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(1)$ gauge theory and the full-index for 3D $\mathcal{N}=4$ $U(1)$ vector multiplet. The terms in the next line describe the 3D $\mathcal{N}=4$ hypermultiplets. The terms in the last line count monopole operator of dimension $\Delta(m)=\frac{(N_f+2)|m|}{2}$.

The dual interface involves a pair of 4D $\mathcal{N}=4$ $U(1)$ gauge theories living in a half-space obeying Neumann b.c. $\mathcal{N}'$. Unlike the original interface, each of 4D theories has a defect that couples a 3D $\mathcal{N}=4$ fundamental twisted hypermultiplet to the associated 4D $U(1)$ gauge theories. In addition, the interface has a 3D $\mathcal{N}=4$ $U(1)^{N_f-1}$ twisted vector multiplet coupled to both 4D gauge theories through the 3D $\mathcal{N}=4$ bifundamental twisted hypermultiplets.

The half-index for the dual interface should take the form

$$\begin{aligned}&\mathbb{III}^{\text{4D } U(1)+\text{thyper}|\widetilde{(1)^{N_f-1}}|\text{4D } U(1)+\text{thyper}}_{\mathcal{N}'\mathcal{N}'}(t,x_\alpha,z_\alpha;q)\\ &=\underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\oint\frac{ds_1}{2\pi i s_1}}_{\mathbb{III}^{\text{4D } U(1)}_{\mathcal{N}'}}\cdot\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^{N_f-1}}{(q^{\frac{1}{2}}t^2;q)_\infty^{N_f-1}}\sum_{m_1,\ldots,m_{N_f-1}\in\mathbb{Z}}\oint\prod_{i=1}^{N_f-1}\frac{ds_i}{2\pi i s_i}\underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\oint\frac{ds_{N_f}}{2\pi i s_{N_f}}}_{\mathbb{III}^{\text{4D } U(1)}_{\mathcal{N}'}}\\ &\times\underbrace{\frac{(q^{\frac{3}{4}}ts_1^{\pm}z_1^{\pm};q)_\infty}{(q^{\frac{1}{4}}t^{-1}s_1^{\pm}z_1^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D tHM}}(s_1z_1)}\cdot\frac{(q^{\frac{3}{4}+\frac{|m_1|}{2}}ts_1^{\pm}s_2^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1|}{2}}t^{-1}s_1^{\pm}s_2^{\mp};q)_\infty}\cdot\prod_{i=2}^{N_f-2}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_{i+1}|}{2}}ts_i^{\pm}s_{i+1}^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_{i+1}|}{2}}t^{-1}s_i^{\pm}s_{i+1}^{\mp};q)_\infty}\frac{(q^{\frac{3}{4}+\frac{|m_{N_f-1}|}{2}}ts_{N_f-1}^{\pm}s_{N_f}^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{N_f-1}|}{2}}t^{-1}s_{N_f-1}^{\pm}s_{N_f}^{\mp};q)_\infty}\cdot\underbrace{\frac{(q^{\frac{3}{4}}ts_{N_f}^{\pm}z_2^{\pm};q)_\infty}{(q^{\frac{1}{4}}t^{-1}s_{N_f}^{\pm}z_2^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D tHM}}(s_1z_1)}\\ &\times q^{\frac{|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|}{4}}\cdot t^{|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|}\cdot\prod_{i=1}^{N_f-1}\left(\frac{x_i}{x_{i+1}}\right)^{m_i}\cdot\left(\frac{x_{N_f}}{x_1}\right)^{m_1+m_{N_f-1}},\end{aligned}\tag{7.9}$$

where z_α is the fugacities for the flavor symmetry of defect 3D twisted hypermultiplets, and x_α is the fugacities for the topological symmetry of the 3D $U(1)^{N_f-1}$ twisted vector multiplet. The contributions in the second line are the square of half-index of Neumann b.c. $\mathcal{N}'$ for 4D $U(1)$ gauge theory and the full-index for 3D $U(1)^{N_f-1}$ twisted vector multiplet. The terms in the third and fourth lines correspond to the contributions of 3D $\mathcal{N}=4$ twisted hypermultiplets. The terms in the last line count monopole operator of dimension $\Delta(m)=\frac{1}{2}(|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|)$.

It is expected that the half-indices (7.8) and (7.9) agree with each other. We have confirmed the matching for $N_f = 1, 2, 3$ up to certain orders of q (see Appendix A 2 b).

D. 4D $U(N)$|3D SQED$_{N_f}$|4D $U(M)$

We next turn to the interface 4D $U(N)$|3D SQED$_{N_f}$|4D $U(M)$ where $N, M > 1$. The corresponding quiver diagram and brane construction are drawn in Fig. 15. Unlike the interface discussed in Sec. VII C, the dual interface may include singular boundary conditions for 4D $U(N)$ and $U(M)$ gauge theories which are characterized by the Nahm poles of rank $(N-1)$ and rank $(M-1)$. We calculate the half-indices for a pair of dual interfaces and find the matching.

The interface involves a pair of 4D $\mathcal{N}=4$ $U(N)$ and $U(M)$ SYM theories satisfying Neumann b.c. $\mathcal{N}$ corresponding to the NS5-branes. Both of 4D theories couple to 3D $\mathcal{N}=4$ SQED$_{N_f}$ through 3D bifundamental hypermultiplets. The magnetic fluxes for 3D SQED$_{N_f}$ can be labeled by an integer m.

The half-index of the 4D $U(N)$|3D SQED$_{N_f}$|4D $U(M)$ interface is

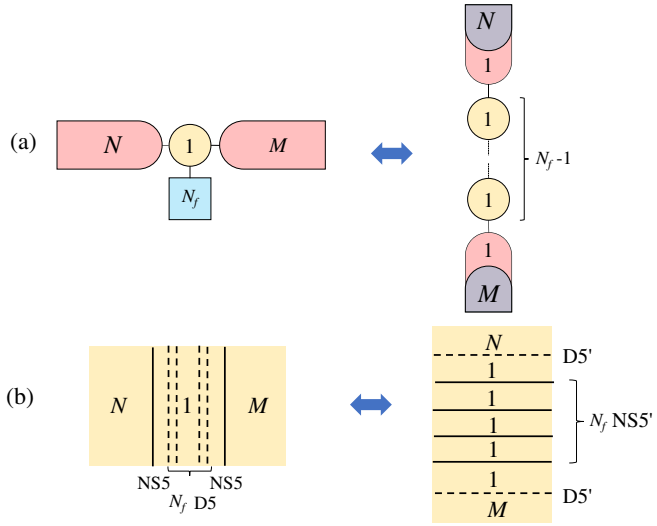

FIG. 15. (a) The quiver diagrams of the interface 4D $U(N)$|3D SQED$_{N_f}$|4D $U(M)$ and its mirror. (b) The corresponding brane configurations.

$$
\begin{aligned}
&\mathbb{III}^{\text{4D } U(N)|\text{SQED}_{N_f}|\text{4D } U(M)}_{\mathcal{NN}}(t, x_\alpha, z_\alpha; q) \\
&= \underbrace{\frac{1}{N!}\frac{(q)_\infty^N}{(q^{\frac{1}{2}}t^{-2};q)_\infty^N}\oint\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\sum_{m\in\mathbb{Z}}\oint\frac{ds_{N+1}}{2\pi i s_{N+1}}}_{\mathbb{III}^{\text{4D } U(N)}_{\mathcal{N}}} \\
&\times \underbrace{\frac{1}{M!}\frac{(q)_\infty^M}{(q^{\frac{1}{2}}t^{-2};q)_\infty^M}\oint\prod_{i=N+2}^{N+M+1}\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\frac{\left(\frac{s_i}{s_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t^{-2}\frac{s_i}{s_j};q\right)_\infty}}_{\mathbb{III}^{\text{4D } U(M)}_{\mathcal{N}}}\prod_{i=1}^{N}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_i^{\pm}s_{N+1}^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_i^{\pm}s_{N+1}^{\mp};q)_\infty}\cdot\prod_{\alpha=1}^{N_f}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_{N+1}^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_{N+1}^{\pm}x_\alpha^{\pm};q)_\infty} \\
&\cdot\prod_{i=N+2}^{N+M+1}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_{N+1}^{\pm}s_i^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_{N+1}^{\pm}s_i^{\mp};q)_\infty}q^{\frac{(N+M+N_f)|m|}{4}}t^{-(N+M+N_f)|m|}z_1^m z_2^{-m},
\end{aligned}
\tag{7.10}
$$

where the fugacities x_α with $\prod_{\alpha=1}^{N_f}x_\alpha = 1$ are associated with the $SU(N_f)$ flavor symmetry of SQED$_{N_f}$, while the fugacities z_α are associated with the topological symmetry of SQED$_{N_f}$. The terms from the second to fourth line describe gauge multiplets in the interface, including the half-indices of Neumann b.c. $\mathcal{N}$ for 4D $\mathcal{N}=4$ $U(N)$ and $U(M)$ gauge theories and the full-index for 3D $\mathcal{N}=4$ $U(1)$ vector multiplet. The terms in the fifth line correspond to the 3D $\mathcal{N}=4$ hypermultiplets. The terms in the last line describe monopole operator with the R-charge $\Delta(m) = \frac{(N+M+N_f)|m|}{2}$.

Under the action of S-duality, one finds the dual interface which also has a pair of 4D $\mathcal{N}=4$ gauge theories. The D5′-brane intersecting with N D3-brane breaks $U(N)\times U(1)$ gauge symmetry down to $U(1)$, while the other D5′-brane intersecting with M D3-branes breaks $U(M)\times U(1)$ gauge symmetry down to $U(1)$. The reductions of 4D gauge symmetries are described by the boundary conditions specified by the two Nahm poles of rank $(N-1)$ and rank $(M-1)$. Correspondingly, extra contributions would appear at the defects from the broken gauge theories [12].

According to the presence of NS5′-branes, the surviving 4D $\mathcal{N}=4$ $U(1)$ gauge theories further satisfy Neumann b.c. $\mathcal{N}'$. They are couple to 3D $\mathcal{N}=4$ $\widetilde{U(1)^{N_f-1}}$ quiver gauge theory via 3D $\mathcal{N}=4$ bifundamental twisted hypermultiplets.

The half-index should take the form

$$
\begin{aligned}
&\mathbb{III}^{\text{4D } U(N)\to U(1)|\widetilde{(1)^{N_f-1}}|\text{4D } U(M)\to U(1)}_{\mathcal{N}'\mathcal{N}'}(t,x_\alpha,z_\alpha;q)\\
&=\underbrace{\frac{(q)_\infty}{(q^{\frac12}t^2;q)_\infty}\oint\frac{ds_1}{2\pi i s_1}}_{\mathbb{III}^{\text{4D}U(1)}_{\mathcal{N}'}}\cdot\underbrace{\prod_{k=1}^{N-1}\frac{(q^{\frac{k+1}{2}}t^{-2(k-1)};q)_\infty}{(q^{\frac{k}{2}}t^{-2k};q)_\infty}}_{\mathbb{III}^{\text{4D}U(N-1)}_{\text{Nahm}'}}\cdot\frac{(q^{\frac12}t^{-2};q)_\infty^{N_f-1}}{(q^{\frac12}t^2;q)_\infty^{N_f-1}}\sum_{m_1,\ldots,m_{N_f-1}\in\mathbb{Z}}\oint\prod_{i=2}^{N_f}\frac{ds_i}{2\pi i s_i}\underbrace{\prod_{l=1}^{M-1}\frac{(q^{\frac{l+1}{2}}t^{-2(l-1)};q)_\infty}{(q^{\frac{l}{2}}t^{-2l};q)_\infty}}_{\mathbb{III}^{\text{4D } U(M-1)}_{\text{Nahm}'}}\\
&\cdot\underbrace{\frac{(q)_\infty}{(q^{\frac12}t^2;q)_\infty}\oint\frac{ds_{N_f+1}}{2\pi i s_{N_f+1}}}_{\mathbb{III}^{\text{4D } U(1)}_{\mathcal{N}'}}\frac{(q^{\frac34+\frac{N-1}{4}}t^{1-(N-1)}s_1^\pm z_1^\pm;q)_\infty}{(q^{\frac14+\frac{N-1}{4}}t^{-1-(N-1)}s_1^\pm z_1^\pm;q)_\infty}\cdot\frac{(q^{\frac34+\frac{|m_1|}{2}}ts_1^\pm s_2^\mp;q)_\infty}{(q^{\frac14+\frac{|m_1|}{2}}t^{-1}s_1^\pm s_2^\mp;q)_\infty}\cdot\prod_{i=2}^{N_f-1}\frac{(q^{\frac34+\frac{|m_i-m_{i+1}|}{2}}ts_i^\pm s_{i+1}^\mp;q)_\infty}{(q^{\frac14+\frac{|m_i-m_{i+1}|}{2}}t^{-1}s_i^\pm s_{i+1}^\mp;q)_\infty}\\
&\times\frac{(q^{\frac34+\frac{|m_{N_f-1}|}{2}}ts_{N_f}^\pm s_{N_f+1}^\mp;q)_\infty}{(q^{\frac14+\frac{|m_{N_f-1}|}{2}}t^{-1}s_{N_f}^\pm s_{N_f-1}^\mp;q)_\infty}\cdot\frac{(q^{\frac34+\frac{M-1}{4}}t^{1-(M-1)}s_{N_f+1}^\pm z_2^\pm;q)_\infty}{(q^{\frac14+\frac{M-1}{4}}t^{-1-(M-1)}s_{N_f+1}^\pm z_2^\pm;q)_\infty}q^{\frac{|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|}{4}}t^{|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|}\\
&\times\prod_{\alpha=1}^{N_f-1}\left(\frac{x_\alpha}{x_{\alpha+1}}\right)^{m_\alpha}\cdot\left(\frac{x_{N_f}}{x_1}\right)^{m_1+m_{N_f-1}},
\end{aligned}
\tag{7.11}
$$

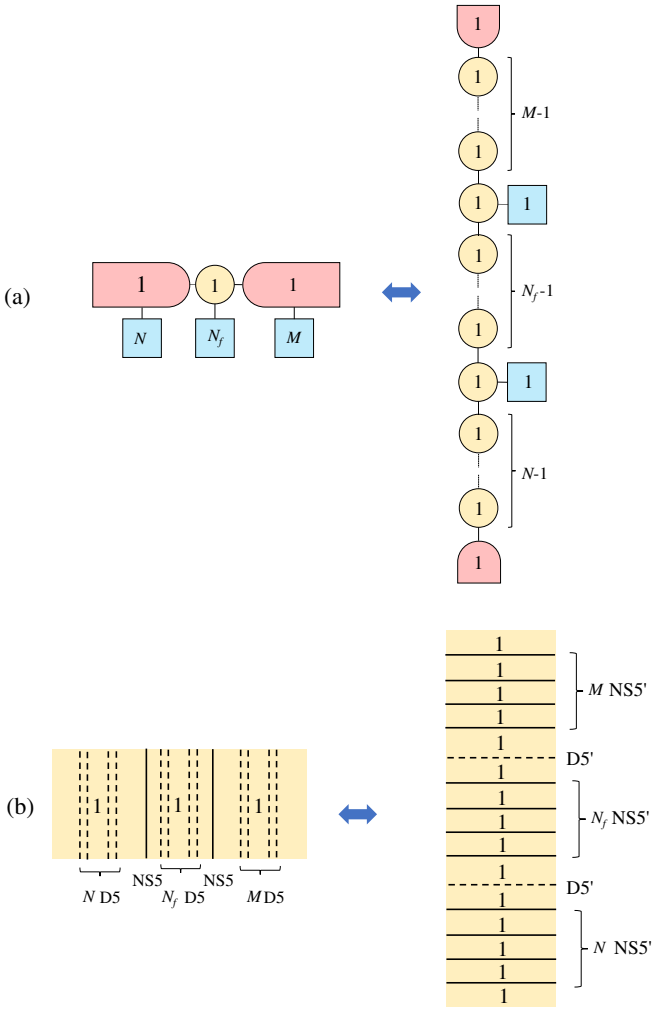

FIG. 16. (a) The quiver diagrams of the interface 4D $U(1)+N$ hypers$|$3D SQED$_{N_f}|$4D $U(1)+M$ hypers and its mirror. (b) The corresponding brane configurations.

where z_1, z_2 are associated with the extra local operators at the two defects and x_α are associated with the topological symmetry for 3D $U(1)^{N_f-1}$ quiver gauge theory. The terms from the second to third line includes the square of half-indices of Neumann b.c. $\mathcal{N}'$ for 4D $\mathcal{N}=4$ $U(1)$ gauge theory, the two half-indices of Nahm$'$ b.c. of rank $(N-1)$ and rank $(M-1)$, and the full-index for 3D $\mathcal{N}=4$ $U(1)$ vector multiplet. The terms from the fourth to fifth line describe 3D $\mathcal{N}=4$ twisted hypermultiplets. The contributions in the last line count bare monopole of dimension $\Delta(m)=\frac12(|m_1|+|m_{N_f-1}|+\sum_{i=1}^{N_f-2}|m_i-m_{i+1}|)$.

We expect that the half-indices (7.10) and (7.11) agree with each other. We have shown the matching of indices for $(N,M,N_f)=(2,3,1)$, (2,3,2), and (2,3,3) up to certain orders of q in Appendix A 2 b.

E. 4D $U(1)+N$ hypers$|$3D SQED$_{N_f}|$4D $U(1)+M$ hypers

We investigate the interface with a pair of 4D $\mathcal{N}=4$ $U(1)$ gauge theories where one of them has N D5-brane defects and the other has M D5-brane defects and the both 4D theories obey the Neumann b.c. $\mathcal{N}$ with a coupling to 3D $\mathcal{N}=4$ SQED$_{N_f}$.[2] We denote this interface by 4D $U(1)+N$ hypers$|$3D SQED$_{N_f}|$4D $U(1)+M$ hypers. The corresponding quiver diagram and brane construction are shown in Fig. 16. The N and M D5-branes introduce N and M fundamental hypermultiplets living at the defects.

[2]Although we have restricted to the Abelian interface in this section, the generalization to the non-Abelian interface is straightforward by using the results so far.

In this case, there are a 4D $\mathcal{N}=4$ $U(1)$ gauge theory with N defects which couple the 4D theory to the N fundamental 3D hypermultiplets and another 4D $\mathcal{N}=4$ $U(1)$ gauge theory with M defects which couple the 4D theory to the M fundamental 3D hypermultiplets. A pair of 4D gauge theories obeys Neumann b.c. $\mathcal{N}$ and interacts with 3D $\mathcal{N}=4$ SQED_{N_f} through a coupling given by 3D $\mathcal{N}=4$ bifundamental hypermultiplets. The interface has 4D $U(1)\times U(1)$ gauge symmetry and 3D $U(1)$ gauge symmetry. We can label the magnetic fluxes for 3D $U(1)$ gauge theory by an integer m.

Then the half-index for the interface 4D $U(1)+N$ hypers$|$3D $\mathrm{SQED}_{N_f}|$4D $U(1)+M$ hypers takes the form

$$
\begin{aligned}
&\mathbb{III}_{\mathcal{N}\mathcal{N}}^{\text{4D } U(1)+N \text{ hypers}|\mathrm{SQED}_{N_f}|\text{4D } U(1)+M \text{ hypers}}(t,x_\alpha,z_\alpha;q)\\
&=\underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds_1}{2\pi i s_1}}_{\mathbb{III}_{\mathcal{N}}^{\text{4D } U(1)}}\cdot\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\sum_{m\in\mathbb{Z}}\oint\frac{ds_2}{2\pi i s_2}\cdot\underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}\oint\frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}}^{\text{4D } U(1)}}\\
&\times\prod_{\alpha=1}^{N}\underbrace{\frac{(q^{\frac{3}{4}}t^{-1}s_1^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}}t^{-1}s_1^{\pm}x_\alpha^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D HM}}(s_1x_\alpha)}\cdot\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_1^{\pm}s_2^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_1^{\pm}s_2^{\mp};q)_\infty}\cdot\prod_{\alpha=N+1}^{N+N_f}\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_2^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_2^{\pm}x_\alpha^{\pm};q)_\infty}\\
&\times\frac{(q^{\frac{3}{4}+\frac{|m|}{2}}t^{-1}s_2^{\pm}s_3^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts_2^{\pm}s_3^{\mp};q)_\infty}\cdot\prod_{\alpha=N+N_f+1}^{N+N_f+M}\underbrace{\frac{(q^{\frac{3}{4}}t^{-1}s_3^{\pm}x_\alpha^{\pm};q)_\infty}{(q^{\frac{1}{4}}ts_3^{\pm}x_\alpha^{\pm};q)_\infty}}_{\mathbb{I}^{\text{3D HM}}(s_3x_\alpha)}\\
&\times q^{\frac{(N+N_f+M)|m|}{4}}\cdot t^{-(N+N_f+M)|m|}\cdot\left(\frac{z_1}{z_2}\right)^m,
\end{aligned}
\tag{7.12}
$$

where we have used $\{x_\alpha\}_{\alpha=1,\ldots,N}$ for the fugacities of the $SU(N)$ flavor symmetries of the defect hypermultiplets, $\{x_\alpha\}_{\alpha=N+1,\ldots,N+N_f}$ for the fugacities of the $SU(N_f)$ flavor symmetry of 3D SQED_{N_f}, $\{x_\alpha\}_{\alpha=N+N_f+1,\ldots,N+N_f+M}$ for the fugacities of the $SU(M)$ flavor symmetries of the defect hypermultiplets. The fugacities z_α are associated to the topological symmetry of 3D SQED_{N_f}. The contributions in the second line are the two half-indices of Neumann b.c. $\mathcal{N}$ for 4D $U(1)$ gauge theory and the full-index for 3D $U(1)$ vector multiplet. The terms appearing in the next two lines are the contributions from the 3D hypermultiplets. The terms in the last line count bare monopole of dimension $\Delta(m)=\frac{(N+N_f+M)|m|}{2}$.

From the S-dual configuration of the brane setup in Fig. 16, we can read off the dual interface. It has a pair of 4D $\mathcal{N}=4$ $U(1)$ gauge theories which satisfy Neumann b.c. $\mathcal{N}'$ and couple to 3D quiver gauge theory

$$
\begin{array}{ccccccccc}
(1)^{N-1} & - & (1) & - & \widetilde{(1)^{N_f-1}} & - & (1) & - & (1)^{M-1}\\
 & & | & & & & | & & \\
 & & [1] & & & & [1] & &
\end{array}
$$

through the 3D bifundamental twisted hypermultiplets. We can label the magnetic fluxes for the 3D quiver gauge theory by $N+N_f+M-1$ integers $m_1,\ldots,m_{N+N_f+M-1}$.

The half-index for the dual configuration is given by

$$
\begin{aligned}
&\mathbb{III}_{\mathcal{N}'\mathcal{N}'}^{\text{4D } U(1)|\ (1)^{N-1}-(1)-\widetilde{(1)^{N_f-1}}-(1)-(1)^{M-1}\ |\text{4D } U(1)}(t,x_\alpha,z_\alpha;q)\\
&\quad\text{(the middle } (1)\text{'s each have a } [1] \text{ flavor attached)}\\
&=\underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\oint\frac{ds_1}{2\pi i s_1}}_{\mathbb{III}_{\mathcal{N}'}^{\text{4D } U(1)}}\frac{(q^{\frac{1}{2}}t^{-2};q)_\infty^{N+N_f+M-1}}{(q^{\frac{1}{2}}t^2;q)_\infty^{N+N_f+M-1}}\sum_{m_1,\ldots,m_{N+N_f+M-1}\in\mathbb{Z}}\oint\prod_{i=2}^{N+N_f+M}\frac{ds_i}{2\pi i s_i}
\end{aligned}
$$

$$\times \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2;q)_\infty}\oint\frac{ds_{N+N_f+M+1}}{2\pi i s_{N+N_f+M+1}}\frac{(q^{\frac{3}{4}+\frac{|m_1|}{2}}ts_1^{\pm}s_2^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_1|}{2}}t^{-1}s_1^{\pm}s_2^{\mp};q)_\infty}}_{\mathbb{III}^{4D\ U(1)}_{\mathcal{N}'}}\cdot\prod_{i=2}^{N+N_f+M-1}\frac{(q^{\frac{3}{4}+\frac{|m_{i-1}-m_i|}{2}}ts_i^{\pm}s_{i+1}^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{i-1}-m_i|}{2}}t^{-1}s_i^{\pm}s_{i+1}^{\mp};q)_\infty}\cdot\frac{(q^{\frac{3}{4}+\frac{|m_N|}{2}}ts_{N+1}^{\pm}z_1^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_N|}{2}}t^{-1}s_{N+1}^{\pm}z_1^{\pm};q)_\infty}$$

$$\times\frac{(q^{\frac{3}{4}+\frac{|m_{N+N_f}|}{2}}ts_{N+N_f+1}^{\pm}z_2^{\pm};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{N+N_f}|}{2}}t^{-1}s_{N+N_f+1}^{\pm}z_2^{\pm};q)_\infty}\cdot\frac{(q^{\frac{3}{4}+\frac{|m_{N+N_f+M-1}|}{2}}ts_{N+N_f+M}^{\pm}s_{N+N_f+M+1}^{\mp};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_{N+N_f+M-1}|}{2}}t^{-1}s_{N+N_f+M}^{\pm}s_{N+N_f+M+1}^{\mp};q)_\infty}$$

$$\times q^{\frac{|m_1|}{4}+\frac{|m_N|}{4}+\frac{|m_{N+N_f}|}{4}+\frac{|m_{N+N_f+M-1}|}{4}+\sum_{i=1}^{N+N_f+M-2}\frac{|m_i-m_{i+1}|}{4}}$$

$$\times t^{|m_1|+|m_N|+|m_{N+N_f}|+|m_{N+N_f+M-1}|+\sum_{i=1}^{N+N_f+M-2}|m_i-m_{i+1}|}$$

$$\times\prod_{\alpha=1}^{N+N_f+M}x_\alpha^{m_{N+N_f+M-\alpha+1}-m_{N+N_f+M-\alpha}},\tag{7.13}$$

where $m_0=m_{N+N_f+M}\equiv 0$. The fugacities x_α and z_α are now associated to the topological and flavor symmetries for 3D quiver gauge theory $\overset{(1)^{N-1}}{}-\overset{(1)}{\underset{|}{\underset{[1]}{}}}-\overset{\widetilde{(1)^{N_f-1}}}{}-\overset{(1)}{\underset{|}{\underset{[1]}{}}}-\overset{(1)^{M-1}}{}$. The terms from the second to fourth line are the two half-indices of Neumann b.c. $\mathcal{N}$ for 4D $U(1)$ gauge theory and the full-index for 3D $U(1)^{N+N_f+M-1}$ twisted vector multiplet. The terms appearing in the next two lines describe the contributions from 3D twisted hypermultiplets. The remaining terms count bare monopole of dimension

$$\Delta(m)=\frac{|m_1|}{2}+\frac{|m_N|}{2}+\frac{|m_{N+N_f}|}{2}+\frac{|m_{N+N_f+M-1}|}{2}+\sum_{i=1}^{N+N_f+M-2}\frac{|m_i-m_{i+1}|}{2}.\tag{7.14}$$

We expect that the half-index (7.12) is equal to (7.13). In fact, we have checked that they match up to certain orders of q for $(N,M,N_f)=(2,2,0)$, (2,2,1), (3,3,0) (see Appendix A 2 b).

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APPENDIX: SERIES EXPANSIONS OF INDICES

We only show the several terms in the expansions of indices by using Mathematica.

1. 3D full-indices

a. Wilson lines in SQED$_1$

We have checked that the 3D full-index (3.7) for the SQED$_1$ with a Wilson line operator $\mathcal{W}_n$ of charge n and the 3D full-index (3.8) for the twisted hypermultiplet with a vortex line $\mathcal{V}_n$ agree up to $\mathcal{O}(q^{10})$ for $n=1,\ldots,5$.

$\mathbb{I}^{\text{3D SQED}_1}_{\mathcal{W}_n}$	$\mathbb{I}^{\text{tHM}}_{\mathcal{V}_n}$	Series expansions
$n=1$	$n=1$	$tq^{\frac{1}{4}}+(x+x^{-1})q-t^2(x+x^{-1})q^{\frac{3}{4}}+t^{-1}(x^{-2}+1+x^2)q^{\frac{7}{4}}+(x^{-1}+x)q^2+\cdots$
$n=2$	$n=2$	$t^2q^{\frac{1}{2}}+t(x^{-1}+x)q^{\frac{7}{4}}-t^{-3}(x^{-1}+x)q^{\frac{9}{4}}+t(x^{-1}+x)q^{\frac{11}{4}}+(x^{-2}+1+x^2)q^3+\cdots$
$n=3$	$n=3$	$t^3q^{\frac{3}{4}}+t^2(x^{-1}+x)q^{\frac{5}{2}}-t^4(x^{-1}+x)q^3+t^2(x^{-1}+x)q^{\frac{7}{2}}-t^4(x^{-1}+x)q^4+\cdots$
$n=4$	$n=4$	$t^4q+t^3(x^{-1}+x)q^{\frac{4}{13}}-t^5(x^{-1}+x)q^{\frac{15}{4}}+t^3(x^{-1}+x)q^{\frac{17}{4}}-t^5(x^{-1}+x)q^{\frac{19}{4}}+\cdots$
$n=5$	$n=5$	$t^5q^{\frac{5}{4}}+t^4(x^{-1}+x)q^4-t^6(x^{-1}+x)q^{\frac{9}{2}}+t^4(x^{-1}+x)q^5-t^6(x^{-1}+x)q^{\frac{11}{2}}+\cdots$

(A1)

b. Abelian mirror symmetry

We have confirmed that 3D $\mathcal{N}=4$ full-indices for the following Abelian gauge theories agree up to $\mathcal{O}(q^5)$.

Theories	Mirror theories	Series expansions
$T[SU(2)]$	$T[\widetilde{SU(2)}]$	$1+t^{-2}(1+\frac{x_1}{x_2}+\frac{x_2}{x_1}+t^4(1+\frac{z_1}{z_2}+\frac{z_2}{z_1}))q^{\frac{1}{2}}+\cdots$
SQED_3	$[1]-\widetilde{(1)^2}-[1]$	$(1+(t^{-2}+t^2x_1^{-2}x_2^{-1}x_3^{-1}(x_1+x_2)(x_1+x_3)(x_2+x_3)))q^{\frac{1}{2}}+t^{-3}z_1^{-1}z_2^{-1}(z_1^2+z_2^2)q^{\frac{3}{4}}+\cdots$
SQED_4	$[1]-\widetilde{(1)^3}-[1]$	$1+(t^{-2}+x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}t^2(x_2x_3x_4(x_2+x_3+x_4)+x_1^2(x_3x_4+x_2(x_3+x_4))$ $+x_1(x_2^2(x_3+x_4)+x_3x_4(x_3+x_4)+x_2(x_3^2+3x_3x_4+x_4^2))))q^{\frac{1}{2}}+\cdots$

(A2)

c. Non-Abelian mirror symmetry

We have confirmed that the 3D $\mathcal{N}=4$ full-indices for the following mirror pairs of non-Abelian gauge theories agree up to $\mathcal{O}(q^3)$.

Theories	Mirror theories	Series expansions
$(2)-[4]$	$(1)-\widetilde{(2)}-(1)$ $\quad\quad\vert$ $\quad\quad[2]$	$1+t^{-2}x_1^{-2}x_2^{-1}x_3^{-1}x_4^{-1}z_1^{-1}z_2^{-1}(t^4x_1^2(x_3x_4+x_2(x_3+x_4))z_1z_2+x_1(t^4x_2^2(x_3+x_4)z_1z_2$ $+t^4x_3x_4(x_3+x_4)z_1z_2+x_2(t^4x_3^2z_1z_2+t^4x_4^2z_1z_2+x_3x_4(z_1^2+z_1z_2+3t^4z_1z_2+z_2^2))))q^{\frac{1}{2}}$
$T[SU(3)]$	$T[\widetilde{SU(3)}]$	$1+(t^2x_1^{-1}x_2^{-1}x_3^{-1}(x_1+x_2)(x_1+x_3)(x_2+x_3)+t^{-2}z_1^{-1}z_2^{-1}z_3^{-1}(z_1+z_2)(z_1+z_3)(z_2+z_3))q^{\frac{1}{2}}+\cdots$
$(1)-(2)-(1)$ $\quad\vert\quad\vert\quad\vert$ $[1]\quad[2]\quad[1]$	$(1)-\widetilde{(2)}-(1)$ $\quad\vert\quad\vert\quad\vert$ $[1]\quad[2]\quad[1]$	$1+t^{-2}\left(3+t^4\left(3+x_2x_3^{-1}+x_3x_2^{-1}\right)+z_2z_3^{-1}+z_3z_2^{-1}\right)q^{\frac{1}{2}}$ $+\left(t^3\frac{(x_2+x_3)(x_1+x_4)(x_2x_3+x_1x_4)}{x_1x_2x_3x_4}+t^{-3}\frac{(z_2+z_3)(z_1+z_4)(z_2z_3+z_1z_4)}{z_1z_2z_3z_4}\right)q^{\frac{3}{4}}+\cdots$
$(1)-(2)-(1)$ $\quad\vert\quad\vert$ $\quad[2]\quad[1]$	$\widetilde{(2)-(1)}$ $\quad\vert\quad\vert$ $\quad[3]\quad[1]$	$1+t^{-2}\left(3+t^4\frac{(x_1+x_2)^2}{x_1+x_2}+z_1\left(z_2^{-1}+z_3^{-1}\right)+\frac{z_2}{z_3}+\frac{z_3}{z_2}+\frac{z_2+z_3}{z_1}\right)q^{\frac{1}{2}}$ $+\left(t^3\left(\frac{x_1}{x_3}+\frac{x_2}{x_3}+\frac{x_3}{x_1}+\frac{x_3}{x_2}\right)+t^{-3}\left(\frac{z_1}{z_4}+\frac{z_2}{z_4}+\frac{z_3}{z_4}+\frac{z_4}{z_1}+\frac{z_4}{z_2}+\frac{z_4}{z_3}\right)\right)q^{\frac{3}{4}}+\cdots$
$(1)-(2)-[4]$	$(2)-\widetilde{(2)}-(1)$ $\quad\vert\quad\vert$ $[2]\quad[1]$	$1+t^{-2}\frac{t^4x_2x_3x_4(x_2+x_3+x_4)z_1z_2+t^4x_1^2(x_3x_4+x_2(x_3+x_4))z_1z_2+x_1(t^4x_2^2(x_3+x_4)z_1z_2}{t^2x_1x_2x_3x_4z_1z_2}$ $\underline{+t^4x_3x_4(x_3+x_4)z_1z_2+x_2(t^4x_3^2z_1z_2+t^4x_4^2z_1z_2+x_3x_4(z_1^2+(2;3t^4)z_1z_2+z_2^2)))}\,q^{\frac{1}{2}}+\cdots$

(A3)

d. Seiberg-like duality

We have checked that the pairs of full-indices for Seiberg-like dual 3D $\mathcal{N}=4$ gauge theories agree up to $\mathcal{O}(q^3)$.

Ugly	Good	Series expansions
$(2)-[3]$	$\mathrm{SQED}_3+\mathrm{tHM}$	$1+t^{-1}\frac{(z_1^2+z_2^2)}{z_1z_2}q^{\frac{1}{4}}+(\frac{t^2(x_1+x_2)(x_1;x_[3])(x_2+x_3)}{x_1x_2x_3}+t^{-2}\frac{(z_1^2+z_2^2)^2}{z_1^2z_2^2})q^{\frac{1}{2}}+\cdots$
$(3)-[5]$	$(2)-[5]+\mathrm{tHM}$	$1+t^{-1}\frac{(z_1^2+z_2^2)}{z_1z_2}q^{\frac{1}{4}}$ $+(t^2\frac{x_2x_3x_4x_5(x_2+x_3+x_4+x_5)+x_1^2(x_2x_4x_5+x_3x_4x_5+x_2x_3(x_4+x_5))+x_1(x_3x_4x_5(x_3+x_4+x_5)}{x_1x_2x_3x_4x_5}$ $\frac{+x_2^2(x_4x_5+x_3(x_4+x_5))+x_2(x_3^2(x_4+x_5)+x_4x_5(x_4+x_5)+x_3(x_4^2+4x_4x_5+x_5^2))}{}+t^{-2}\frac{(z_1^2+z_2^2)^2}{z_1^2z_2^2})q^{\frac{1}{2}}+\cdots$

(A4)

2. 4D half-indices

a. Hafl-BPS boundary conditions

We have checked that the following pairs of half-indices of dual half-BPS boundary conditions in 4D $\mathcal{N}=4$ gauge theories agree up to certain orders of q.

Half-BPS b.c.	Series expansions	Up to orders
4d $U(1)\vert$3d $U(1)$	$1+\frac{(z_1^2+z_2^2)}{tz_1z_2}q^{\frac{1}{4}}+\frac{(z_1+z_2)^2}{t^2z_1^2z_2^2}q^{\frac{1}{2}}+\frac{(z_1^2+z_2^2)(z_1^4-(-1+t^4)z_1^2z_2^2+z_2^4)}{t^3z_1^3z_2^3}q^{\frac{3}{4}}+\cdots$	$\mathcal{O}(q^{10})$
4d $U(2)\vert$3d $U(1)$	$1+\frac{(z_1+z_2)^2}{t^2z_1z_2}q^{\frac{1}{2}}+\frac{z_1^4-(-2+t^4)z_1^3z_2-2(-2+t^4)z_1^2z_2^2-(-2+t^4)z_1z_2^3+z_2^4}{t^4z_1^2z_2^2}q+\cdots$	$\mathcal{O}(q^5)$
4d $U(3)\vert$3d $U(1)$	$1+\frac{2}{t^2}q^{\frac{1}{2}}+\frac{z_1^2+z_2^2}{t^3z_1z_2}q^{\frac{3}{4}}+\left(-2+\frac{4}{t^4}\right)q-\frac{(-2+t^4)(z_1^2+z_2^2)}{t^5z_1z_2}q^{\frac{5}{4}}+\cdots$	$\mathcal{O}(q^5)$
4d $U(3)\vert$3d $U(2)$	$1+\frac{z_1^2+z_2^2}{tz_1z_2}q^{\frac{1}{4}}+\frac{z_1^4+3z_1^2z_2^2+z_2^4}{t^2z_1^2z_2^2}q^{\frac{1}{2}}+\frac{(z_1^2+z_2^2)(z_1^4-(-3+t^4)z_1^2z_2^2+z_2^4)}{t^3z_1^3z_2^3}q^{\frac{3}{4}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(4)\vert$3d $U(2)$	$1+\frac{(z_1+z_2)^2}{t^2z_1z_2}q^{\frac{1}{2}}+\frac{z_1^4-(-3+t^4)z_1^3z_2-2(-3+t^4)z_1^2z_2^2-(-3+t^4)z_1z_2^3+z_2^4}{t^4z_1^2z_2^2}q+\cdots$	$\mathcal{O}(q^2)$
4d $U(5)\vert$3d $U(2)$	$1+\frac{2}{t^2}q^{\frac{1}{2}}+\frac{(z_1^2+z_2^2)}{t^3z_1z_2}q^{\frac{3}{4}}+\left(-2+\frac{5}{t^4}\right)q+\frac{(-3+t^4)(z_1+z_2^2)}{t^5z_1z_2}q^{\frac{5}{4}}+\cdots$	$\mathcal{O}(q^{\frac{3}{2}})$
4d $U(3)\vert$3d $U(2)\times U(1)$	$1+\frac{\left(z_1^2(z_2+z_3)+z_2z_3(z_2+z_3)+z_1(z_2^2+3z_2z_3+z_3^2)\right)}{t^2z_1z_2z_3}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(1)\vert$3d $(1)-[2]$	$1+\frac{2x_1x_2+t^4(x_1+x_2)^2}{t^2x_1x_2}q^{\frac{1}{2}}+\frac{z_1^2+z_2^2}{t^3z_1z_2}q^{\frac{3}{4}}+\frac{3x_1^2x_2^2-3t^4x_1^2x_2^2+t^8(x_1^2+x_1x_2+x_2^2)^2}{t^4x_1^2x_2^2}q+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)\vert$3d $(1)-[2]$	$1+\frac{2x_1x_2+t^4(x_1+x_2)^2}{t^2x_1x_2}q^{\frac{1}{2}}+\left(-2+\frac{t^4(x_1^2+x_1x_2+x_2^2)^2}{x_1^2x_2^2}+\frac{4+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^4}\right)q+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)\vert$3d $(2)-[4]$	$1+\left(\frac{2}{t^2}+\frac{t^2(x_2x_3x_4(x_2+x_3+x_4)+x_1^2(x_3x_4+x_2(x_3+x_4))}{+x_1(x_2^2(x_3+x_4)+x_3x_4(x_3+x_4)+x_2(x_3^2+4x_3x_4+x_4^2))}\right)q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^{\frac{3}{2}})$
4d $U(3)\vert$3d $(2)-[4]$	$1+\left(\frac{2}{t^2}+\frac{t^2(x_2x_3x_4(x_2+x_3+x_4)+x_1^2(x_3x_4+x_2(x_3+x_4))}{+x_1(x_2^2(x_3+x_4)+x_3x_4(x_3+x_4)+x_2(x_3^2+4x_3x_4+x_4^2))}\right)q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^{\frac{3}{2}})$
4d $U(1)\vert T[SU(3)]$	$1+\frac{3+t^4\left(3+x_1\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{x_2+x_3}{x_1}\right)+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^2}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(2)\vert T[SU(3)]$	$1+\frac{3+t^4\left(3+x_1\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{x_2+x_3}{x_1}\right)+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^2}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$

(A5)

b. Hafl-BPS interfaces

We have confirmed that the following pairs of half-indices of 3D dual interfaces agree up to certain orders of q.

Half-BPS interfaces	Series expansions	Up to orders
4d $U(1)|$3d $U(1)|$4d $U(1)$	$1+\frac{z_1^2+(3+t^4)z_1z_2+z_2^2}{t^2z_1z_2}q^{\frac{1}{2}}+\frac{z_1^4-(-3+t^4)z_1^3z_2+(6-3t^4+tt^8)z_1^2z_2^2-(-3+t^4)z_1z_2^3+z_2^4}{t^4z_1^2z_2^2}q+\cdots$	$\mathcal{O}(q^5)$
4d $U(1)|$3d $U(1)^2|$4d $U(1)$	$1+\frac{z_1^2(z_2+z_3)+z_2z_3(z_2+z_3)+z_1(z_2^2+(4+t^4)z_2z_3+z_3^2)}{t^2z_1z_2z_3}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(2)|$3d $U(2)|$4d $U(2)$	$1+\frac{z_1^2+(3+t^4)z_1z_2+z_2^2}{t^2z_1z_2}q^{\frac{1}{2}}+\frac{z_1^4+4z_1^3z_2+2(5+t^8)z_1^2z_2^2+4z_1z_2^3+z_2^4}{t^4z_1^2z_2^2}q+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d $U(2)^2|$4d $U(2)$	$1+\frac{z_1^2(z_2+z_3)+z_2z_3(z_2+z_3)+z_1(z_2^2+(4+t^4)z_2z_3+z_3^2)}{t^2z_1z_2z_3}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^1)$
4d $U(1)|$3d $U(1)|$4d $U(2)$	$1+\frac{3+t^4}{t^2}q^{\frac{1}{2}}+\frac{z_1^2+z_2^2}{t^3z_1z_2}q^{\frac{3}{4}}+\left(-2+\frac{7}{t^4}+t^4\right)q-\frac{(-3+t^4)(z_1^2+z_2^2)}{t^5z_1z_2}q^{\frac{5}{4}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d $U(1)|$4d $U(2)$	$1+\frac{3+t^4}{t^2}q^{\frac{1}{2}}+\left(-1+t^4+\frac{8+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^4}\right)q+\frac{-(-3+t^4)z_1^2+(16-5t^4+t^{12})z_1z_2-(-3+t^4)z_2^2}{t^6z_1z_2}q^{\frac{3}{2}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d $U(1)|$4d $U(3)$	$1+\frac{3+t^4}{t^2}q^{\frac{1}{2}}+\left(-1+\frac{8}{t^4}+t^4\right)q+\frac{z_1^2+z_2^2}{t^5z_1z_2}q^{\frac{5}{4}}+\frac{17-4t^4+t^{12}}{t^6}q^{\frac{3}{2}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(1)|$3d $U(2)|$4d $U(2)$	$1+\frac{z_1^2+z_2^2}{tz_1z_2}q^{\frac{1}{4}}+\frac{z_1^4+(4+t^4)z_1^2z_2^2+z_2^4}{t^2z_1^2z_2^2}q^{\frac{1}{2}}+\frac{z_1^6+5z_1^4z_2^2+5z_1^2z_2^4+z_2^6}{t^3z_1^3z_2^3}q^{\frac{3}{4}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d $U(2)|$4d $U(3)$	$1+\left(\frac{3}{t^2}+t^2\right)q^{\frac{1}{2}}+\left(\frac{z_1}{t^3z_2}+\frac{z_2}{t^3z_1}\right)q^{\frac{3}{4}}+\left(\frac{9}{t^4}+2t^4\right)q+\frac{4(z_1^2+z_2^2)}{t^5z_1z_2}q^{\frac{5}{4}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(1)|$3d $U(2)|$4d $U(3)$	$1+\frac{z_1^2+(3+t^4)z_1z_2+z_2^2}{t^2z_1z_2}q^{\frac{1}{2}}+\frac{z_1^4+4z_1^3z_2+(9-t^4+t^8)z_1^2z_2^2+4z_1z_2^3+z_2^4}{t^4z_1^2z_2^2}q+\cdots$	$\mathcal{O}(q^2)$
4d $U(1)|$3d $U(2)|$4d $U(4)$	$1+\frac{3+t^4}{t^2}q^{\frac{1}{2}}+\frac{z_1^2+z_2^2}{t^3z_1z_2}q^{\frac{3}{4}}+\left(-1+\frac{8}{t^4}+t^4\right)q+\frac{4(z_1^2+z_2^2)}{t^5z_1z_2}q^{\frac{5}{4}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(1)|$3d $U(2)|$4d $U(5)$	$1+\frac{3+t^4}{t^2}q^{\frac{1}{2}}+\left(-1+t^4+\frac{8+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^4}\right)q+\frac{4z_1^2+(17-4t^4+t^{12})z_1z_2+4z_2^2}{t^6z_1z_2}q^{\frac{3}{2}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(2)|$3d $U(3)|$4d $U(3)$	$1+\frac{z_1^2+z_2^2}{tz_1z_2}q^{\frac{1}{4}}+\frac{z_1^4+(4+t^4)z_1^2z_2^2+z_2^4}{t^2z_1^2z_2^2}q^{\frac{1}{2}}+\frac{z_1^6+5z_1^4z_2^2+5z_1^2z_2^4+z_2^6}{t^3z_1^3z_2^3}q^{\frac{3}{4}}+\cdots$	$\mathcal{O}(q^{\frac{3}{2}})$
4d $U(1)|$3d SQED$_1|$4d $U(1)$	$1+\frac{3+2t^4}{t^2}q^{\frac{1}{2}}+\frac{z_1^2+z_2^2}{t^3z_1z_2}q^{\frac{3}{4}}+\left(-2+\frac{6}{t^4}+3t^4\right)q-\frac{(-3+t^4)(z_1^2+z_2^2)}{t^5z_1z_2}q^{\frac{5}{4}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(1)|$3d SQED$_2|$4d $U(1)$	$1+\left(\frac{3}{t^2}+t^2\left(3+\frac{x_1}{x_2}+\frac{x_2}{x_1}\right)\right)q^{\frac{1}{2}}+\left(-1+\frac{x_1}{x_2}+\frac{x_2}{x_1}+t^4\left(6+\frac{x_1^2}{x_2^2}+\frac{3x_1}{x_2}+\frac{3x_2}{x_1}+\frac{x_2^2}{x_1^2}\right)+\frac{6+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^4}\right)q+\cdots$	$\mathcal{O}(q^3)$
4d $U(1)|$3d SQED$_3|$4d $U(1)$	$1+\left(\frac{3}{t^2}+t^2\left(4+x_1\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{x_2+x_3}{x_1}\right)\right)q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d SQED$_1|$4d $U(3)$	$1+\frac{3+2t^4}{t^2}q^{\frac{1}{2}}+\frac{8+3t^8}{t^4}q+\frac{z_1^2+(17-t^4+4t^{12})z_1z_2+z_2^2}{t^6z_1z_2}q^{\frac{3}{2}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d SQED$_2|$4d $U(3)$	$1+\left(\frac{3}{t^2}+t^2\left(3+\frac{x_1}{x_2}+\frac{x_2}{x_1}\right)\right)q^{\frac{1}{2}}+\left(1+\frac{8}{t^4}+\frac{x_1}{x_2}+\frac{x_2}{x_1}+t^4\left(6+\frac{x_1^2}{x_2^2}+\frac{3x_1}{x_2}+\frac{3x_2}{x_1}+\frac{x_2^2}{x_1^2}\right)\right)q+\cdots$	$\mathcal{O}(q^3)$
4d $U(2)|$3d SQED$_3|$4d $U(3)$	$1+\left(\frac{3}{t^2}+t^2\left(4+x_1\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{x_2+x_3}{x_1}\right)\right)q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^3)$
4d $U(1)_{2HM}|$3d $U(1)|$4d $U(1)_{2HM}$	$1+\frac{3+t^4\left(5+\frac{x_1}{x_2}+\frac{x_2}{x_1}+\frac{x_3}{x_4}+\frac{x_4}{x_3}\right)+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^2}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(1)_{2HM}|$3d SQED$_1|$4d $U(1)_{2HM}$	$1+\left(\frac{3}{t^2}+t^2\left(6+\frac{x_1}{x_2}+\frac{x_2}{x_1}+\frac{x_3}{x_4}+\frac{x_4}{x_3}\right)\right)q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$
4d $U(1)_{3HM}|$3d $U(1)|$4d $U(1)_{3HM}$	$1+\frac{3+t^4\left(7+x_1\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{x_2+x_3}{x_1}+\frac{x_4}{x_5}+\frac{x_5}{x_4}+\frac{x_4}{x_6}+\frac{x_5}{x_6}+\frac{x_6}{x_4}+\frac{x_6}{x_5}\right)+\frac{z_1}{z_2}+\frac{z_2}{z_1}}{t^2}q^{\frac{1}{2}}+\cdots$	$\mathcal{O}(q^2)$

(A6)

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