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Magnetic small-angle neutron scattering

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Small-angle neutron scattering (SANS) is one of the most important techniques for microstructure determination, being utilized in a wide range of scientific disciplines, such as materials science, physics, chemistry, and biology. The reason for its great significance is that conventional SANS is probably the only method capable of probing structural inhomogeneities in the bulk of materials on a mesoscopic real-space length scale, from roughly 1 – 300 nm. Moreover, the exploitation of the spin degree of freedom of the neutron provides SANS with a unique sensitivity to study magnetism and magnetic materials at the nanoscale. As such, magnetic SANS ideally complements more real-space and surface-sensitive magnetic imaging techniques, e.g., Lorentz transmission electron microscopy, electron holography, magnetic force microscopy, Kerr microscopy, or spin-polarized scanning tunneling microscopy. In this review article we summarize the recent applications of the SANS method to study magnetism and magnetic materials. This includes a wide range of materials classes, from nanomagnetic systems such as soft magnetic Fe-based nanocomposites, hard magnetic Nd–Fe–B-based permanent magnets, magnetic steels, ferrofluids, nanoparticles, and magnetic oxides, to more fundamental open issues in contemporary condensed matter physics such as skyrmion crystals, noncollinear magnetic structures in noncentrosymmetric compounds, magnetic/electronic phase separation, and vortex lattices in type-II superconductors. Special attention is paid not only to the vast variety of magnetic materials and problems where SANS has provided direct insight, but also to the enormous progress made regarding the micromagnetic simulation of magnetic neutron scattering.

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I. INTRODUCTION

Small-angle neutron scattering (SANS) is a particularly powerful and unique technique, which allows to investigate microstructural (density and composition) as well as magnetic inhomogeneities in the volume of materials and on a mesoscopic length scale between a few and a few hundred nanometers. This is a size regime in which many macroscopic material properties are realized. From the historical point of view, experimental and theoretical progress in the domain of small-angle scattering is closely connected to the development of laboratory small-angle x-ray scattering (SAXS) methods (Guinier and Fournet, 1955). Since SANS and its x-ray counterpart are well developed and widely acknowledged in diverse fields of science, such as materials science, physics, chemistry, and biology, there exists an enormous body of research literature. The standard references for nuclear (nonmagnetic) SANS and SAXS are the well-known textbooks by Feigin and Svergun, 1987; Gille, 2014; Glatter and Kratky, 1982; Guinier and Fournet, 1955; and Svergun et al., 2013. For a selection of reviews on various topics of small-angle scattering, for instance, on polymers, disordered and porous materials, colloidal systems, ferrofluids, magnetic materials, superconductors, ceramics, biological structures, and precipitates in metallic alloys and composites, we refer to Allen, 2005; Avdeev and Aksenov, 2010; Avdeev et al., 2015; Bates, 1988; Chen and Lin, 1987; Chen et al., 1988; Eskildsen et al., 2011; Fitzsimmons et al., 2004; Fratzl, 2003; Fritz and Glatter, 2006; Gerold and Kostorz, 1978; Hammond, 2010; Hayter, 1988; Higgins and Stein, 1978; Hollamby, 2013; Jacrot, 1976; Kostorz, 1991, 2014; Laver, 2012; Martin and Hurd, 1987; Mehrichenko and Wignall, 2007; Michels, 2014; Michels et al., 2014; Michaels and Weißmüller, 2008; Page, 1988; Pauw, 2013; Pedersen, 1997; Radlinski et al., 2004; Schellen and Hendricks, 1978; Schmatz et al., 1974; Schmidt, 1991; Stuhrmann, 2004; Svergun and Koch, 2003; Thiyagarajan, 2003; Wagner and Kohlbrecher, 2005; Wiedemann, 2002, 2010; and Wignall and Mehrichenko, 2005.

Only with the advent of high-brilliance neutron sources and the concomitant development of the first dedicated SANS instruments (Ibel, 1976; Schmatz et al., 1974), did it become possible to explore magnetism and superconductivity on a mesoscopic length scale by means of SANS. With magnetic SANS playing a pivotal role, the rapidly evolving progress in the field of magnetism and superconductivity (Bader, 2006; Bauer and Pfleiderer, 2010; Brandt, 1995; Furrer and Waldmann, 2013; Hellman et al., 2017; Nagaosa and Tokura, 2013; Nisoli et al., 2013; Sellmyer and Skomski, 2006; Skomski, 2003) is naturally accompanied by the quest to resolve ever finer details of the magnetic microstructure. As such, SANS ideally complements well-known and established methods for characterizing and analyzing the static and dynamic spin structure of nanomaterials, including neutron diffraction and spectroscopy (Chatterji, 2006; Furrer et al., 2009), Lorentz and Kerr microscopy (Hubert and Schäfer, 1998), magnetic force microscopy (Koblischka and Hartmann, 2003; Meyer et al., 2004), spin-polarized scanning tunneling microscopy (Wiesendanger, 2016, 2009), or x-ray magnetic circular dichroism in combination with photoelectron emission microscopy (Bauer, 2014; Cheng and Keavney, 2012; Locatelli and Bauer, 2008).

In this review article we provide a summary of the recent applications of the SANS method to study magnetism and magnetic materials. This covers many of the most important classes of magnetic materials and addresses a wide range of topics from fundamental questions in condensed matter physics to applied materials science. The review is organized as follows: Sec. II revisits the basics of “classical” diffuse magnetic SANS, including a summary of recent theoretical and experimental progress regarding the spin structures of polycrystalline bulk ferromagnets and its relation to the conventional particle-matrix approach. We provide a compilation of the various unpolarized, half-polarized, and polarized SANS cross sections. Sections III and IV highlight the recent discoveries concerning the investigation of Nd–Fe–B-based permanent magnets and magnetic steels, whereas Sec. V covers magnetic nanoparticles and ferrofluids. In Sec. VI, we review the recent progress made in using full-scale micromagnetic simulations for the understanding of the fundamentals of magnetic SANS on multiphase systems. Section VII is concerned with the application of the magnetic SANS method to study complex magnetic systems which exhibit nanoscale magnetic inhomogeneity. These include magnetically/electronically phase-separated complex oxides and metal alloys, which have recently been extensively studied with SANS. Sections VIII and IX summarize the state-of-the-art of SANS research on skyrmion lattices, long-range noncollinear magnetic structures, and vortex lattices in type-II superconductors. Finally, Sec. X provides a brief summary and gives an outlook on future developments and chal-
lenges. In order to keep the sections self contained, each chapter provides an introductory paragraph for the reader.

II. MAGNETIC SANS: BASICS

We begin this section with a description of a typical SANS setup in Sec. II.A. In Sec. II.B the basic expressions for the various unpolarized and spin-polarized elastic SANS cross sections $d\Sigma/d\Omega$ will be displayed. We focus on the two most relevant scattering geometries which have the applied magnetic field $H_0$ either perpendicular or parallel to the wave vector $k_0$ of the incoming neutron beam. In the first Born approximation (Messiah, 1990), the magnetic contribution to $d\Sigma/d\Omega$ is fully determined by the three Cartesian Fourier components $M_{x,y,z}(\mathbf{q})$ of the magnetization vector field $M_{x,y,z}(\mathbf{r})$ of the sample. Using the continuum theory of micromagnetics (Sec. II.C), the functions $M_{x,y,z}(\mathbf{q})$ can be computed for bulk ferromagnets in the small-misalignment approximation, in this way providing closed-form expressions for any desired $d\Sigma/d\Omega$ as a function of momentum-transfer vector $\mathbf{q}$, applied magnetic field, magnetic interaction parameters (exchange, anisotropy, magnetostatics), and microstructural quantities such as particle size, shape, and texture; selected experimental data will be discussed in order to underline the theoretical approach. Finally, Sec. II.D establishes the connection to the conventional particle-matrix description of magnetic SANS, which assumes homogeneously magnetized domains. It is emphasized that the SANS cross sections which are introduced in this section are the ones for diffuse magnetic SANS, while the well-known equations for elastic magnetic Bragg diffraction, relevant for the discussion of spiral magnetic structures, skyrmions, or vortex lattices in superconductors, are introduced in Sec. VIII.

A. Description of the SANS Setup

Figure 1 depicts the typical SANS setup along with schematics of the two most commonly used scattering geometries. By means of a mechanical velocity selector or chopper-based time-of-flight methods the incoming wavelength band (typically $\lambda \sim 3 - 30 \text{Å}$) is selected from a cold neutron beam [energy range: $\sim 0.1 - 10 \text{meV} \sim 1 - 120 \text{K}$ (Schöber, 2014)], provided by a spallation or a reactor source. The mean wavelength and wavelength resolution can be tuned $|\Delta \lambda / \lambda | \sim 1 - 30 \%$ (FWHM), depending on the rotational speed and tilting angle of the selector or the duty cycle and frame overlap of the chopper system. In the evacuated pre-sample flight path a set of apertures collimates the beam. A particular strength of the SANS technique is that experiments can be conducted under rather flexible sample environments (e.g., temperature, electric and magnetic field, pressure, neutron polarization, time-resolved data acquisition). The typical size of the irradiated area of sample is of the order of $1 \text{cm}^2$.

Two-dimensional position-sensitive detector arrays, moving along rails in an evacuated post-sample flight path (sample-to-detector distance: $\sim 1 - 40 \text{m}$), count the scattered neutrons during acquisition times ranging between a few minutes and a few hours. The recorded neutron counts (in each pixel element) are corrected for detector dead time, dark current and efficiency, sample transmission and background scattering and are normalized to incident-beam flux. A solid-angle correction is applied to the data which corrects for the planar geometry of the detector (Glinka et al., 1998; Karge et al., 2017). The size of an individual pixel element of the detector is typically $\leq 10 \text{mm} \times 10 \text{mm}$, so that the related resolution effects become negligible. The scattering cross section of the sample is obtained by comparing the corrected signal to a reference sample (e.g., water, polystyrene, porous silica, vanadium single crystal) of known cross section. The data-reduction procedure provides the macroscopic differential scattering cross section $d\Sigma/d\Omega$ of the sample.
in absolute units (typically cm$^{-1}$) and as a function of the magnitude and orientation of the momentum-transfer or scattering vector $q$ (see Fig. 1). In order to conveniently present the neutron data, one often carries out a so-called azimuthal averaging procedure, whereby the data at a constant magnitude of $q$ are integrated within a certain angular range (e.g., over $0 \leq \theta \leq 2\pi$); this yields $d\Sigma/d\Omega$ as a function of $|q| = q$. The uncertainty in the cross sections determined by this procedure is estimated to be $5 - 10\%$ (Glinka et al., 1998; Rennie et al., 2013).

When $\psi$ denotes the scattering angle [incident beam $k_0 \parallel e_x$, see Fig. 1(a)], and the azimuthal angle $\theta$ is used to specify the orientation of $q$ on the two-dimensional detector [with $q = 2k_0 \sin(\psi/2)$], the scattering vector is found to be

$$q = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = q \begin{pmatrix} -\sin(\psi/2) \\ \cos(\psi/2) \sin \theta \\ \cos(\psi/2) \cos \theta \end{pmatrix} = k_0 \begin{pmatrix} \cos \psi - 1 \\ \sin \psi \sin \theta \\ \sin \psi \cos \theta \end{pmatrix}.\tag{1}$$

For small-angle scattering $\psi \lesssim 5 - 10^\circ$, so that the magnitude of the component of $q$ along the incident-beam direction, $q \sin(\psi/2)$, is much smaller than the other two components. The three-dimensional scattering vector is therefore approximated by a two-dimensional one. This approximation, which violates the condition for elastic scattering, the data-reduction procedure, the performance of SANS instruments, or the treatment of multiple scattering see Allen and Berk, 1994; Barker and Pedersen, 1995; Chen and Lin, 1987; Dewhurst, 2008; Dewhurst et al., 2016; Glinka et al., 1998; Kohlbrecher and Wagner, 2000; May, 1994; Mazumder et al., 2001; Mildner and Cubitt, 2012; Mülbauer et al., 2016; Pedersen et al., 1990; Saroun, 2007; and Schelten and Schmatz, 1980.

The neutrons incident on the sample may be polarized by means of a (supermirror transmission) polarizer and the initial neutron polarization can be reverted by $180^\circ$ using a (radio-frequency) spin flipper (Bazhenov et al., 1993; Keller et al., 2000) (see Fig. 1). In order to discriminate the neutron spin state after interaction with the sample, the $^3$He spin filter (Batz et al., 2005) acts as neutron spin analyzer and, correspondingly, is installed behind the sample (sometimes inside the detector housing). Magnetic guide fields of the order of 1 mT serve to maintain the polarization on the path between polarizer and $^3$He filter. Progress in the development of $^3$He spin filters (Petoukhov et al., 2006) allows one to perform routinely uniaxial (also called longitudinal or one-dimensional) neutron-polarization analysis on a SANS instrument, for instance, at SANS-1 and KWS-1 at the Heinz Maier-Leibnitz Zentrum, at D22 and D33 at the Institut Laue-Langevin, or at NG3 and NG7 at the NIST Center for Neutron Research. We emphasize that the above described setup of supermirror transmission polarizer (P), rf spin flipper (F), and $^3$He spin analyzer (A), represents the most commonly installed configuration for uniaxial polarization analysis at SANS instruments. There exist, of course, many other neutron instrumentation devices for polarizing neutron beams and for turning the neutron-spin direction (Williams, 1988).

In uniaxial polarization analysis (Moon et al., 1969), it becomes possible to measure four intensities that connect two neutron-spin states. The externally applied magnetic field at the sample position defines the quantization axis for both incident and scattered polarization, whereby the scattered neutron may undergo a spin-reversing event due to the magnetic interaction with the sample. Following Moon et al., 1969, the four spin-resolved scattering cross sections are the two non-spin-flip quantities $d^\Sigma_{\uparrow \uparrow}/d\Omega$ and $d^\Sigma_{\downarrow \downarrow}/d\Omega$, and the two spin-flip cross sections $d^\Sigma_{\uparrow \downarrow}/d\Omega$ and $d^\Sigma_{\downarrow \uparrow}/d\Omega$. When the rf flipper is off (inactive), we measure, depending on the spin state of the $^3$He filter, the non-spin-flip or the spin-flip cross section $d^\Sigma_{\uparrow \uparrow}/d\Omega$ or $d^\Sigma_{\downarrow \downarrow}/d\Omega$. Likewise, when the flipper is on, we either measure $d^\Sigma_{\uparrow \downarrow}/d\Omega$ or $d^\Sigma_{\downarrow \uparrow}/d\Omega$. The corresponding expressions for the cross sections are denoted as the POLARIS equations (see Sec. II.B.4).
SANS experiments with a polarized incident beam only and no detection of the polarization of the scattered neutrons provide access to the half-polarized cross sections (denoted SANSPOL) \( \frac{d\Sigma}{d\Omega}^+ \) and \( \frac{d\Sigma}{d\Omega}^- \), which combine non-spin-flip and spin-flip scattering contributions. In particular (see Sec. II.B.3),

\[
\begin{align*}
\frac{d\Sigma^+}{d\Omega} &= \frac{d\Sigma^{++}}{d\Omega} + \frac{d\Sigma^{+-}}{d\Omega}, \\
\frac{d\Sigma^-}{d\Omega} &= \frac{d\Sigma^{-+}}{d\Omega} + \frac{d\Sigma^{-+}}{d\Omega}.
\end{align*}
\]

The difference between “spin-up” and “spin-down” SANSPOL cross sections yields information on the polarization-dependent nuclear-magnetic and chiral scattering terms (see Sec. V.A). As demonstrated, e.g., in Keller et al., 2000 on an FeSO₄ glass ceramic, this difference allows one to highlight weak magnetic contributions relative to strong nuclear scattering (or vice versa). Finally, the unpolarized SANS cross section is obtained as (see Sec. II.B.2)

\[
\frac{d\Sigma}{d\Omega} = \frac{1}{2} \left( \frac{d\Sigma^+}{d\Omega} - \frac{d\Sigma^-}{d\Omega} \right) = \frac{1}{2} \left( \frac{d\Sigma^{++}}{d\Omega} + \frac{d\Sigma^{+-}}{d\Omega} + \frac{d\Sigma^{-+}}{d\Omega} + \frac{d\Sigma^{-+}}{d\Omega} \right).
\]

We note that both the unpolarized as well as the half-polarized cross sections can be measured directly (see below). For more information on polarized neutron scattering (and on spherical neutron polarimetry), we refer to the classic papers by Blume, 1963; Brown, 2006; Halpern and Johnson, 1939; Izyumov and Maleev, 1962; Maleev, 1961; Maleev et al., 1963; Marshall and Lowde, 1968; Mezei, 1986; Moon et al., 1969; Okorokov et al., 1978; Schärf and Capellmann, 1993; Schweizer, 2006; Shull et al., 1951; and Tasset, 1989 and to the textbooks by Hicks, 1995; Lovesey, 1984; Squires, 1978; and Williams, 1988.

Although most of the magnetic SANS discussion in this review article is treated within the elastic approximation, quasielastic/inelastic scattering contributions are relevant for certain aspects of Sec. VII and Sec. VIII. We refer to Grigoriev et al., 2015; Maleev, 1965; and Okorokov et al., 1986 for a detailed discussion of inelastic SANS.

B. SANS cross sections

1. General considerations

We restrict the attention to the two commonly used scattering geometries with externally applied magnetic field \( \mathbf{H}_0 \) either perpendicular [Fig. 1(a)] or parallel [Fig. 1(b)] to the incoming neutron beam \( \mathbf{k}_0 \). We adopt a Cartesian laboratory coordinate system with corresponding unit vectors \( \mathbf{e}_x \), \( \mathbf{e}_y \), and \( \mathbf{e}_z \); the field \( \mathbf{H}_0 \) is assumed to be always parallel to \( \mathbf{e}_z \). For the perpendicular scattering geometry (\( \mathbf{k}_0 \perp \mathbf{H}_0 \)), the angle \( \theta \) on the two-dimensional detector is then measured between \( \mathbf{H}_0 \) and the momentum-transfer vector \( \mathbf{q} \approx \{0, q_y, q_z\} = q\{\sin \theta, \cos \theta, 0\} \), whereas for \( \mathbf{k}_0 \parallel \mathbf{H}_0, \theta \) is the angle between \( \mathbf{e}_x \) and \( \mathbf{q} \approx \{q_x, q_y, 0\} = q\{\cos \theta, \sin \theta, 0\} \).

The discrete atomic structure of condensed matter is generally of no relevance for SANS, such that the magnetization state of the sample can be represented by a continuous magnetization vector field which is defined at each position \( \mathbf{r} \) inside the material. Magnetic SANS is then a consequence of nanoscale variations in both the orientation and/or magnitude of the magnetization.

\[
\mathbf{M}(\mathbf{r}) = \{M_x(\mathbf{r}), M_y(\mathbf{r}), M_z(\mathbf{r})\}
\]

represents the three-dimensional Fourier transform of the magnetization

\[
\mathbf{M}(\mathbf{q}) = \{\tilde{M}_x(\mathbf{q}), \tilde{M}_y(\mathbf{q}), \tilde{M}_z(\mathbf{q})\}
\]

where \( i^2 = -1 \), and \( \mathbf{q} = \{q_x, q_y, q_z\} \) is the wave vector. With \( \mathbf{H}_0 \parallel \mathbf{e}_z \), \( \tilde{M}_z \) denotes the longitudinal magnetization Fourier coefficient, whereas \( \tilde{M}_x \) and \( \tilde{M}_y \) are the transversal components, giving rise to so-called spin-misalignment scattering. For many magnetization configurations it turns out that the functions \( \tilde{M}_{x,y,z} \) are real-valued, but, for instance, for structures lacking space inversion symmetry they may pick up an imaginary part (Michels et al., 2016); complex-conjugated quantities are marked by a superscript asterisks (*). The nuclear SANS cross section, which is due to nanoscale density and/or compositional fluctuations, is characterized by the Fourier transform \( \tilde{N}(\mathbf{q}) \) of the continuous scattering-length density \( N(\mathbf{r}) \).

For the understanding of magnetic neutron scattering, the Halpern-Johnson or magnetic-interaction vector

\[
\mathbf{Q} = \tilde{\mathbf{q}} \times (\tilde{\mathbf{q}} \times \tilde{\mathbf{M}}(\mathbf{q})) = \tilde{\mathbf{q}} \left( \tilde{\mathbf{q}} \cdot \tilde{\mathbf{M}}(\mathbf{q}) \right) - \tilde{\mathbf{M}}(\mathbf{q}), \quad (7)
\]

where \( \tilde{\mathbf{q}} \) is the unit scattering vector, is of utmost importance (Halpern and Johnson, 1939); it is a manifestation of the dipolar origin of magnetic neutron scattering and it emphasizes the fact that only the component of \( \mathbf{M} \) which

\[\footnote{The case of a smooth modulation of localized or itinerant spins on top of a discrete atomic lattice is discussed in Sec. VIII.}\]
is perpendicular to $\mathbf{q}$ is relevant for magnetic scattering.\footnote{We note that different symbols for the Halpern-Johnson vector such as $\mathbf{M}_1$, $\mathbf{Q}_1$, $\mathbf{S}_1$, or $\mathbf{q}$, as in the original paper by Halpern and Johnson, 1939, can be found in the literature.}

For $\mathbf{k}_0 \perp \mathbf{H}_0$ and $\mathbf{k}_0 \parallel \mathbf{H}_0$, we obtain, respectively,

$$
\mathbf{Q}_\perp = \begin{cases} 
-\widetilde{M}_x \\
-\widetilde{M}_x \cos^2 \theta + \widetilde{M}_y \sin \theta \cos \theta \\
\widetilde{M}_y \sin \theta \cos \theta - \widetilde{M}_z \sin^2 \theta 
\end{cases}, \hspace{1cm} (8)
$$

$$
\mathbf{Q}_\parallel = \begin{cases} 
-\widetilde{M}_x \sin^2 \theta + \widetilde{M}_y \sin \theta \cos \theta \\
\widetilde{M}_x \sin \theta \cos \theta - \widetilde{M}_y \cos^2 \theta \\
-\widetilde{M}_z 
\end{cases}. \hspace{1cm} (9)
$$

By assuming perfect neutron optics and that the incident neutron polarization is along $\mathbf{e}_z$, i.e., $\mathbf{P} = \mathbf{e}_z$, the elastic non-spin-flip and spin-flip cross sections can be formally written as (Moon \textit{et al.}, 1969)

$$
\frac{d\Sigma^{\pm \pm}}{dQ} \sim |\mathbf{N}|^2 \pm \left( \mathbf{N} \mathbf{Q}_z^* + \mathbf{N}^* \mathbf{Q}_z \right) + |\mathbf{Q}_z|^2, \hspace{1cm} (10)
$$

$$
\frac{d\Sigma^{\pm \mp}}{dQ} \sim |\mathbf{Q}_z|^2 + |\mathbf{Q}_y|^2 \pm i \left( \mathbf{Q}_z \mathbf{Q}_y^* - \mathbf{Q}_y \mathbf{Q}_z^* \right). \hspace{1cm} (11)
$$

Several comments are required (Blume, 1963): It is seen that the transversal components $\mathbf{Q}_x$ and $\mathbf{Q}_y$ give rise to spin-flip scattering, while the longitudinal component $\mathbf{Q}_z$ results in non-spin-flip scattering. We also note that the nuclear coherent scattering, the nuclear incoherent scattering which is due to isotope disorder, as well as 1/3 of the nuclear-spin incoherent scattering are all non-spin-flip scattering; the remaining 2/3 of the nuclear-spin incoherent scattering reverses the neutron spin, but, since its magnitude is usually small relative to the coherent magnetic and nuclear SANS of nonhydrogenated samples relevant here (Stuhrmann, 2004) and since it only gives rise to a constant $q$-independent scattering contribution, we ignore it in the spin-resolved channels. Furthermore, if we set $\theta = 0^\circ$ in Eq. (8), which corresponds to the case that $\mathbf{q} \parallel \mathbf{P}$, we see that $\mathbf{Q}_\perp = \{-\widetilde{M}_x, -\widetilde{M}_y, 0\}$, so that nuclear coherent and magnetic scattering are fully separated. In the case $\mathbf{k}_0 \parallel \mathbf{H}_0$ [Eq. (9)], spin-flip scattering probes only the transversal magnetization components $\mathbf{M}_{x,y}$.

In the equations for the cross sections that follow, $V$ denotes the scattering volume, $K = 8\pi^3 V^{-1} b_H^2$, where $b_H = 2.70 \times 10^{-15} \text{ m}^{-1} \mu_B$, is a constant (with $\mu_B$ the Bohr magneton), which relates the atomic magnetic moment $\mu_a$ to the atomic magnetic scattering length $b_m$ (Moon \textit{et al.}, 1969): 

$$
b_m = \frac{\gamma_n \gamma_0 \mu_a}{2 \mu_B} f(\mathbf{q}) \cong 2.70 \times 10^{-15} \text{ m} \frac{\mu_a}{\mu_B} f(\mathbf{q}) \cong b_H \mu_a,
$$

where $\gamma_n = 1.913$ denotes the neutron magnetic moment expressed in units of the magnetic moment, $r_0 = 2.818 \times 10^{-15} \text{ m}$ is the classical radius of the electron, and $f(\mathbf{q})$ is the normalized atomic magnetic form factor; note that $f \equiv 1$ along the forward direction. By inserting Eqs. (8) and (9) into the expressions for the non-spin-flip and spin-flip cross sections, Eqs. (10) and (11), and by noting the relations between the various cross sections, Eqs. (3) and (4), one can conveniently express the SANS cross sections in terms of the Cartesian Fourier components $\mathbf{M}_{x,y}$ of the magnetization. We use the following abbreviations for the magnetic and nuclear-magnetic interference terms (subscripts $\perp$ and $\parallel$ refer to the respective scattering geometry): $CT_{yz} = \widetilde{M}_y \widetilde{M}_x^* + \widetilde{M}_x \widetilde{M}_y^*$, $CT_{xy} = \widetilde{M}_x \widetilde{M}_y^* + \widetilde{M}_y \widetilde{M}_x^*$, $CT_{yz} \overline{\mathbf{N}} = 0$, and $CT_{\widetilde{M} \overline{\mathbf{N}}}$.

In actual SANSPOL and POLARIS experiments the neutron optics do not work perfectly and polarization corrections become necessary. The incident beam polarization efficiency is denoted by $P = I^+/I^+ + I^-$, where $I^\pm$ are, respectively, the number of neutrons with spins aligned antiparallel and parallel with respect to $\mathbf{H}_0$ and $\epsilon^\pm$ is the efficiency of the spin flipper ($\epsilon^+ = 0$ for flipper off and $\epsilon^- = \epsilon \equiv 1$ for flipper on); note that $P = 1/2$ for an unpolarized beam. The half-polarized SANS cross sections can be obtained directly and corrected for non-ideal neutron polarization provided that the parameters $P$ and $\epsilon$ are known from reference measurements. For the spin-resolved (POLARIS) cross sections, it is necessary to measure all four partial cross sections in order to correct for spin leakage between the different channels (Wildes, 2006). Such corrections can for example be accomplished by means of the $\text{Be}r\text{SANS}$ (Keiderling, 2002; Keiderling \textit{et al.}, 2008), Pol-Corr (Krycka \textit{et al.}, 2012a), and GRAS$_\text{sanP}$ (Dewhurst, 2016) software tools. The relevant expressions are as follows:

2. Unpolarized SANS

$$
\frac{d\Sigma_{\perp}}{dQ} = K \left( b_H^2 |\mathbf{N}|^2 + |\mathbf{M}_x|^2 + |\mathbf{M}_y|^2 \cos^2 \theta \right. \\
\left. + |\mathbf{M}_z|^2 \sin^2 \theta - CT_{yz} \sin \theta \cos \theta \right), \hspace{1cm} (12)
$$

$$
\frac{d\Sigma_{\parallel}}{dQ} = K \left( b_H^2 |\mathbf{N}|^2 + |\mathbf{M}_x|^2 \sin^2 \theta + |\mathbf{M}_y|^2 \cos^2 \theta \right. \\
\left. + |\mathbf{M}_z|^2 - CT_{xy} \sin \theta \cos \theta \right). \hspace{1cm} (13)
$$
3. Half-polarized SANS (SANSPOL)

\[ \frac{d\Sigma^\pm}{d\Omega} = K \left( b_H^2 |\vec{N}|^2 + |\vec{M}_z|^2 + |\vec{M}_y|^2 \cos^2 \theta \\
+ |\vec{M}_z|^2 \sin^2 \theta - CT_{yz} \sin \theta \cos \theta \\
+(2P-1)(2\epsilon^\pm - 1)b_H^{-1} CT_{N\vec{M}_i} \sin^2 \theta \right. \\
\left. - (2P-1)(2\epsilon^\pm - 1)b_H^{-1} CT_{N\vec{M}_i} \sin \theta \cos \theta \\
+i(2P-1)(2\epsilon^\pm - 1) \right), \quad (14) \]

where the chiral function \( \chi \) is given by\(^3\)

\[ \chi(q) = \left( \vec{M}_x \vec{M}_y - \vec{M}_y \vec{M}_x \right) \cos^2 \theta \\
- \left( \vec{M}_x \vec{M}_z - \vec{M}_z \vec{M}_x \right) \sin \theta \cos \theta. \quad (15) \]

\[ \frac{d\Sigma^{\pm\pm}}{d\Omega} = K \left( b_H^{-2} |\vec{N}|^2 + |\vec{M}_z|^2 \sin^2 \theta + |\vec{M}_y|^2 \cos^2 \theta \\
+ |\vec{M}_z|^2 - CT_{xy} \sin \theta \cos \theta \\
+(2P-1)(2\epsilon^\pm - 1)b_H^{-1} CT_{N\vec{M}_i} \right) \quad (16) \]

Note that \( \chi = 0 \) for \( k_0 \parallel H_0 \).

4. Polarized SANS (POLARIS)

\[ \frac{d\Sigma^{\pm\pm}}{d\Omega} = K \left( b_H^{-2} |\vec{N}|^2 + |\vec{M}_z|^2 \sin^2 \theta + |\vec{M}_y|^2 \cos^2 \theta \\
+ |\vec{M}_z|^2 - CT_{xy} \sin \theta \cos \theta \\
+(2P-1)(2\epsilon^\pm - 1)b_H^{-1} CT_{N\vec{M}_i} \right). \quad (17) \]

\[ \frac{d\Sigma^{\pm\mp}}{d\Omega} = K \left( |\vec{M}_z|^2 + |\vec{M}_y|^2 \cos^2 \theta + |\vec{M}_z|^2 \sin^2 \theta \cos^2 \theta \\
+ CT_{yz} \sin \theta \cos \theta + i\chi \right). \quad (18) \]

\[ \frac{d\Sigma^{\mp\pm}}{d\Omega} = K \left( |\vec{M}_z|^2 \sin^2 \theta + |\vec{M}_y|^2 \cos^2 \theta - CT_{xy} \sin \theta \cos \theta \right) \quad (19) \]

\[ \frac{d\Sigma^{\mp\mp}}{d\Omega} = K \left( |\vec{M}_z|^2 \sin^2 \theta + |\vec{M}_y|^2 \cos^2 \theta - CT_{xy} \sin \theta \cos \theta \right). \quad (20) \]

We reemphasize that the nuclear-spin incoherent scattering is ignored in the spin-flip cross sections.

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\(^3\) We note that in the neutron-diffraction community the chiral term is sometimes denoted with the symbol “C”.

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C. Magnetic SANS theory

The above expressions for the SANS cross sections depend on the Fourier components \( \vec{M}_{x,y,z} \) of the magnetization. The main task is to derive expressions for these functions based on a particular microstructural model. In this section we briefly summarize the recent developments regarding the analytical computation of the cross sections using the theory of micromagnetics (Honecker and Michels, 2013; Metlov and Michels, 2015, 2016; Mettus and Michels, 2015; Michels et al., 2016). Micromagnetics is a phenomenological continuum theory which has been developed in order to compute the magnetization vector field \( \mathbf{M} \) of an arbitrarily-shaped ferromagnetic body, provided that the applied magnetic field, the geometry of the ferromagnet, and the magnetic material’s parameters are known (Aharoni, 1996; Brown Jr., 1963; Kronmüller and Fähnle, 2003). The characteristic length scale which is addressed by micromagnetic calculations ranges between a few nanometers and a few hundreds of nanometers—a size regime that overlaps with the resolution range of the SANS technique. Pioneering work in this direction was performed by Kronmüller et al., 1963 who calculated the magnetic SANS due to spin disorder related to the strain fields of dislocations.

In an attempt to describe the magnetic SANS of a polycrystalline magnetic material, Michels et al., 2016 have considered the magnetization response to a spatially varying (local) saturation magnetization \( M_s = M_s(r) \) and magnetic anisotropy field \( \mathbf{H}_p = \mathbf{H}_p(r) \). Spatial variations in \( M_s \), e.g., at internal phase boundaries, give rise to magnetostatic stray fields which in turn result in a nanoscale magnetization nonuniformity (representing a contrast for magnetic SANS). Likewise, the field \( \mathbf{H}_p(r) \) is a source of spin disorder, since it increases the magnitude of the transversal magnetization components. The static equations of micromagnetics for the bulk can be conveniently written as (Aharoni, 1996; Brown Jr., 1963; Kronmüller and Fähnle, 2003)

\[ \mathbf{M}(r) \times \mathbf{H}_{\text{eff}}(r) = 0. \quad (21) \]

Equation (21) expresses the fact that at static equilibrium the torque on the magnetization \( \mathbf{M}(r) \) due to an effective magnetic field \( \mathbf{H}_{\text{eff}}(r) \) vanishes everywhere inside the material. The effective field,

\[ \mathbf{H}_{\text{eff}} = \mathbf{H}_0 + \mathbf{H}_d + \mathbf{H}_p + \mathbf{H}_{\text{ex}} + \mathbf{H}_{\text{DM}}, \quad (22) \]

is composed of a uniform applied magnetic field \( \mathbf{H}_0 \), the magnetostatic field \( \mathbf{H}_d(r) \), the magnetic anisotropy field \( \mathbf{H}_p(r) \), the exchange field \( \mathbf{H}_{\text{ex}} \), and of the field \( \mathbf{H}_{\text{DM}} \), which is due to the Dzyaloshinski-Moriya interaction (DMI). In the approach-to-saturation regime, when the sample consists of a single magnetic domain and small spin deviations from the mean magnetization are considered \( (M_x \ll M_s \text{ and } M_y \ll M_s; \mathbf{M}_s \parallel \mathbf{e}_z) \), the balance-of-torques equation can be linearized and a closed-form
solution for the transversal Fourier components can be found (Michels et al., 2016); these allow one to compute any magnetic SANS cross section.

As an example, Fig. 2 displays for \( \mathbf{k}_0 \perp \mathbf{H}_0 \) and an unpolarized beam the spin-misalignment SANS cross section \( d\Sigma_M/d\Omega \) arising due to transversal magnetization components. The term spin-misalignment SANS cross section refers to the cross section which remains at a particular value of the applied magnetic field when the total (nuclear and magnetic) \( d\Sigma /d\Omega \) in the saturated state is subtracted (compare Eq. (23) below). At the largest fields [Fig. 2(c) and 2(d)], \( d\Sigma_M /d\Omega \) exhibits maxima roughly along the diagonals of the detector—the so-called “clover-leaf” anisotropy—previously observed in the Fe-based two-phase alloy NANOPERM (compare, e.g., Fig. 3 in Michels et al., 2006). The position of the maxima in \( d\Sigma_M /d\Omega \) depend on \( q \) and \( H_0 \) (see also Fig. 11 in Michels et al., 2014). Reducing the field [Fig. 2(a) and 2(b)], one observes an elongation of the spin-misalignment scattering along the field direction, with a “flying-saucer-type” pattern taking over at small \( q \) and \( H_0 \). The sharp spike in Fig. 2(a) is due to the magnetostatic interaction; it was first predicted by Weissmüller et al., 1999 and experimentally observed by Péring et al., 2014 on a Nd–Fe–B-based permanent magnet (see Fig. 3).

The corresponding two-dimensional correlation functions \( c(y,z) \) are displayed in Fig. 2(e)–2(h) at the same fields as \( d\Sigma_M /d\Omega \) (Mettus and Michels, 2015). While \( d\Sigma_M /d\Omega \) at small fields [Fig. 2(a) and 2(b)] is enhanced parallel to \( \mathbf{H}_0 \), the correlation function exhibits maxima in the direction perpendicular to \( \mathbf{H}_0 \); the range of the correlations extends to several hundreds of nanometers [Fig. 2(e) and 2(f)]. Increasing the field results in the suppression of the correlations and at the largest field \( d\Sigma_M /d\Omega \) possesses a nearly fourfold anisotropy with maxima along the detector diagonals and minima along the horizontal and vertical axes [Fig. 2(d)], which translate into the corresponding extrema in \( c(y,z) \) [Fig. 2(h)]. In nuclear SANS, negative values of the distance distribution function \( p(r) \) are attributed to distances that connect regions with opposite sign of the scattering-length density contrast more frequently than regions with the same sign (Glatter and Kratky, 1982). However, for magnetic SANS, such an easily accessible interpretation of the correlation function of the spin-misalignment SANS

FIG. 2 (a)–(d) Contour plots of normalized \( d\Sigma_M /d\Omega \) at applied magnetic fields as indicated (\( \mathbf{k}_0 \perp \mathbf{H}_0 \); \( \mathbf{H}_0 \) is horizontal); the scattering of the saturated state has been subtracted. (e)–(f) Corresponding two-dimensional correlation functions \( c(y,z) \) (= 2D Fourier transforms of \( d\Sigma_M(q_y,q_z)/d\Omega \)). The DMI has not been taken into account. Image taken from Mettus and Michels, 2015.

FIG. 3 (a) Spike anisotropy observed in the total unpolarized \( d\Sigma /d\Omega \) of a sintered Nd–Fe–B-based permanent magnet in the remanent state (\( \mathbf{k}_0 \perp \mathbf{H}_0 \)). (b) \( d\Sigma /d\Omega \) versus azimuthal angle \( \theta \) at \( q = 0.10 \pm 0.02 \text{mm}^{-1} \). Data taken from Péring et al., 2014.
cross section in terms of a specific magnetization distribution is not straightforward; this is mainly related to the fact that \( c(y,z) \) being the Fourier transform of \( d\Sigma_{M}/d\Omega \) does not directly represent the correlations in the magnetic microstructure (as does the autocorrelation function), but also includes the magnetodipolar interaction of the neutrons with the sample (via the trigonometric functions and the cross term in the cross section) (Erokhin et al., 2016).

It has recently been suggested that the DMI is of relevance for the magnetic SANS of materials containing many lattice imperfections (Michels et al., 2016), e.g., due to the breaking of inversion symmetry at internal interfaces, which may cause a chiral term, Eq. (15). The defect-induced symmetry breaking can be characterized from measurement of the spin-flip cross sections, Eqs. (18), according to \(-2i\chi(q) = \frac{d\Sigma_{+}}{d\Omega} - \frac{d\Sigma_{-}}{d\Omega}\). Using the expressions for the Fourier components (Michels et al., 2016), the function \(-2i\chi(q)\) is plotted in Fig. 4. At small fields, two extrema parallel and antiparallel to the field axis are predicted [Fig. 4(a)], whereas at larger fields additional maxima and minima appear approximately along the detector diagonals [Fig. 4(b) and 4(c)]. The strong field dependency of \(-2i\chi(q,H_{0})\) may be employed in order to experimentally determine the DM constant. Note that \(-2i\chi(q)\) describes an asymmetry arising in the elastic SANS cross section due to the effect of the anti-symmetric exchange interaction on the static magnetic microstructure. We refer to Sec. VIII for further studies which address the inelastic and critical scattering related to the DM term.

As shown in Honecker et al., 2013, near magnetic saturation and for a two-phase particle-matrix-type ferromagnet, the unpolarized \( d\Sigma/d\Omega \) for \( k_{0} \perp H_{0} \) can be evaluated by means of micromagnetic theory. As an example, the azimuthally-averaged field-dependent SANS cross section of a zero-magnetostriction nanocomposite from the NANOPERM family of alloys along with the fits to the micromagnetic theory is displayed in Fig. 5(a). It is seen that the entire \((q,H_{0})\)-dependence of \( d\Sigma/d\Omega \) can be excellently described by the micromagnetic prediction. From a global fit of the entire data set to the micromagnetic theory one obtains a value of \( A = 4.7 \pm 0.9 \) pJ/m for the volume-averaged exchange-stiffness constant [ comparable inset in Fig. 5(a)].

In addition to the exchange constant, the analysis provides the square magnitudes of the Fourier coefficients of the magnetic anisotropy field \( S_{H} \propto \langle |H_{p}(q)|^{2} \rangle \) and of the longitudinal magnetization \( S_{M} \propto \langle |M_{z}(q)|^{2} \rangle \). The obtained results for these functions are shown in Fig. 5(b). It is immediately seen that over the displayed \( q \)-range \( \langle |M_{z}|^{2} \rangle \) is orders of magnitude larger than \( |H_{p}|^{2} \), suggesting that jumps \( \Delta M \) in the magnetization at internal interfaces are the dominating source of spin disorder in these alloys. Numerical integration of \( S_{M}(q) \) and \( S_{H}(q) \) over the whole \( q \)-space, i.e., \( \langle 2\pi^{2}\beta_{s}^{2} \rangle^{-1} \int_{0}^{\infty} S_{H,M} q^{2} dq \) yields, respectively, the mean-square anisotropy field \( \langle |H_{p}(q)|^{2} \rangle \) and the mean-square longitudinal magnetization fluctuation \( \langle |M_{z}|^{2} \rangle \) (Honecker et al., 2013). For the data shown in Fig. 5(b) we obtain the following lower bounds: \( \mu_{0}|\langle |H_{p}(q)|^{2} \rangle|^{1/2} \approx 10 \) mT and \( \mu_{0}|\langle |M_{z}|^{2} \rangle|^{1/2} \approx 50 \) mT. This finding qualitatively supports the notion of dominant spin-misalignment scattering due to magnetostatic fluctuations.

D. Relation to conventional particle-matrix approach

Magnetic SANS of bulk magnetic materials (e.g., single-phase elemental ferromagnets, Nd–Fe–B-based permanent magnets, or steels) is to a large extent determined by long-range magnetization fluctuations due to defect-related spin-misalignment. Away from magnetic saturation, all three magnetization Fourier components \( M = \{M_{x}, M_{y}, M_{z}\} \) govern the magnetic SANS cross sections, whereas in the saturated state, when \( M = \{0, 0, M_{z} = M_{z}(r)\} \), the cross sections are determined by
nuclear SANS and by the Fourier transform $\hat{M}_s(q)$ of the spatially-dependent saturation magnetization $M_s(r)$. It is important to realize that the spin-misalignment SANS cross section depends primarily on the magnetic interactions (but also on the underlying grain microstructure), while the SANS at saturation is entirely determined by the geometry of the microstructure, in other words, $|\hat{N}|^2$ and $|\hat{M}_s|^2$ depend only on the size, shape, position of particles, and on the scattering-length density contrast of the particles relative to the matrix. For instance, for the perpendicular and parallel scattering geometry, the unpolarized cross sections at saturation reduce, respectively, to

$$\frac{d\Sigma_{\mathrm{sat}}}{d\Omega} = K \left( b_H^2 |\hat{N}|^2 + |\hat{M}_s|^2 \sin^2 \theta \right),$$

and

$$\frac{d\Sigma_{\parallel}}{d\Omega} = K \left( b_H^2 |\hat{N}|^2 + |\hat{M}_s|^2 \right).$$

The magnetic structure factor of a saturated microstructure consisting of a distribution of $i = 1, \ldots, N_p$ particles with saturation magnetization $M_{s,i}$ in a matrix of saturation magnetization $M_m$ can be written as (Schlömann, 1967)

$$|\hat{M}_s(q)|^2 = \frac{1}{8\pi} \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} (M_{s,i} - M_{s,j})(M_m - M_{s,j}) \times V_{p,i} V_{p,j} F_i(q) F_j^*(q) e^{-iq(r_i - r_j)},$$

where $V_{p,i}$, $F_i$, and $r_i$ represent, respectively, the particle volume, the form factor, and the position vector of particle “i”. An analogous expression describes the corresponding nuclear SANS (see, e.g., Chen and Lin, 1987).

In the monodisperse and dilute limit, we have

$$|\hat{M}_s(q)|^2 = \frac{N_p}{8\pi^3} (\Delta M)^2 V_p^2 F(q)^2,$$

where $\Delta M = M_{s,i} - M_{s,j}$. Inserting Eq. (26) into Eqs. (23) and (24), the magnetic SANS cross section at saturation (e.g., for $k_0 \perp H_0$) takes on the familiar form (Wiedenmann, 2001)

$$\frac{d\Sigma_{\mathrm{sat}}}{d\Omega} = \frac{N_p}{V} \Delta \eta^2_m V_p^2 |F(q)|^2 \sin^2 \theta,$$

where $\Delta \eta^2_m = b_H^2 (\Delta M)^2$ denotes the magnetic scattering-length density contrast. Equation (27) represents the well-known expression—embryonic the particle-matrix concept—which is employed in many magnetic SANS investigations, even in situations where the material under study is not fully saturated. As the derivation has shown, Eq. (27) relies on the special assumption of homogeneously magnetized domains ($M_x = M_y = 0$), and for various reasons (see, e.g., the discussion in Michels, 2014) Eq. (27) does not describe the magnetic SANS of bulk ferromagnets (unless fully saturated). However, as discussed in Sec. V, the magnetic SANS of nearly uniformly magnetized nanoparticles may be described by means of the particle-matrix approach by employing special profiles for the magnetic scattering-length densities (Disch et al., 2012).

Analytical expressions for particle form factors $|F(q)|^2$ have been derived for an extensive number of particle shapes and there exist also a few closed-form results for the structure factor (Feigin and Svergun, 1987; Glatter and Kratky, 1982; Guinier and Fournet, 1955; Pedersen, 1997). We refer to the review article by Pedersen, 1997 for a detailed discussion of this topic. Likewise, several software packages (Breßler et al., 2015; Butler et al., 2016; Kline, 2006) provide collections of particle form factor models (including particle-size distributions) and structure factors to analyze SANS data.

Before closing this section we would like to briefly introduce two important limiting expressions for the scattering curve, which allows one to obtain information about the structure size and the internal particle surface area. Generally, when there are two phases of uniform scattering-length density and with discontinuous (sharp) interfaces, the scattering in the limit of large $q$ (much larger than the inverse of the characteristic structure scale) obeys (Ciccarelli et al., 1988; Debye et al., 1957; Debye and Bueche, 1949; Porod, 1982a)

$$\frac{d\Sigma}{d\Omega} (q) \approx 2\pi(\Delta \eta)^2 \frac{S}{V} q^{-4},$$

where $S$ denotes the particle surface area. Equation (28) is known as the Debye-Porod law, and it is valid not only for single particles, but also for densely packed systems. This expression for the high-$q$ limit can be supplemented by one for the scattering near the origin of reciprocal space, the so-called Guinier approximation (Guinier, 1994): when the scattering is from a set of noninterfering discrete objects then, in the limit of low $q < 1.3/R_g$,

$$\frac{d\Sigma}{d\Omega} (q) \approx \frac{d\Sigma}{d\Omega} (0) e^{-q^2 R_g^2 / 2}.$$

For identical scatterers, $R_g$ denotes the individual radius of gyration. In fact, for dilute monodisperse systems, the Guinier plot (ln$(d\Sigma/d\Omega)$ versus $q^2$) should be a linear function, whose slope yields $R_g$. As pointed out by Svergun and Koch, 2003, linearity of the Guinier plot can be considered as a test of the sample homogeneity and deviations indicate attractive or repulsive interparticle interactions leading to interference effects (Rothwell, 1968). Furthermore, when the particle-size distribution is nonuniform, $R_g$ needs to be replaced with the ratio of moments of the size distribution (see Feigin and Svergun, 1987 and Kostorz, 1982 for details).

The Porod and Guinier laws have been derived for non-magnetic particle-matrix-type assemblies in the context of the early theoretical developments of the technique of
small-angle x-ray scattering and their application to magnetic materials should be considered with special care. They are certainly applicable to systems consisting of saturated magnetic particles in a nonmagnetic matrix or, likewise, to pores in a saturated matrix. On the other hand, when the smoothly-varying magnetization profiles of micromagnetics (Sec. VI) are at the origin of the related magnetic scattering—implies the absence of a sharp interface in the magnetic microstructure—the asymptotic power-law behavior of the cross section differs from the $q^{-4}$-behavior [see, e.g., Fig. 7(b)], in agreement with theoretical predictions and experimental observations (Mettus and Michels, 2015). This statement does of course not preclude the existence of sharp interfaces in the nuclear grain microstructure of a magnetic material: there may well exist sharp particle-matrix interfaces, but the corresponding spin distribution which decorates these interfaces and which gives rise to the magnetic SANS cross section might be continuous over the defects.

III. Nd–Fe–B BASED PERMANENT MAGNETS

This section addresses the properties of Nd–Fe–B-based permanent magnets as seen by magnetic SANS. The major challenge in this field remains the understanding of how the details of the microstructure (e.g., average grain size and shape, distribution of Nd-rich intergranular phases, crystallographic texture, interfacial chemistry) correlate with the macroscopic magnetic properties. In order to tackle this issue, a multiscale characterization approach is generally adopted, which comprises a suite of both experimental and theoretical state-of-the-art methods including high-resolution electron microscopy, electron backscattering diffraction, three-dimensional atom-probe analysis, Lorentz and Kerr microscopy, or atomistic and continuum micromagnetic simulations (Gutfleisch et al., 2011; Hono and Sepehri-Amin, 2012; Liu et al., 2013; Sepehri-Amin et al., 2015, 2013; Woodcock et al., 2012). The SANS technique has made an important contribution here, since it provides—quite uniquely—information on variations of both the magnitude and orientation of the magnetization vector on a nanometer length scale and from within the volume of the material. We begin the discussion in this section by introducing the concept of the correlation function of the spin-misalignment SANS cross section and we then discuss selected SANS results obtained on Nd–Fe–B-based nanocomposites and sintered magnets.

![Image of the meaning of the correlation length $l_C$ of the spin misalignment. (left) Computed spin misalignment at a spherical pore ($2R = 12$ nm) in a ferromagnetic iron matrix (two-dimensional cut out of a three-dimensional simulation). Shown is the magnetization component $\mathbf{M}_x(r)$ perpendicular to $H_0 \parallel \mathbf{e}_z$; thickness of arrows is proportional to the magnitude of $\mathbf{M}_x$. Solid gray lines: magnetodipolar field distribution. The correlation length $l_C$ is a measure for the size of the inhomogeneously magnetized region around the defect; $l_C$ consists of a field-independent contribution $R$, which specifies the structural size of the defect, and of a field-dependent exchange length $l_H$, which transmits the perturbation at the pore-matrix interface into the surrounding crystal lattice. (right) Corresponding magnetization Fourier components $|\widetilde{M}_x(q)|^2$ and $|\widetilde{M}_y(q)|^2$ projected into the plane $q_x = 0$. Bright colors correspond to “high” values and dark colors to “low” values of the Fourier components. Pixels in the corners of the images have $q \gtrsim 0.4$ nm$^{-1}$. Logarithmic color scale is used. Image taken from Erokhin et al., 2015.

A. Correlation function of the spin-misalignment SANS cross section

Using azimuthally-averaged data for the spin-misalignment SANS cross section $d\Sigma_M/d\Omega$ it becomes possible to compute the correlation function $C(r)$ of the spin-misalignment SANS cross section, according to (Mettus and Michels, 2015)

$$C(r) \sim \int_0^\infty \frac{d\Sigma_M(q)}{d\Omega}(q) j_0(qr) q^2 dq,$$

where $j_0(x) = \sin(x)/x$ denotes the zeroth-order spherical Bessel function. By means of the extrapolated value of the correlation function at the origin, $C(r = 0)$, one can determine the correlation length $l_C$ of the spin misalignment. Figure 6 illustrates the meaning of $l_C$, which specifies the range over which perturbations in the spin...
structure around a lattice defect (e.g., pore, grain boundary, dislocation, vacancy) are transmitted by the exchange interaction into the surrounding crystal lattice. A convenient definition for $l_C$ is

$$C(r = l_C) = C(0)e^{-1}, \quad (31)$$

which yields the exact correlation length for exponentially decaying correlations. Note, however, that this definition does not imply that the correlations do decay exponentially. Equation (31) is merely a convenient way to define a characteristic length which can be related to the magnetic microstructure and which can be computed model independently.

In several studies (Bick et al., 2013a; Honecker and Michels, 2013; Michels, 2010; Michels et al., 2003; Michels and Weissmüller, 2008) it was found that $l_C(H_i)$ data can be well described by

$$l_C(H_i) = R + \sqrt{\frac{2A}{\mu_0 M_s (H_i + H^*)}}, \quad (32)$$

where the field-independent parameter $R$ is of the order of the defect size and the second field-dependent term on the right hand side represents a modified exchange length $l_H$ of the field (see Eq. (33) for the definition of $l_H$). Equation (32) is a phenomenological prediction based on micromagnetic theory, which embodies the convolution relationship between the magnetic anisotropy-field microstructure $H_B(r)$ and micromagnetic response functions which decay with $l_H$ (Weissmüller et al., 1999, 2001). The “correlation length” $R$ of the magnetic anisotropy field appears to be the average size over which the direction and/or magnitude of $H_B$ changes. For a statistically isotropic polycrystalline material, where each crystallite is a single crystal with magnetocrystalline anisotropy only, the parameter $R$ is sensibly related to the average crystallite size (Honecker and Michels, 2013; Michels, 2010). The field $H^*$ is expected to model the influence of the magnetodipolar interaction and of the magnetic anisotropy (Bick et al., 2013a). For soft magnetic materials with low crystalline anisotropy and at large applied magnetic fields (when the magnetostatic interaction may be negligible), one may ignore the field $H^*$, so that Eq. (32) simplifies to

$$l_C(H_i) = R + l_H(H_i) = R + \sqrt{\frac{2A}{\mu_0 M_s H_i}}, \quad (33)$$

The latter equation has been found to excellently describe the field-dependent spin-misalignment correlations in nanocrystalline Co and Ni (Michels et al., 2003). By contrast, for (uniaxial) hard magnets, the anisotropy field $H_K = 2K_u/M_s$, which for Nd$_2$Fe$_{14}$B single crystal is about 8 T at 300 K (Woodcock et al., 2012), is expected to cut down the size of spin inhomogeneities. Likewise, jumps $\Delta M$ of the magnitude of the magnetization at internal phase boundaries, which in Fe-based nanocomposites can be as large as 1.5 T (Michels et al., 2006), give rise to magnetic stray-field torques that produce spin disorder in the surrounding magnetic phase (compare Fig. 6); such kind of perturbations also decrease the size of gradients in the magnetization (Honecker et al., 2013). It is interesting to note that at $H_i = 0$ and for $H^* = H_K = 2K_u/(\mu_0 M_s)$, Eq. (32) reduces to

$$l_C = R + \sqrt{\frac{A}{K_u}}, \quad (34)$$

where $\sqrt{A/K_u}$ is the domain-wall parameter.

### B. Selected results on Nd–Fe–B magnets

A chronological assessment of SANS studies on Nd–Fe–B-based permanent magnets starts with the early work of Fujii et al., 1987, who investigated the role of domain walls and grain boundaries on the magnetic microstructure of sintered Nd$_{15}$Fe$_{77}$B$_8$ and Nd$_{15}$Fe$_{76}$Al$_1$B$_8$. Despite of this pioneering approach of using the SANS method for studying rare-earth-based intermetallic compounds, only recently (more precisely in the last five years) has a deeper understanding of the capabilities (and limitations) of the SANS technique been reached. It was Takeda et al., 2012 who continued with SANS research on sintered Nd–Fe–B magnets by analyzing the temperature dependence of SANS patterns, with special attention to the correlation between the average structure and the coercivity. The effect of the grain-boundary diffusion process on the magnetization reversal of isotropic sintered (Périgo et al., 2016) and hot-deformed (textured) nanocrystalline (Saito et al., 2015; Yano et al., 2014, 2012) Nd–Fe–B magnets has been investigated.

We start by considering a melt-spun isotropic Nd–Fe–B-based nanocomposite, which consists of hard magnetic Nd$_2$Fe$_{14}$B particles (size: $\sim 22$ nm) and Fe$_3$B crystallites (size: $\sim 29$ nm) (Bick et al., 2013a,b). It is important to mention that for this particular alloy the difference $\Delta M$ in the saturation magnetizations of the Nd$_2$Fe$_{14}$B phase and the Fe$_3$B crystallites is rather small, $\mu_0 \Delta M \approx 0.01$ T (Schrefl et al., 1993). Consequently, the related longitudinal magnetic SANS $\propto [M_z]^2 \propto (\Delta M)^2$ is negligible as compared to the nuclear SANS $|\bar{N}|^2$.

Figure 7(a) displays the total unpolarized $d\Sigma/d\Omega$ of the Nd–Fe–B nanocomposite. A strong field dependence between the largest applied field of 10 T and the coercive field of $\mu_0 H_C = -0.55$ T is observable. Since nuclear SANS is field independent and since SANS due to $|M_z|^2$ fluctuations is negligible (for this particular alloy), it is evident that the dominating contribution to $d\Sigma/d\Omega$ is due to transversal spin misalignment. In order to obtain the corresponding spin-misalignment SANS...
FIG. 7 (a) Azimuthally-averaged total SANS cross section $d\Sigma/d\Omega$ of Nd$_2$Fe$_{14}$B/Fe$_3$B as a function of momentum transfer $q$ and applied magnetic field $H_0$ ($T = 300$ K) ($k_0 \perp H_0$) (log-log scale). Solid circles (●): applied-field values (in Tesla) decrease from bottom to top: 10, 6, 1, $-0.25$, $-0.55$; (□): $-1$ T; (◊): $-3$ T. Inset: room-temperature magnetization curve of Nd$_2$Fe$_{14}$B/Fe$_3$B. (b) Applied-field dependence of the spin-misalignment SANS cross section $d\Sigma_M/d\Omega$ of nanocrystalline Nd$_2$Fe$_{14}$B/Fe$_3$B. Solid circles (●): field values (in Tesla) decrease from bottom to top: 6, 1, $-0.25$, $-0.55$; (□): $-1$ T; (◊): $-3$ T. The $d\Sigma_M/d\Omega$ data displayed in (b) were obtained by subtracting the 10 T data shown in (a) from the $d\Sigma/d\Omega$ at lower fields. Dashed line: $d\Sigma_M/d\Omega \propto q^{-5.5}$. Image taken from Bick et al., 2013a.

FIG. 8 Field dependence of the correlation function $C(r)$ of the spin misalignment of nanocrystalline Nd$_2$Fe$_{14}$B/Fe$_3$B (log-linear scale). The field values follow the course of a hysteresis loop, starting from a large positive field and then reducing the field to negative values (see insets). Dotted line (extrapolating the 6 T data to $r = 0$): $C(r) = 4.58 - 0.043 r^2$. Image taken from Bick et al., 2013a.

tween $n \sim 5 - 6$ at all fields investigated [see Fig. 7(b)], the total unpolarized SANS reveals significantly lower values for $n$, which approach the Porod value of $n = 4$ at 10 T (Bick et al., 2013a).

Fourier transformation of the $d\Sigma_M/d\Omega$ data according to Eq. (30) yields the correlation function $C(r)$ of the spin misalignment (see Fig. 8). The field-dependent correlations in Fig. 8 do not decay exponentially, in agreement with the absence of an $n = 4$ power-law exponent in $d\Sigma_M/d\Omega$. Furthermore, the $C(r)$ seem to approach the origin $r = 0$ with zero slope (compare dotted line in Fig. 8), which is in agreement with the notion of magnetic SANS from continuously varying magnetization profiles and with the absence of a sharp interface in the magnetic microstructure (Porod, 1982a). For comparison, for the example of a uniform sphere (of radius $R$) and with a sharp interface, one finds the well-known expression (valid for $r \leq 2R$)

$$c_s(r) = 1 - \frac{3r}{4R} + \frac{r^3}{16R^3},$$

from which one can recognize that the first derivative of $c_s(r)$ evaluated at $r = 0$ is related to the surface $S$ of the particle, $c'_s(0) = -3/(4R) = -S/(4V)$. Such properties of the correlation function (derived for the nuclear small-angle scattering of uniform particles) do not hold for magnetic SANS of bulk ferromagnets.

The values of the correlation length $\xi_C$ [determined by means of Eq. (31)] are plotted in panel (A) of Fig. 9 as a function of the applied magnetic field, which is usually the control parameter in magnetic SANS experiments. For the Nd–Fe–B nanocomposite (with $\Delta M \geq 0$), we
FIG. 9 (A) Applied-field dependence of the correlation length $l_C$ of the spin misalignment of nanocrystalline Nd$_2$Fe$_{14}$B/Fe$_7$B. Solid line: fit of the data to Eq. (32), where $R = 10.9 \text{ nm}$ and $\mu_0 H^* = +0.60 \text{ T}$ are treated as adjustable parameters, and the quantities $A = 12.5 \text{ pJ/m}$ and $\mu_0 M_s = 1.6 \text{ T}$ are held fixed. In addition to $l_C(H_0)$-data obtained at the instrument Quokka (ANSTO), results obtained at the SANS instruments KWS 1 (JCNS) and D11 (ILL) are also shown. Dashed horizontal line: average radius of the Nd$_2$Fe$_{14}$B particles ($R = 11 \text{ nm}$). Dotted vertical line: coercive field $\mu_0 H_c = -0.55 \text{ T}$. Image taken from Bick et al., 2013a. (B) Normalized correlation function $C(r)$ of the spin misalignment for a textured (hot-deformed) and isotropic Nd$_2$Fe$_{14}$B/α-Fe nanocomposite in the remanent state. The $C(r)$ of the textured sample is computed by using $d\Sigma_M/d\Omega$ averaged along the vertical and horizontal directions ($\pm 7.5^\circ$ sector averages) as well as using the full circular ($2\pi$) average of $d\Sigma_M/d\Omega$; the $C(r)$ of the isotropic sample is computed using the corresponding $2\pi$-averaged $d\Sigma_M/d\Omega$ (see inset). Solid horizontal line: $C(r) = e^{-1}$. Data taken from Michels et al., 2017.

expect that $l_C$ describes the spatial extent of magnetization inhomogeneities, mainly within the soft magnetic Fe$_7$B grains, that are caused by the jump in the magnetic materials parameters (exchange constant, direction and magnitude of magnetic anisotropy) at the interface between the Nd$_2$Fe$_{14}$B particles and the surrounding Fe$_7$B crystallites. As can be seen in panel (A) of Fig. 9, $l_C$ approaches a constant value of about 12.5 nm at the largest positive fields and increases with decreasing applied field to take on a maximum value of about 18.5 nm at the experimental coercive field of $\mu_0 H_c = -0.55 \text{ T}$. Further increase of $H_c$ towards more negative values results again in a decrease of $l_C$ towards $\sim 12.5 \text{ nm}$. From the fit of the $l_C(H_0)$ data to Eq. (32) (solid line in panel (A) of Fig. 9), we obtain $R = 10.9 \text{ nm}$ (close to the experimental average grain radius of the Nd$_2$Fe$_{14}$B phase) and $\mu_0 H^* = +0.60 \text{ T}$, which is close to the absolute value of the experimental coercive field. At the remanent state, the penetration depth of the spin disorder into the Fe$_7$B phase amounts to $\sim 5 - 6 \text{ nm}$.

In a recent comparative study of the magnetic microstructure of textured and isotropic Nd$_2$Fe$_{14}$B/α-Fe nanocomposites, evidence for a correlated crystallographic and spin texture was found (Michels et al., 2017). Specifically, the analysis of the neutron data suggested that the spin-misalignment scattering of the textured sample—prepared via melt-spinning and subsequent hot-deformation—is dominated by spin components along one direction perpendicular to the easy $c$-axis (pressing direction) of the Nd$_2$Fe$_{14}$B grains. This anisotropy in the magnetization distribution is accompanied by the presence of a crystallographic texture along these directions, as revealed by x-ray diffraction synchrotron data. Panel (B) of Fig. 9 illustrates the anisotropy of the correlations by depicting the correlation function of the textured nanocomposite: the correlation length along the horizontal ($l_C \approx 42 \text{ nm}$) and vertical ($l_C \approx 53 \text{ nm}$) directions differ considerably, which indicates differences in the magnetic interactions along these directions. This conjecture is supported by electronmicroscopy and three-dimensional atom-probe tomography work (Liu et al., 2014), which reports anisotropic properties of the grain-boundary phase in hot-deformed nanocrystalline Nd–Fe–B magnets.

Sintered Nd–Fe–B-based permanent magnets, prepared via the powder metallurgical route, are presently more important from the economic perspective. Here, the research focus is to reduce the amount of heavy rare-earth metals (Tb, Dy), which are added in order to guarantee the temperature stability of the magnet ($H_c$) at the operating temperature of the device (e.g., $\sim 150 \text{ °C}$ for electromotor applications). In this context, the effect of the grain-boundary diffusion process (GBDP) on the bulk magnetic microstructure of Tb-doped Nd–Fe–B-based sintered magnets has been studied by Périgo et al., 2016 by means of magnetic-field-dependent unpolarized SANS. In the GBDP (Sepehri-Amin et al., 2010), the Nd–Fe–B magnet is exposed at elevated temperatures to a fine powder or a vapor containing high-magnetic-anisotropy-inducing heavy-rare-earth elements such as Tb or Dy, which then diffuse (preferentially along liquid grain boundaries) into the bulk of the material, in this way locally increasing the coercivity. Compared to the Tb-free sample, Périgo et al., 2016 observe in the GBDP specimen a 15% reduced correlation length and a 16% increased $H_c$ (see Fig. 10). The origin of the reduced correlation length is related to the increased local magnetic anisotropy field of the Tb-enriched interfaces, which represent possible nucleation sites for reversed magnetic domains: the presence of Tb results in less magnetic disorder near/across the grain boundaries. Qualitatively, this finding agrees with the work of Saito et al., 2015 who reported reduced spatial fluctuations of magnetic moments due to the GBDP in Nd-Cu-infiltrated Nd–Fe–B nanocrystalline magnets.

SANS has also allowed the experimental observation of magnetic poles (due to $\nabla \cdot \mathbf{M} \neq 0$) existing in the bulk of magnetic materials. The signature of the magneto-static interaction is a characteristic spike or flying-saucer-type pattern in $d\Sigma_M/d\Omega$ with sharp maxima for angles $\theta = 0^\circ$ and $\theta = 180^\circ$. Such an angular anisotropy has
been predicted by Weissmüller et al., 1999 and experimentally observed by Pérgo et al., 2014 on an isotropic Nd–Fe–B-based permanent magnet (see Fig. 3). The spike anisotropy underlines the importance of the magnetodipolar interaction for understanding magnetic SANS.

Future neutron work will address the role of the intergranular Nd-rich layers for the coercivity mechanism of sintered magnets. Here, the combination of micromagnetic simulations (see Sec. VI) with experimental SANS data might provide important insights into the signature of the grain boundaries in the magnetic SANS cross section.

IV. MAGNETIC STEELS

This section aims to illustrate how magnetic SANS has contributed to the fast paced development of steels for applications under extreme conditions, such as those imposed by irradiation with fast neutrons ($E > 1$ MeV).

A. New steels for application under extreme conditions

The advancement, innovative application, and societal impact of steel have not ceased over the centuries. Currently, additional motivation for the development of new steels arises, for example, from extreme requirements, such as those imposed by nuclear fission (Zinkle and Was, 2013) and fusion (Knaster et al., 2016) environments. Recent progress in computational alloy design indeed indicates potential for the development of new steels.

The favorable property combinations of steels are partly achieved by manipulating the solid-state transformation from the high-temperature face centered cubic (fcc) phase. Microstructures formed as a result include martensite and ferrite, which became eponyms for classes of steels. These steels have a body-centered cubic (bcc) lattice and exhibit ferromagnetism. It is also possible, for instance by alloying with Ni, to stabilize the fcc phase to below room temperature. The resulting austenitic steels are paramagnetic and also experience wide application. It is, however, the ferromagnetism of the former steels that allows magnetic and nuclear scattering to be separated and additional information to be gained. By contrast, the ferromagnetism of a sample is challenging for transmission electron microscopy (TEM).

The application of SANS to ferromagnetic steels is partly at variance with other applications of magnetic SANS. Therefore, specific features of the approach will be introduced in Sec. IV.B.

A first field of application discussed here is related to low-alloy reactor pressure vessel (RPV) steels. Historically, mechanical testing of neutron-irradiated RPV steels indicated advancing embrittlement, obviously an issue of utmost significance for safe reactor operation. While the dominant embrittlement mechanisms were gradually understood (Odette and Lucas, 1998), special long-term irradiation effects are currently of interest (Altstadt et al., 2014). New insight on the nature and distribution of irradiation-induced nanostructures derived from the application of magnetic SANS will be addressed in Sec. IV.C.

Components of future fission reactors and fusion devices will have to withstand more severe conditions, including neutron exposures up to 200 displacements per atom (dpa), a factor of 1000 more than RPV-typical exposures. 8 – 18 % Cr steels were found to be promising candidates. The contributions of magnetic SANS to an improved understanding of the Fe-Cr system and the irradiation behavior of advanced Cr steels will be considered in Sec. IV.D.

Advanced oxide dispersion strengthened (ODS) steels exhibit exceptional irradiation resistance due to a high density of internal point defect sinks and traps, e.g., grain boundaries and particle-matrix interfaces (Odette et al., 2008). Information on the type and size distribution of oxide nanoparticles gained from the application of magnetic SANS will be highlighted in Sec. IV.E.

Before entering into a detailed discussion of the classes of steels introduced above, other contributions of magnetic SANS to the development of new steels should be mentioned. These include ultra-high-strength steels combining the intermetallic strengthening mechanisms associated with maraging steels and alloy carbide strengthening (Delagnes et al., 2012; Eidenberger et al., 2011; Perrott et al., 2012; Zhang et al., 2011). The reported results indicate that the complex precipitation processes require the usage of combinations of complementary methods.
B. Prerequisites

We consider a two-phase material consisting of a dilute assembly of nanoparticles randomly dispersed in a ferromagnetic matrix. The basis for the separation of nuclear (“nucl”) and magnetic (“mag”) SANS is the application of a saturating magnetic field perpendicular to the unpolarized incident neutron beam (compare also Eq. (23) in Sec. II.D).

\[
\frac{d\Sigma}{d\Omega}(q, \theta) = \frac{d\Sigma_{\text{nucl}}}{d\Omega}(q) + \frac{d\Sigma_{\text{mag}}}{d\Omega}(q) \sin^2 \theta, \quad (36)
\]

where \(\theta\) denotes the angle enclosed by the direction of the applied field and the momentum transfer vector \(q\) [for conditions of validity of Eq. (36), see Bischof et al. (2007)]. The subsequent analysis procedure is similar for magnetic and nuclear SANS; the scattering cross sections of a suitable reference sample or a background is subtracted from the cross sections of the sample of interest. An inverse problem is solved for the difference scattering curve to obtain the size distribution of the additional scatterers. For the case of neutron-irradiated materials, the respective unirradiated sample is usually taken as reference. The outcome then is the distribution of irradiation-induced nanostructures.

In order to calculate the size distribution in absolute units, the contrast, i.e., the square of the magnetic, Eq. (37), or nuclear, Eq. (38), difference scattering length density between scatterer (“S”) and matrix (“Ma”) must be known.

\[
\Delta \eta_m = \langle \eta_m(S) - \eta_m(Ma) \rangle = \sum_{i \in S} n_{i,S} b_{m,i,S} - \sum_{i \in Ma} n_{i,\text{Ma}} b_{m,i,\text{Ma}}, \quad (37)
\]

\[
\Delta \eta_n = \langle \eta_n(S) - \eta_n(Ma) \rangle = \sum_{i \in S} n_{i,S} b_{n,i,S} - \sum_{i \in Ma} n_{i,\text{Ma}} b_{n,i,\text{Ma}}, \quad (38)
\]

Here, \(n_i, \eta_{m,i},\) and \(\eta_{n,i}\) are the atomic density, magnetic, and nuclear scattering length density of atomic species \(i\), respectively. Contrary to the nuclear scattering length, the magnetic scattering length of an atom depends on the local environment. We provide the scattering lengths of Cr, Cu, Fe, Mn, and Ni in Table I for later reference.

The \(A\)-ratio was originally introduced as the ratio of the scattering cross sections perpendicular and parallel to the applied saturating magnetic field (Frisingus and Buenemann, 1979):

\[
A = \frac{d\Sigma_{\perp}}{d\Sigma_{\parallel}} = 1 + \frac{d\Sigma_{\text{mag}}}{d\Sigma_{\text{nucl}}} = 1 + \frac{(\Delta \eta_m)^2}{(\Delta \eta_n)^2}. \quad (39)
\]

In general, \(A\) is a function of \(q\). As shown by Eq. (39), if the magnetic and nuclear scatterers are the same objects, \(A\) can be expressed as the ratio of contrasts. Given the matrix structure and composition, Eqs. (37) to (39) describe a relationship between a measurable quantity of SANS and the structure and composition of the scatterers. The problem of extracting the composition of the scatterers from \(A\) is obviously underdetermined, except for the simplest systems and for nonmagnetic scatterers (magnetic holes). The contrast of magnetic holes is still used with some profit in the case of magnetic scatterers. Indeed, the volume fraction of scatterers obtained this way is a lower bound for all possible volume fractions, provided that \(0 < \langle b_m \rangle_S < 2 \langle b_m \rangle_{\text{Ma}}\).

A trial-and-error procedure to specify or narrow down the composition in the case of nonmagnetic scatterers is depicted in Fig. 11. As indicated therein by numbers 1 to 5, the procedure consists of the following steps: 1 - Calculate the experimental \(A\)-ratio from the separated magnetic and nuclear scattering. 2 - Calculate the size distribution of precipitates from magnetic and nuclear scattering and check consistency between both. 3 - Calculate the total volume fraction \(f\) of scatterers in absolute units from magnetic scattering (known contrast) and in relative units from nuclear scattering (unknown contrast). 4 - Assume a trial composition of the precipitates, calculate \(A_{\text{tr}}\), compare with \(A_{\text{exp}}\) (\(\rightarrow\) error). 5 - Use the same trial composition of precipitates along with \(f\) to calculate the needed bulk concentrations. Compare with the bulk composition (\(\rightarrow\) error). Take into account solubility limits if applicable. 6 - Update the trials and iterate to reach self-consistency.

Alternatively, if the assumption of a dilute assembly of scatterers is violated, the Porod invariant can be used to calculate the volume fraction \(f\) of scatterers (Porod, 1982b):

\[
\int_0^\infty d\Sigma_{\text{mag},\text{nucl}} dq^2 dq = 2\pi^2 f (1 - f) (\Delta \eta_{m,n})^2. \quad (40)
\]

Equation (40) is applicable independent of the structure of the two-phase medium and the degree of dilution of the system. Care has to be exercised for the proper extrapolation of the integrand.

Even at magnetic saturation, magnetic effects such as spin misalignment may give rise to significant deviations

<table>
<thead>
<tr>
<th>(i)</th>
<th>(b_n) (fm)</th>
<th>((\Delta \eta_n/n_{\text{nucl}})^2)</th>
<th>(b_m) (fm)</th>
<th>((\Delta \eta_m/n_{\text{mag}})^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cr</td>
<td>-3.5</td>
<td>3.635</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>Cu</td>
<td>0</td>
<td>36</td>
<td>7.718</td>
<td>3.0</td>
</tr>
<tr>
<td>Fe</td>
<td>6</td>
<td>0</td>
<td>9.450</td>
<td>0</td>
</tr>
<tr>
<td>Mn</td>
<td>(1.9)</td>
<td>(36)</td>
<td>-3.73</td>
<td>173</td>
</tr>
<tr>
<td>Ni</td>
<td>1.6</td>
<td>19.4</td>
<td>10.3</td>
<td>0.72</td>
</tr>
</tbody>
</table>
from Eq. (36) (Weissmüller et al., 2001) questioning the validity of the separation into magnetic and nuclear contributions. Bischof et al. (2007) derived corrections and analyzed a set of heat treatments of a martensitic steel to conclude that spin-misalignment corrections are particularly important for nuclear scattering in nanostructured steels. This correction requires magnetic-field-dependent SANS measurements.

Finally, it is important to note that spin-polarized neutrons allow the sign of the product $\Delta \eta_m - \Delta \eta_n$ to be determined. This is impossible using conventional SANS and, therefore, may help to identify the type of scatterers in some cases, as underpinned in Secs. IV.C and IV.D below.

**C. Fe-Cu alloys and reactor pressure vessel steels**

Once the outstanding role of low levels of impurity Cu on the irradiation embrittlement of RPV steels was empirically recognized (Steele, 1975), both the investigation of simple model systems, such as binary Fe-Cu alloys, and the application of new techniques with nm-scale sensitivity, such as SANS, entered the scene. SANS studies of the same model systems contributed to a deeper understanding of phase separation processes (Kampmann and Wagner, 1986) and provided input for the emerging multiscale modeling approach to irradiation damage. The utilization of Cu precipitation to replace carbide hardening of steels (Fine et al., 2007), and the self-healing effect of Cu precipitation on steels (He et al., 2010) raised renewed interest.

Kampmann and Wagner (1986) studied Cu precipitation in thermally-aged Fe$_{1-x}$Cu$_x$ ($x = 0.0138$ and 0.0064). The observed Cu precipitates were concluded to be nonmagnetic, which allowed the precipitate number density to be estimated in absolute units and models of the decomposition kinetics to be critically evaluated. He et al. (2010) performed time-resolved SANS on Fe$_{1-x}$Cu$_x$ ($x = 0.0098$) during isothermal in situ aging at 550°C in order to study Cu-segregation-assisted self-healing of deformation-induced defects. Assuming nonmagnetic precipitates at the measurement temperature, they derived the volume fractions of co-existing bcc, 9R, and fcc Cu precipitates as a function of aging time. Schober et al. (2010) combined magnetic SANS and atom probe tomography (APT) for thermally-aged Fe$_{1-x}$Cu$_x$ ($x = 0.0099$) in order to address the issue of the Fe fraction in Cu precipitates, for which levels between 0% and 50% had been reported before.

Several SANS studies of neutron-irradiated Fe-Cu alloys were reported. Most notably, Miller et al. (2003) recognized that nuclear SANS fed with the contrast derived from APT allows the Fe content of Cu precipitates to be cross-checked without assumptions on the precipitate magnetism. They found that, for irradiated Fe$_{1-x}$Cu$_x$ ($x = 0.008$), the alloy Cu content needed to reconcile SANS with APT significantly exceeds the actual Cu content. Hence APT must have overestimated the Fe fraction in the Cu precipitates. Very recently, Shu et al. (2018), essentially based on temperature-dependent magnetic SANS, finally confirmed the presence of nonmagnetic Cu precipitates containing little or no Fe in neutron-irradiated and thermally-aged Fe$_{1-x}$Cu$_x$ ($x = 0.008$).

Regarding RPV steels, early SANS studies revealed irradiation-induced scatterers with a mean radius of approximately 1 nm for Cu impurity levels $\gtrsim$ 0.1 wt%. It is now well understood that (i) coherent bcc Cu-rich precipitates (CRPs) are the dominant type of nanofeatures formed as a result of neutron irradiation of Cu-bearing
steels at reactor ambient temperatures, 270 – 300 °C. CRPs may contain Mn, Ni, Si, Fe, and vacancies. (ii) Coherent or semicoherent Mn-Ni(-Si) precipitates (MNPs) or their nonequilibrium precursors are the dominant type of nanofeatures in low-Cu, Mn-Ni-alloyed RPV steels (Odette, 1995). (iii) So-called matrix damage mainly consisting of dislocation loops and sub-nm vacancy clusters evolves slowly in all kinds of RPV steels. The contrast of the loops is too weak, and the vacancy clusters are too small, to be efficiently detected by SANS.

Increasing neutron exposure (fluence) gives rise to an increase of the volume fraction of CRPs at almost constant size. The effect of neutron flux (fluence rate) on the formation of CRPs is elucidated in Fig. 12. SANS, even in combination with APT, is not capable of unambiguously clarifying the average composition of multiconstituent precipitates in RPV steels (Carter et al., 2001; Hyde et al., 2014). Approaches aimed to isolate individual effects are crucial. Recent insight into the roles of impurity Cu and alloying Mn and Ni derived from magnetic SANS is highlighted below.

Spin-polarized neutrons (Ulbricht, 2006) indicated $\Delta n_{\alpha}, \Delta n_{\alpha'} > 0$ for the nanofeatures formed in a neutron-irradiated RPV steel containing 0.2 wt% Cu and 0.8 wt% Ni. This excludes the dominance of coherent Ni-clusters ($b_{n,Ni} > b_{n,Fe}$) and supports CRPs instead (see Table I).

For two neutron-irradiated Fe-Mn-Ni(-Cu) model alloys containing 1.2 wt% Mn (both), 0.7 wt% Ni (both) and 0.1 wt% Cu (second alloy only), $A < 1.4$ (the value for voids) was observed for the lowest exposure of 0.051 dpa (Bergner et al., 2010). This indicates the dominance of Mn ($b_{n,Mn} < 0$, see Table I) in the early stage of solute cluster formation. These clusters are presumably related to the formation of “late blooming” MNPs at high fluences (Odette, 1995). Applying a combination of methods, Spruoster et al. (2016) confirmed that the MNPs identified after high-fluence irradiations are well-defined phases.

Böhmert et al. (2004) and Wagner et al. (2012) reported pronounced linear correlations of the square root of precipitate volume fraction and irradiation hardening of RPV steels. The volume fractions were derived from magnetic SANS assuming nonmagnetic scatterers for the sets of RPV steels. This type of correlation is rationalized in the framework of the dispersed-barrier hardening model, which considers the obstruction of dislocation glide by randomly-dispersed irradiation-induced obstacles, e.g., CRPs or MNPs.

**D. Fe-Cr alloys and ferritic-martensitic Cr steels**

As mentioned in Sec. IV.A, 8 – 18 % Cr steels are promising candidates for nuclear applications. The high-temperature $\gamma$-phase field in the Fe-Cr system extends to 11.9 wt% Cr. The martensitic transformation is only available below this level. The microstructure of Cr steels may be martensitic or ferritic depending on the cooling rate (transformable or ferritic-martensitic steels). At higher Cr contents, the steels are always ferritic (ferritic stainless steels).

Cr contents in excess of the solubility limit of Cr in Fe below 500 °C give rise to phase separation into Fe-rich $\alpha$ and Cr-rich $\alpha'$. The formation of $\alpha'$ is responsible for the so-called 475 °C embrittlement during thermal aging and for the embrittlement during neutron irradiation of $\geq 9$ % Cr steels. $\alpha$ and $\alpha'$ are both bcc with only minor differences of the lattice parameters. The wide miscibility gap and small misfit strains make the Fe-Cr system particularly attractive for the investigation of phase separation by spinodal decomposition. SANS is well suited because both magnetic and nuclear scattering lengths sufficiently differ from Cr to Fe. Moreover, magnetic and nuclear contrasts are of similar magnitude, contrary to Fe-Cu with $\Delta n_{\mbox{m}} \gg \Delta n_{\mbox{m}}$ (see Table I).

SANS was applied to investigate spinodal decomposition in Fe$_{1-x}$Cr$_x$ (typically $x > 0.25$) (Bley, 1992; Katano and Izumi, 1984; Ujihara and Osamura, 2000; Vintaykin and Kolontsov, 1968; Xu et al., 2016). These studies have in common that no external magnetic field was applied to the samples; for a comment see LaSalle and Schwartz (1986).

The determination of the solubility limit of Cr in $\alpha$ below 500 °C (Fe solvus line of the Fe-Cr phase diagram) is challenging because $\alpha$-$\alpha'$ phase separation via nucleation and growth is extremely slow. Bonny et al. (2008) concluded from reviewed experimental data that the solvus line according to the standard phase diagram must be shifted to higher Cr concentrations. Bergner et al. (2009), using Eq. (40), derived the volume fraction of $\alpha'$ from SANS experiments performed on industrial-purity Fe$_{1-x}$Cr$_x$ ($x = 0.125$) neutron-irradiated at 300 °C and found that a quasi-steady state of Cr precipitation had been reached. The measured $A$-ratio, $A = 2.05 \pm 0.1$, was independent of $q$ and agreed with the value for $\alpha'$ particles, $A = 2.1$ (Mathon et al., 2003) (see Fig. 13). In view of the technical importance, a strict estimation of the solvus line requires additional attention.

Fe-Cr alloys also attracted interest as model systems aiming to explore the irradiation behavior of Cr steels. A set of neutron-irradiated Fe$_{1-x}$Cr$_x$ alloys ($x = 0.025 – 0.125$) was characterized by means of magnetic SANS and APT. The fraction of 41.5 % Fe in $\alpha'$ found by APT was inconsistent with nuclear SANS (Bergner et al., 2013). Further clarification is required.

Briggs et al. (2017) reported results of a combined APT/SANS investigation of $\alpha'$ precipitation in neutron-irradiated Fe-Cr-Al alloys to conclude about the role of Al. As pointed out by the authors, no magnetic field was applied and the measured total scattering was assumed to represent nuclear scattering, which gives rise to an over-estimation of the volume fraction of $\alpha'$. In retrospect, it
Advanced oxide dispersion-strengthened steels

Advanced oxide dispersion-strengthened (ODS) Cr steels, also referred to as nanostructured transformable steels (< 12% Cr) or nanostructured ferritic alloys (> 12% Cr), are distinguished by their excellent radiation resistance (Odette et al., 2008; Ukai and Fujiwara, 2002). The standard fabrication route is based on powder metallurgy including the steps of mechanical alloying by means of high-energy ball milling of a prealloy powder blended with typically 0.2 – 0.5 wt% yttria powder and consolidation, mainly by means of hot isostatic pressing (HIP).

As oxide nanoparticles are typically nonmagnetic, the volume fraction can be derived in absolute units from magnetic SANS. The A-ratio is particularly useful to check hypotheses on the type of oxides dominating in macroscopic samples (see Fig. 11). The selection of a suitable reference sample or baseline for SANS is an issue: Y-free counterparts produced in parallel with ODS steels (Alinger et al., 2004; Pareja et al., 2015) do not only differ from the ODS samples with respect to the absence of Y-bearing oxides (intended difference), but also with respect to other features, such as the presence of Cr- and Ti-bearing oxides (side effect).

Magnetic SANS was applied to examine advanced ODS steels across all development stages and production steps. The effect of milling parameters on the size distribution and type of oxide nanoparticles was addressed by investigating as-milled powders (Alinger et al., 2004; Mathon et al., 2012) or consolidated samples with the consolidation parameters kept constant (Hilger et al., 2016). Alinger et al. (2004) studied ODS alloys with the composition and processing parameters varied systematically and found that both Y (in the form of added yttria powder) and Ti (as constituent of the pre-alloy) are required for a high number density of oxide nanoparticles to be formed during HIP. Mathon et al. (2012), supplemented by Hilger et al. (2016), tabulated A-ratios for different kinds of Y-, Cr-, Ti-containing equilibrium and nonequilibrium oxides in Fe-Cr alloys. The comparison of measured (2.0 – 2.8) and calculated A-ratios proved useful to rule out a number of reported candidates for the dominant type of nanoparticles, such as Y₂O₃ (A = 3.2). It is worth noting that the measured A-ratio is a good indicator of the particles’ Y:Ti ratio because of the different nuclear scattering lengths of Y (bₙ = 7.75 fm) and Ti (bₙ = –3.37 fm). However, the variation of the dominant types of nanoparticles as a function of the processing parameters is still not exhaustively understood.

Wang et al. (2012) performed in situ annealing at continuously increasing temperatures up to 1400°C in SANS experiments on a 14% Cr nanostructured ferritic alloy. No magnetic field was applied, raising questions on the role of magnetic scattering. Nonetheless, the nanoclusters were found to persist up to 1400°C indicating unusual thermal stability. The robust measures of particle

FIG. 13 Effect of neutron exposure on A(q) for neutron-irradiated Fe₁₋ₓCrₓ (x = 0.125). The measured magnetic scattering is shown in the inset. The unirradiated reference was subtracted before calculating A(q). Theoretical A-ratios are indicated.
radius and volume fraction derived from SANS give rise to pronounced correlations with hardness and yield stress (Alinger et al., 2004; Zhong et al., 2016).

F. Summary on magnetic steels

Magnetic SANS has contributed significantly to the understanding of the behavior of ferromagnetic RPV steels, high-Cr steels, and advanced ODS steels in extreme environments. It is the ferromagnetism that allows magnetic and nuclear scattering to be separated, the number density of scatterers to be estimated, and the $A$-ratio as an indicator of composition to be exploited. The possibility of Fe fractions in nm-sized precipitates substantially in excess of equilibrium levels would impede the derivation of volume fractions and number densities from SANS results. However, such excess levels seem to be generally unfounded. By contrast, it is suggested to use SANS in combination with APT in order to calibrate the Fe fraction in cases of doubt. A number of material-specific open issues have been identified above. As material trends move towards nanostructured steels, strict interpretation of SANS results increasingly requires spin-misalignment scattering to be addressed. Samples of complex shape or in situ treatments may prevent magnetic saturation from being reached in SANS experiments. Refined strategies to estimate volume fractions and number densities under such circumstances are desirable.

V. MAGNETIC NANOPARTICLES AND FERROFLUIDS

In this section we review the use of SANS for the investigation of magnetic nanoparticles and ferrofluids. Magnetic nanoparticles mostly consist of an inorganic magnetic nanoparticle core surrounded by a nonmagnetic stabilizing shell of either organic ligands, polymers, or inorganic materials. Below the critical size limit, the nanoparticle demagnetization energy is not sufficient to compensate the domain-wall energy, resulting in a single-domain state. Single-domain nanoparticles have different magnetization relaxation characteristics than the bulk material, leading to phenomena such as superparamagnetism, covered in detail in Bedanta and Kleemann, 2009. The distinct magnetization relaxation behavior of nanoparticles gives rise to diverse applications, e.g., in information technologies or medical applications such as medical imaging and magnetic hyperthermia. Colloidal dispersions of single-domain magnetic nanoparticles in a carrier liquid (Bader, 2006; Pankhurst et al., 2003) are referred to as ferrofluids and find application in high-vacuum gears, seals, loudspeakers, sensors, etc (Joseph and Mathew, 2014). Important parameters regarding the preparation and application of ferrofluids include the magnetic material of the nanoparticles and the stability against precipitation.

The synthesis of magnetic nanoparticles can be carried out by either top-down techniques (starting with the bulk material) or bottom-up approaches (building up the material from atomic or molecular species) (Joseph and Mathew, 2014; Lu et al., 2007; Park et al., 2007; Xia et al., 2009), whereas the latter are generally preferred for enhanced sample homogeneity on the nanoscale. Co-precipitation has been widely applied as a large-scale bottom-up technique, at the expense of moderate particle-size distribution. Large-scale synthesis of monodisperse nanoparticles (e.g., ferrites) with defined shape is nowadays routinely achieved by thermal decomposition of metal oleates in the presence of organic stabilizing ligands (Park et al., 2004, 2005). Stabilization of the prepared nanoparticles in dispersion is important in order to avoid aggregation and precipitation and depends strongly on the nature of the carrier liquid (polar or nonpolar) and the stabilizing approach (steric stabilization using surfactants, polymers, or electrostatic stabilization) (Joseph and Mathew, 2014).

Major challenges addressed by SANS include the intraparticle magnetization, i.e., the spatial magnetization distribution within magnetic nanoparticles, and interparticle structure formation, i.e., aggregate or superstructure formation induced by dipolar interparticle interactions. Moreover, SANS gives insight into microstructural aspects of magnetohydrodynamics and magnetoviscosity of ferrofluids.

We start the discussion in Sec. V.A by introducing the relevant cross-section relations in the framework of the particle-matrix approach. Sec. V.B will review the contribution of magnetic SANS to the investigation of the intraparticle magnetic morphology, including the spatially-resolved magnetization profile in nearly uniformly magnetized nanoparticles, whereas Sec. V.C will focus on the application of magnetic SANS to the structure formation in more concentrated ferrofluids. In Sec. V.D we will summarize recent developments concerning the magnetic structure of shape-anisotropic nanoparticles, such as oriented nanowires in porous alumina matrices.

A. Particle-matrix approach

As already stated in Secs. I and II, magnetic SANS is sensitive to a length scale relevant for magnetic nanoparticles in a nonmagnetic matrix. The particle-matrix approach, consisting of nuclear and magnetic form-factor contributions of the individual nanoparticles as well as a structure factor describing potential interparticle interactions, is therefore widely used to describe SANS data from these systems.

To apply a form-factor model $F(q, \{R, \ldots\})$, the different averaging procedures to obtain a macroscopic cross-
section from the microscopic cross sections for individual particles are of particular importance. This intrinsic property of SANS is connected to the limited size of the coherence volume in the neutron beam (Felber et al., 1998). A system of scatters in the particle-matrix approach is called diluted, if the neutron coherence volume contains on the average only a single particle. If, on the other side, one aims for the investigation of structure formation that volume has to be large enough so that the neutron can probe particle-particle interferences (see Sec. V.C). To a great extent, and with some exceptions (see, e.g., Grigoriev et al., 2010), this coherence volume is defined by the instrument setup. Assuming a diluted system of nanoparticles in a neutron beam with an average typical flux of the modern instruments of the order of \(10^7 \text{s}^{-1} \text{cm}^{-2}\), different neutrons will scatter on different particles in a nonconstructive way. At the 2D detector (Fig. 1), the sum of the squared amplitudes is then related to the count rate, and normalized by the scattering volume and incident flux to obtain the macroscopic SANS cross section.

In most experimental situations, the individual particles have slightly different properties, e.g., different radii, and a global average over a particle-size distribution function \(\omega(R)\) has to be part of the data-fitting procedure; in other words, the sum of the squared amplitudes is often substituted by an integral assuming \(\omega(R)\) to be a continuous function, and the nuclear SANS cross section is expressed as

\[
\frac{d\Sigma}{d\Omega} = \int \Delta\eta_n [F_n(q, R)]^2 \omega(R) dR, \tag{41}
\]

where \(\Delta\eta_n\) denotes the nuclear scattering-length density (SLD) contrast between particle and matrix (generally assumed to be independent of \(R\)). For magnetic SANS, described by a magnetic form factor \(F_m(q, R)\) and the magnetic SLD contrast \(\Delta\eta_m\), a similar expression is valid. As shown in Sec. II.D, the magnetic SLD contrast \(\Delta\eta_m\) between magnetic particle and nonmagnetic material can be written as \(\eta_m = b_H M_s\), where \(b_H = \frac{2\pi a}{\lambda}\mu_{B}^{-1}\) and \(M_s\) is the saturation magnetization (in \(\text{A/m}\)) of the particle. For single-domain (Stoner-Wohlfarth) particles exhibiting stable ferromagnetism, Eq. (41) has to be solved directly by employing a particular form-factor model (or even a structure factor); in case of inhomogeneously magnetized nanoparticles the magnetization distribution has to be computed using micromagnetic theory and the SANS cross sections are given by Eqs. (12)–(20).

In the following we will discuss how the magnetic SANS cross section of superparamagnetic (SPM) or pseudo-SPM particles in a nonmagnetic matrix can be computed; here, an additional averaging procedure over the orientation distribution \(p(\varphi)\) of the individual magnetic particle moment misaligned by the angle \(\varphi\) relative to the field direction becomes relevant. For a single-domain particle, one can assume an orientation distribution based on the thermal and magnetic-energy competition. If \(d\Omega = \sin \varphi d\varphi d\phi\) is the solid angle increment in a polar system where the \(z\)-direction is given by the external field \(\mathbf{H}_0\), then the orientational average over \(\varphi\) is described by the probability distribution

\[
p(\varphi) \propto \exp \frac{\mu_0 H_0 M_s V_p \cos \varphi}{kT}, \tag{42}
\]

where \(M_s\) is the saturation magnetization of the particle with volume \(V_p\). In this model a single neutron interacts with the magnetization vector of the particle which is lying on a cone with solid angle \(\omega = (\varphi, \phi)\) around the external-field axis. Because the \(\phi\) component is independent from any external field, the distribution is random and will not introduce any anisotropy. This is different for the \(\varphi\) component; it directly reflects the influence of the external magnetic field (and temperature) via Eq. (42), leading to a general scattering-intensity pattern of the type

\[
I(q) = A(q) + B(q) \sin^2 \theta. \tag{43}
\]

In case of moderately polydisperse systems, the double average \(\int \int \omega(R)p(\varphi) dR d\varphi\) can be divided in two separate parts. If \([\cdot \cdot]\) denotes the directional average, \([x] = \int x(\varphi)p(\varphi) d\varphi\), it allows the analytical calculation of all the contributions in Eqs. (12)–(20). The required magnetization averages are:

\[
\begin{align*}
\langle \bar{M}_x \rangle = \tilde{M}_y = 0 & \text{ and } \langle \bar{M}_z \rangle = F_m L(\beta) \\
\langle \bar{M}_x^2 \rangle = \langle \bar{M}_y^2 \rangle &= F_m^2 \frac{L(\beta)}{\beta} \\
\langle \bar{M}_z^2 \rangle &= F_m^2 \left(1 - 2\frac{L(\beta)}{\beta}\right),
\end{align*}
\]

so that the SANSPOL cross section in the \(\mathbf{k}_0 \perp \mathbf{H}_0\) scattering geometry [Eq. (14)] evaluates to (for details see, e.g., Heinemann et al., 2004a; Heinemann and Wiedenmann, 2003; Wiedenmann, 2001, 2005):

\[
\frac{d\Sigma_{\pm}^{+}}{d\Omega} = \int \omega(R) dR \left( F_n^2 + 2F_m^2 \frac{L(\beta)}{\beta} + \left\{ F_m^2 \left(1 - 3\frac{L(\beta)}{\beta}\right) + (2P - 1)(2e^\pm - 1)F_m F_n L(\beta)\right\} \sin^2 \theta \right), \tag{45}
\]

where \(F_m\) and \(F_n\) are the magnetic and nuclear form factors, \(L(\beta) = \coth(\beta) - 1/\beta\) is the Lande function with \(\beta = \mu_0 H_0 M_s V_p/(kT)\), and \(\theta\) is the angle between the magnetic field and the q-vector. One can observe that \(A(q)\) in Eq. (43) now contains not only the nuclear SANS but also magnetic contributions; these originate from the distribution of the magnetic moments around the external and decrease with increasing field \((L(\beta)/\beta \to 0\) for \(\beta \to \infty\) and \(L(\beta)/\beta \to 1/3\) for \(\beta \to 0\)). Magnetic field \((\mathbf{H}_0)\) or temperature \((T)\) variation change \(\beta\) and thereby the fraction of \(A(q)\) and \(B(q)\) (Heinemann et al., 2004a; Heinemann and Wiedenmann, 2003). For SANSPOL the
B(q) part now also contains nuclear magnetic interference contributions. This is not the case for unpolarized SANS and can be used to improve the data analysis. If the nuclear $F_n(q, R)$ and magnetic $F_m(q, R)$ form factors depend on the same particle-size distribution, the particle-matrix approach from Eq. (45) leads to a simple equation for the intensity differences [flipper on ($\epsilon^+ = \epsilon$) minus flipper off ($\epsilon^- = 0$)]:

$$d\Sigma^+ - d\Sigma^- = 2(2P - 1)\epsilon \times \int L(H_0, R^3)F_n(R)F_m(R)\sin^2 \theta \omega(R)dR. \quad (46)$$

In this cross term only particles with nuclear and magnetic contrast will contribute; this offers the possibility to extract contributions from the magnetic scattering without the influence of properties from purely nonmagnetic scatterers like e.g. micelles. For SANSPOL experiments, the cross-term intensity Eq. (46) is proportional to $L(\beta)$ and to the magnetic contrast (included in $F_m$) which is assumed to be field independent for single domain particles. This allows a straightforward testing of the Langevin behavior by magnetic field variation experiments.

Figure 14, panel (A) displays field-dependent SANS data obtained on a Co-based ferrofluid. The cross term Eq. (46) was fitted under the constraint that all structural contributions like form factors are field independent and the overall intensity scales with the magnetic contrast via the Langevin function only. Taking the Langevin behavior hypothesis as settled, one can apply a Bayesian method to refine any structural model by magnetic contrast variation (Tatchev et al., 2004). An additional field will contribute to a better $a$ posteriori information about the model and can be used to reduce the uncertainty levels for the model parameter. For clarity, Fig. 14, panel (A) shows only two fields, but the model was fitted with four different fields (including the saturation field) for the final confidence intervals. In principle this is already possible with unpolarized neutrons but as one can see from Eq. (45), the nonsaturated parts give rise to an isotropic contribution leading to bigger error margins in the nuclear-magnetic scattering separation. In case of a second population of nonmagnetic scatters like micelles, it will become very demanding to separate these contributions without the use of polarized neutrons. By analyzing the cross term to separate the structural properties from the magnetic contributions, it is possible to fit contributions from the additional nonmagnetic parts (see Fig. 14, panel (B)).

With POLARIS (SANS with 1D polarization analysis) one can go one step further by analyzing the spin-flip only contributions from Eq. (18) in more detail. We have shown that for pure superparamagnetic behavior the resulting SANS cross section can be written as a sum of two contributions, Eq. (43), where $A(q) = 0$ for the $I^- - I^+$ term. Therefore, it becomes possible to extract $A(q)$ and $B(q)$ because of their known angular dependence. Deviation from superparamagnetic behavior based on a more complex internal spin structure of the nanoparticles would lead to additional terms with different angular dependence due to the occurrence of coherent scattering arising from transversal magnetization components $M_x$ and $M_y$ [compare, e.g., Eq. (18)]. Although this could be observed already within the SANSPOL frame, it would be very hard to extract these additional contributions unambiguously. POLARIS, and in particular the spin-flip term, opens the possibility to observe these contributions. Using the same averaging procedure as in Eq. (44), one obtains for the macroscopic spin-flip cross section:

$$d\Sigma^{\pm} \propto \int d\varphi p(\varphi) \left( |\tilde{M}_x|^2 + |\tilde{M}_y|^2 \cos^2 \theta + |\tilde{M}_z|^2 \sin^2 \theta \cos^2 \theta - 2\tilde{M}_y \tilde{M}_z \sin \theta \cos^3 \theta \right). \quad (47)$$

The $\Sigma^{++}$ part is identical to the $\Sigma^{++}$ part and contains only magnetic contributions (see, e.g., Honecker et al., 2010 and Wiedenmann, 2005). Depending on the model chosen for the distribution of magnetic moments, not necessarily following a Langevin behavior, one can try to identify newly occurring anisotropic scattering patterns (see, e.g., Krycka et al., 2010a,b, 2014, 2015, 2012b; and Michels et al., 2015). In all cases a model-based fitting of the full 2D detector pattern or full-fledged extraction of the different angle-dependent contributions, like in SANSPOL, is preferable.

Moreover, it must be emphasized that all additional corrections for the imperfect analyzers, flippers, and the different transmissions for up and down neutrons have to be taken into account very carefully to minimize spin leakage (Wildes, 2006). Because the nuclear scattering...
can be dominating, even small leakages could lead to misinterpretation of the obtained data. Although in principle POLARIS allows one to unambiguously distinguish between magnetic and nuclear coherent scattering, these experimental complications make such experiments more demanding and error vulnerable.

B. Intraparticle magnetization

The spin structure of magnetic nanoparticles can be regarded as a superspin of coupled atomic spins within the particle core. Depending on particle volume, magnetic anisotropy, and thermal energy, Néel relaxation of the superspin is either blocked or dominant, giving rise to superparamagnetism. The existence of a magnetically dead or canted layer towards the particle surface has been widely reported, mainly based on theory and observations from macroscopic magnetization (Coey, 1971; Kodama and Berkowitz, 1999; Kodama et al., 1996). The lower saturation magnetization than observed in the bulk materials is generally attributed to surface spin disorder (Curiale et al., 2009; Dutta et al., 2009; Kovács et al., 2009), leading to the generally accepted model of magnetic nanoparticles consisting of a superspin core and a surface region of canted or disordered spins. In contrast to spatially averaging and macroscopic integral probes, magnetic SANS has the unique strength to disentangle the spatial distribution of magnetization within the nanoparticle by analysis of the nuclear and magnetic particle sizes (indicating surface spin disorder) as well as the magnetic SLD \( \eta_m \) (corresponding to the absolute magnetization).

In a first approach, the magnetic form factor can be regarded similar or equal to the nuclear inorganic nanoparticle core. In this case, the magnetic contrast obtained by (polarized) SANS is beneficial for structural characterization of magnetic nanoparticles in a nonmagnetic matrix or stabilizing material such as an organic ligand shell (Butter et al., 2004; Hoell et al., 2002), an inorganic oxidized surface layer (Butter et al., 2004), or a silica shell (Bonini et al., 2004, 2006, 2007). Avdeev and Aksenov, 2010 established a modified basic functions approach that relies on a detailed contrast variation using H and D-solvents. This approach allowed for a consistent determination of the nuclear particle morphology for dilute polydisperse nanoparticle dispersions stabilized by different fatty acids (Avdeev et al., 2007).

Heinemann and Wiedenmann further applied the additional contrast variation provided by polarized SANS to the structural characterization of magnetic nanoparticles with a nonmagnetic ligand shell next to excess surfactant (forming nonmagnetic micelles) and larger aggregates (composed of magnetic nanoparticles with nonmagnetic surfactant) (Heinemann et al., 2004a,b; Wiedenmann et al., 2006a; Wiedenmann, 2005). By analyzing the radius of gyration of polydisperse maghemite nanoparticles, Avdeev et al., 2009 report a difference between the nuclear and magnetic radii of maghemite nanoparticles in dilute dispersion which is attributed to surface spin canting or disorder effects. Significant variation between nuclear and magnetic structures are attributed to aggregates. The nuclear signal appears affected by van der Waals interactions at small \( q \), whereas there is negligible dipolar magnetic interaction in the superparamagnetic particle ensemble. In order to exclude the influence of interparticle correlation, a decrease of the particle number density in the magnetic fluids is suggested (Nagorny et al., 2010).

With the availability of nearly monodisperse nanoparticles in large quantities, the discrimination of surface spin disorder has been enabled with enhanced precision. Using the POLARIS technique, Kryck et al., 2010a studied the 3D magnetization distribution in dense arrays of 9 nm iron oxide nanoparticles. Assuming a field and temperature-independent structure factor, the significant difference of form factors in \( M_\parallel \) and \( M_\perp \) indicates surface spin canting in a 1–1.5 nm-thick surface region. Further analysis of the vectorial shell magnetization in \( M_\perp/M_\parallel \) using a core-shell form factor model suggests an average canting angle of 23–31° (at 1.2 T and 300 K) (Krycka et al., 2014). The significant thickness of the canted surface layer found in these studies is likely a result of dipolar interactions in the dense nanoparticle arrays. The spatial magnetization distribution in noninteracting nanoparticles was investigated by Disch et al., 2012 in dilute dispersions of highly monodisperse maghemite nanoparticles using SANSPOL; surface spin disorder was revealed in a narrow surface layer that depends on the particle shape and is slightly thicker for nanocubes (0.5 nm) than for nanospheres (0.3 nm). More recently, Zákutná et al. reported surface spin disorder in noninteracting ferrite nanoparticles with a surface layer thickness of 0.3 nm in saturation, increasing up to 0.7 nm with decreasing applied magnetic field (Zákutná, 2018).

In addition to surface spin disorder, polarized SANS revealed an unexpectedly low magnetic scattering contrast in the particle core (Avdeev et al., 2009; Butter et al., 2004; Disch et al., 2012). It is noteworthy that the determination of the quantitative \( \eta_m \) does not rely on calibration of the differential cross section to absolute units, and thus cannot be mistaken for low particle-number densities. Krycka et al., 2010b derive \( \eta_m \) relative to the known nuclear SLD \( \eta_n \) from the ratio \( F_m/F_n \), assuming a phenomenological structure factor in assemblies of iron oxide nanoparticles. The magnetic SLD \( \eta_m \) is a quantitative measure of the magnetization. As an example, Fig. 15 displays the field dependence of the core \( \eta_m \) in maghemite nanocubes and nanospheres, revealing a spontaneous magnetization of only 76% of the bulk material, independent of the particle shape (Disch et al., 2012). The low magnetization even in the particle
core thus indicates spin disorder in the entire nanoparticle, suggesting that the commonly observed lower macroscopic magnetization in nanoparticles as compared to the bulk material is not solely related to surface spin disorder, but to a large extent a result of low magnetization in the entire nanoparticle.

Given that SANS is sensitive to the continuous magnetization distribution on the nm scale, further characterization of the atomic-scale magnetization or spin disorder in magnetic nanoparticles requires the combination with theory or different techniques. Combination of POLARIS with an energy-balance model was reported for the discussion of the average spin-canting angle, thereby assuming, among others, a perfect magnetite Fe$_3$O$_4$ composition and spin-canting solely on the tetrahedral Fe sites of the spinel structure (Krycka et al., 2014). The study is under controversial debate for the different preconditions assumed (Krycka et al., 2015; Michels et al., 2015). Combining SANSPOL with nuclear-resonant x-ray scattering, Herlitschke et al., 2016 recently demonstrated a significant degree of spin disorder in $^{57}$Fe-enriched maghemite nanoparticles that is related to a deviation of about half of the iron atomic spins in both tetrahedral and octahedral sites from perfect ferrimagnetic order. Neutron diffraction is sensitive to the atomic magnetic moments (Golosovsky et al., 2001), and, applied to maghemite nanoparticles, revealed a slightly larger disorder on the octahedral Fe site (Yusuf et al., 2006). Specifically for magnetite/maghemite nanoparticles, the reduced magnetization has recently been associated with antiphase boundaries as observed using HR-TEM and diffraction techniques (Nedelkoski et al., 2017; Wetterskog et al., 2013).

Enhancement of the magnetic properties of iron oxide nanoparticles was achieved by the introduction of a silica shell, resulting in a significant increase of $\eta_m$ and in a decrease of the lower magnetized surface region as compared to the noncoated particles (Lee et al., 2015). Similarly, magnetic nanocrystals embedded in silicate glasses were found to consist of a magnetic core with equal magnetic and nuclear size, surrounded by a nonmagnetic shell (Raghuvanshi et al., 2014, 2015).

Further progress in the application of magnetic SANS to the study of intraparticle magnetization includes the investigation of more complex magnetic morphologies such as multidomain nanoparticles consisting of aggregates of magnetic crystallites (Bender et al., 2017; Denis et al., 2015), magnetic nanosponges (Bonini et al., 2008), or antiferromagnetic nanoparticles with magnetic shell arising from surface spin disorder (Manna et al., 2012). For core-shell particle systems, magnetic SANS allows one to access the different magnetic contributions in core and shell (Iljiri et al., 2005), and the potential spin disorder at core-shell interfaces.

C. Interparticle correlations

Interactions between magnetic nanoparticles typically originate in magnetic dipole-dipole, steric, or van der Waals interactions and can be induced by increased particle concentration or application of magnetic, electric, or flow fields. As a result, the evolving structure factor indicates in the first approximation the attractive and repulsive interparticle interactions, a scenario that limits the single-particle scattering and therefore the particle-matrix approach. Structures reported for dispersions of magnetic nanoparticles or ferrofluids range from short-range ordered aggregates via chain-like structures to pseudocrystalline ordering in concentrated ferrofluids upon application of a magnetic field. For short-range correlations, models such as hard-spheres or sticky hard-spheres can be applied in combination with different approaches for the structure factor $S(q)$ along with the form factor $F(q)$. The $I \propto F^2(q)S(q)$ approach is applied in many situations where a simple expression for $S(q)$ is plausible. If the characteristic properties of the single nanoparticles give rise to pseudocrystalline ordering, even classical Bragg-scattering patterns including superstructure reflections are observed on the 2D detector (see Fig. 16). Here, the structural information (superlattice symmetry) can be derived directly from the reflection positions, i.e., without complex fitting of a particular structure factor.

Magnetic SANS has been widely applied to the investigation of structure formation and stabilization of ferrofluids. Avdeev et al., 2015 report the importance of both carrier polarity and stabilization technique for the stability of ferrofluids; the stability of sterically, electro-
statically, and double-sterically stabilized ferrofluids has been reviewed in detail. In general, the higher the surfactant efficiency, the more stable is the system at surfactant excess (Petrenko et al., 2016). Double-layer stabilization enables the use of carrier media with high polarity such as water for ferrofluids otherwise dispersed in nonpolar solvents, however, at the expense of stability (Avdeev et al., 2006). Attractive interparticle interactions, revealed in sterically-stabilized nonpolar benzene-based ferrofluids, can be directly related to the magnetoviscous properties. In comparison, for moderately polar pentanol-based ferrofluids repulsive interparticle interactions are observed without any magnetoviscous effect (MVE) (Avdeev and Aksenov, 2010).

The MVE is related to an increased viscosity upon application of a magnetic field as a result of interparticle interactions in ferrofluids. Using magnetic RheoSANS, Pop and Odenbach established the correlation between the ferrofluid microstructure and the MVE as a result of aggregate/chain formation in the ferrofluid, whereas the macroscopically observed shear-thinning effect is induced by breakage of these chains upon increased shear rates (Pop et al., 2004; Pop and Odenbach, 2006). In the context of structural relaxation of magnetic nanoparticle aggregates, the potential of a rotating sample in a static magnetic field instead of using a rotating magnetic field was demonstrated for the investigation of the q-dependent diffusion time \( \tau_q \), applicable to relaxation times of 3 – 300 ms (Wandersman et al., 2009). An analogous electroviscous effect was reported upon application of DC or AC electric fields, where changes in the viscosity were related to aggregate formation in transformer-oil-based ferrofluids (Kurimsky et al., 2017; Rajnak et al., 2017, 2015).

Upon application of a magnetic field, strong anisotropy of the Brownian motion in solution is observed, resulting in a lowering of the concentration fluctuations along the field, concomitantly expressed by an anisotropic structure factor of an otherwise fluid-like magnetic colloid sample (Gazeau et al., 2002; Mériguet et al., 2006a, b). Formation of chain-like structures that orient in the direction of an applied magnetic field was observed for magnetosomes (Hoell et al., 2004) as well as for cobalt nanoparticles (Barrett et al., 2011). On a larger scale, Jain et al., 2014 observed field-dependent chain formation of 160 nm ferrofluid emulsion droplets. Wiedemann et al., 2002 discussed the field- and concentration-induced structure formation in cobalt ferrofluids in detail; depending on the particle concentration, a transition of isolated particles (< 1 vol.%) to short-range ordered aggregates (5 vol.%) is observed. Upon application of a magnetic field, a combination of chain-line structures and pseudocrystalline ordering is found (Heinemann et al., 2007; Wiedemann et al., 2003; Wiedemann and Heinemann, 2005). A SANSPOL study of colloidal magnetite nanoparticles illustrates how a larger particle size leads to larger dipole-dipole interaction, resulting in 2D nanoparticle sheets with hexagonal symmetry (Klokkenburg et al., 2007). Recently, Fu et al., 2016 reported the field-induced self-organization of iron oxide nanoparticles. The combined SANS and VSANS study gives information on both the symmetry of the superlattice and its extensions in large-scale oriented aggregates. Whereas aggregate formation is already observed at a very low field of 0.02 T, a highly crystalline superlattice emerges beyond 0.1 T that is indexed according to a fcc lattice with the nearest-neighbor direction [110] oriented parallel to the applied field (see Fig. 16).

In a solid matrix, liquid-like short-range order of small \(~ 2\) nm-sized Fe nanoparticles has been identified, revealing ferromagnetic interparticle correlations at temperatures below 100 K (Bellouard et al., 1996). Field-dependent SANS experiments show magnetic interparticle correlations in dense assemblies of Fe and Co nanoparticles with different degree of structural order (Farrell et al., 2006; Ijiri et al., 2005; Sachan et al., 2008). Ridier et al., 2017 have recently revealed the individual-collective crossover depending on the particle size in dense assemblies of nanoparticles; the observation of ferromagnetically correlated clusters of 4.8 nm particles, with a temperature-dependent magnetic correlation length, is opposed to the superparamagnetic behavior of larger 8.6 nm particles and was attributed to the much larger surface anisotropy in the smaller nanoparticles. Applied to granular perpendicular recording media, SANSPOL measurements provide both the magnetic grain size, being smaller than the nuclear grain size, and the interparticle correlation distance (Lister et al., 2009) as well as the switching process upon magnetization reversal (Lister et al., 2010).
D. Anisotropic nanostructures

The application of magnetic SANS has further been expanded to anisometric magnetic nanoparticles such as oriented nanowires. Magnetic nanowires are nowadays routinely prepared using templated techniques with either amorphous silica or anodized alumina matrices as templates. The degree of alignment and order is tunable from 2D powders of aligned nanowires with short-range order towards crystalline arrangements with hexagonally-packed nanowires. With typical dimensions of a few tens of nm in cross section, such structures still fit in the framework of nanomaterials (i.e., with two dimensions smaller than $\sim 100$ nm). Magnetization reversal in aligned nanowire arrays has been studied using both polarized and unpolarized SANS, and spin misalignment is commonly observed (Grigoryeva et al., 2007; Grutter et al., 2017; Günther et al., 2014; Maurer et al., 2014). However, with a typical nanowire length approaching the micrometer range and usually oriented parallel to the neutron beam, arrays of magnetic nanowires act as a grating, and strong multiple scattering has to be taken into account depending on the nanowire length; this effect is discussed in depth in Grigoriev et al., 2010. Due to their relatively large volume, nanowires exhibit stable ferromagnetic properties and are mostly multidomain structures. The particle matrix approach is therefore not applicable to nanowires; indeed, micromagnetic simulations (Vivas et al., 2017) and experimental data (Günther et al., 2014) demonstrate strong deviations from the uniform-particle form-factor model.

VI. MICROMAGNETIC SIMULATIONS

In this section we review a novel micromagnetic simulation methodology which takes site-dependent magnetic parameters (saturation magnetization, magnetic anisotropy) and interactions (exchange and magnetodipolar field) into account. This approach enables studies of the magnetic microstructure of a wide range of polycrystalline magnetic materials such as single-phase nanocrystalline magnets, magnetic nanocomposites, recording media, or magnetic particles in a nonmagnetic matrix (Erokhin et al., 2012a,b, 2015; Löffler et al., 2005; Michels et al., 2014; Ogrin et al., 2006; Saranu et al., 2008; Zighem et al., 2013).

The majority of results which are discussed later on were obtained on nanocomposites. We would like to emphasize that this class of materials is one of the most complicated from the point of view of numerical simulations. The main difficulty is that they consist of at least two phases, and the boundaries between these phases are complicated curved surfaces; a typical example is a hard-soft nanocomposite consisting of magnetically hard (i.e., having a large magnetocrystalline anisotropy) crystal grains surrounded by a magnetically soft matrix. In order to perform accurate and efficient simulations of multiphase nanocomposites, Erokhin et al., 2012a,b have proposed a methodology which combines the advantages of a flexible mesh generation with an effective calculation of the micromagnetic energy. Namely, the whole mesh generation algorithm can be viewed (Michels et al., 2014) as a method to discretize a sample into polyhedra having nearly spherical shape. It allows the use of spherical dipole approximation—equivalent to the point dipole approximation—for the evaluation of the magnetodipolar interaction between the various finite elements.

In the micromagnetic simulations all four standard contributions to the total magnetic free energy are taken into account: energy in the external magnetic field, energy of the magnetocrystalline anisotropy, exchange-stiffness and magnetodipolar interaction energies (the antisymmetric Dzyaloshinski-Moriya interaction can also be implemented). The system energy due to the presence of an external magnetic field and the energy of the magnetocrystalline anisotropy (which can be uniaxial and/or cubic) are calculated in the model in the standard way, namely:

$$E_{\text{ext}} = -\sum_{i=1}^{N} \mathbf{\mu}_i \cdot \mathbf{H}, \quad (48)$$

$$E_{\text{an}}^{\text{un}} = -\sum_{i=1}^{N} K_i^{\text{un}} \Delta V_i (\mathbf{m}_i \cdot \mathbf{n}_i)^2, \quad (49)$$
FIG. 17 (a) Schematics of the two scattering geometries and of the microscopic structure of the nanocomposite sample with simulation volume $V = 250 \times 600 \times 600$ nm$^3$. Blue polyhedrons—Fe particles; yellow-orange-red polyhedrons—the matrix phase. Image taken from Michels et al. (2014). (b) Examples of the spatial distribution of hard magnetic crystallites (soft crystallites not shown) with different aspect ratios of corresponding ellipsoids of revolution. Image taken from Erokhin and Berkov, 2017. (c) Core-shell microstructure used in the micromagnetic simulations of a Nd–Fe–B nanocomposite and example of a polyhedron mesh element distribution in a single grain; blue (yellow) elements correspond to the core (shell), typical mesh-element size is 2 nm.

\[
E_{\text{exch}} = \sum_{i=1}^{N} K_i \Delta V_i \left( m_{i,x}^2 + m_{i,y}^2 + m_{i,z}^2 \right),
\]

where $\mathbf{H}$ is the external field, $\mu_i = \mu_i(r_i)$ and $\Delta V_i$ are the magnetic moment and the volume of the $i$-th finite element (polyhedron), and $\mathbf{m}_i = \mu_i/\mu_i$ denotes the unit magnetization vector. Both the anisotropy constants $K_i$ and the directions of the anisotropy axes $\mathbf{n}_i$ can be site-dependent, as required for a polycrystalline material. The symbols $m_{i,x}$ etc. represent the components of the unit magnetization vectors in the local coordinate system that is attached to the cubic anisotropy axes. Higher-order anisotropy contributions can be taken into account.

The evaluation of the exchange-energy contribution in the model requires a much more sophisticated approach than in the standard finite difference method (FDM), because the continuous integral version of this energy contains magnetization gradients,

\[
E_{\text{exch}} = \int_V A(r) \left[ (\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2 \right] dV,
\]

where $A$ denotes the exchange-stiffness constant, and $V$ is the sample volume. Finding an approximation to Eq. (51) for a disordered system based on some interpolation procedure preserving the smooth behavior of magnetization components—required for the correct evaluation of derivatives in Eq. (51)—is a highly complicated task.

Berkov and Gorn, 2005 have chosen a completely different approach and developed an algorithm for the exchange-energy evaluation based on the summation of the nearest-neighbor contributions, widely used in FDMs. The integral in Eq. (51) is approximated by the following sum:

\[
E_{\text{exch}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \subset \text{n.n.}(i)} \frac{2A_{ij} \Delta V}{a^2} (\mathbf{m}_i \cdot \mathbf{m}_j).
\]

Here, $a$ is the cell size of a regular cubic grid (so that the cell volume is $\Delta V = a^3$), $A_{ij}$ denotes the exchange-stiffness constant between cells $i$ and $j$, and the notation $j \subset \text{n.n.}(i)$ means that the inner summation is performed over the nearest neighbors of the $i$-th cell only. We note in passing that this Heisenberg-like expression is valid only when the angles between neighboring moments are not too large: as shown in Berkov and Gorn, 2005, neglecting this condition can lead to unphysical results.

In the case of a disordered system, the following expression for the exchange-stiffness energy, which is analogous to Eq. (52), has been proposed:

\[
E_{\text{exch}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \subset \text{n.n.}(i)} \frac{2A_{ij} \Delta V_{ij}}{\Delta r_{ij}^2} (\mathbf{m}_i \cdot \mathbf{m}_j),
\]

where $\Delta V_{ij} = (\Delta V_i + \Delta V_j)/2$, $\Delta r_{ij}$ is the distance between the centers of the $i$-th and the $j$-th finite elements with volumes $\Delta V_i$ and $\Delta V_j$, and $A_{ij}$ is the exchange constant. This expression should be corrected taking into account that the number of nearest neighbors for different finite elements may be different. It is done by the introduction of the correction factor $\theta/n_{av}$, where $n_{av}$ is
the average number of nearest neighbors for the particular random realization of the disordered finite element system. The accuracy of this simple correction method is surprisingly good, as shown by tests presented in Erokhin et al., 2012b and Michels et al., 2014.

The energy of the long-range magnetodipolar interaction between magnetic moments and the corresponding contribution to the total effective field are computed using the point-dipole approximation as

\[ E_{\text{dip}} = -\frac{1}{2} \sum_{i=1}^{N} \mu_i \sum_{j \neq i} \frac{3e_{ij}(e_{ij} \cdot \mu_j) - \mu_j}{\Delta r_{ij}^3}, \tag{54} \]

i.e., magnetic moments of finite elements are treated as point dipoles located at the polyhedra centers. This approximation is equivalent to the approximation of spherical dipoles, i.e., it would be exact for spherical finite elements. Hence, for the discretized system, this approximation introduces some computational errors, because the finite elements are polyhedra. However, these errors are small, because the shape of these polyhedra is close to spherical, due to the special algorithm employed for the generation of the mesh. If necessary, these errors can be significantly reduced further, taking into account higher order terms in the multipole expansion.

The summation in Eq. (54) is performed by the particle-mesh Ewald method. The specific implementation of the lattice-based Ewald method for the magnetodipolar interaction for regular and disordered systems of magnetic particles is described by Berkov and Gorn, 1998 and Gorn et al., 2007. The major advantage is the possibility to use FFT for computing the long-range part of the total magnetodipolar field.

For the minimization of the total magnetic energy, obtained as the sum of all the contributions described above, a highly optimized version of a gradient method is used employing the dissipation part of the Landau-Lifshitz equation of motion for magnetic moments (Berkov, 2007; Landau and Lifshitz, 1935). For the termination of the energy minimization, the local torque criterion is taken into account: the iteration process stops, if the maximal torque acting on magnetic moments is smaller than some prescribed value. This condition is more appropriate than the alternative criterion of a sufficiently small energy difference between two subsequent steps.

The methodology described above was successfully employed (Erokhin et al., 2012a) for the explanation of the nontrivial SANS cross section (so-called “clover-leaf” pattern) observed in the Fe-based nanocrystalline alloy NANOPERM (Suzuki and Herzer, 2006). This nanocomposite consists of magnetically hard grains (with a typical size of 12 nm) surrounded by a soft magnetic matrix. Figure 17(a) demonstrates the “sample” used in micromagnetic simulations: each blue mesh element represents a hard crystallite with a homogeneous magnetization with it; warm colors mark mesh elements representing the soft phase. The approach permits the modeling of a relative large sample volume \((250 \times 600 \times 600 \text{nm}^3)\), providing a high statistical accuracy of simulation results.

Further development (Erokhin and Berkov, 2017) of the simulation technique (Erokhin et al., 2012a,b; Michels et al., 2014) was necessary to study the influence of the nonspherical shape of hard grains [Fig. 17(b)] on the magnetic behavior of nanocomposites. In this case, hard crystallites had to be discretized into smaller mesh elements, because the spherical dipole approximation for the magnetodipolar field created by hard grains was insufficient due to their nonspherical shape. Another example of the geometrical flexibility of the methodology is the recent micromagnetic modeling of a Nd–Fe–B nanocomposite, where a core-shell particle model for the description of 20 nm-sized Nd–Fe–B grains (Erokhin et al., 2018) is implemented. This model takes into account changes in magnetic parameters of the Nd–Fe–B crystallites near their surface, which may be imperfect due to the manufacturing process. A typical core-shell microstructure used in these simulations and a polyhedron mesh used for the discretization of a grain are shown in Fig. 17(c). In order to resolve the magnetization distribution inside the shell, the mesh-element size has been set to 2 nm.

Several examples of micromagnetic simulation results for various structures—a nanocomposite of the NANOPERM type, magnetic nanoparticles of different sizes, core-shell structure of Nd–Fe–B—are collected in Fig. 18. In particular, Fig. 18(a) displays the spatial distribution of the magnetization component \(M_{\perp}\) perpendicular to the applied field around two Fe nanoparticles in NANOPERM along with the magnetodipolar field produced by these particles. Generally speaking, the sources of the magnetodipolar field are regions with a nonzero divergence of the magnetization \((\nabla \cdot \mathbf{M} \neq 0)\). For magnetic nanocomposites, the most prominent “magnetic volume charges” \(\rho_{\text{mag}} = -\nabla \cdot \mathbf{M}\) are due to the abrupt changes in the magnetic materials parameters at the phase boundary between particles and matrix, e.g., variations in the magnetization or anisotropy constants. Such jumps in the magnetic material parameters may give rise to an inhomogeneous spin structure which decorates each nanoparticle. Exactly this situation, for a magnetization jump of \(\Delta M = 1200 \text{kA/m} \pm 1.5 \text{T}\) at the interface between the Fe particle and the amorphous magnetic matrix in NANOPERM, can be observed in Fig. 18(a), where correlations between the magnetodipolar field and the magnetization distribution in the soft phase can be clearly seen.

Another very important question in the development of nanomaterials is the determination of the critical single-domain size for magnetic nanoparticles of various shapes. This parameter is crucial for the production of high-performance permanent magnets based on these materials. The system of small-sized magnetic grains behaves
FIG. 18 (a) Results of a micromagnetic simulation for the spin distribution around two selected Fe nanoparticles (blue circles), which are assumed to be in a single-domain state. The external magnetic field $H$ is applied horizontally in the plane. The magnetization component $M_\perp$ perpendicular to $H$ is shown by red arrows; thickness of arrows is proportional to the magnitude of $M_\perp$. Blue lines visualize the magnetodipolar field. Image taken from Michels 	extit{et al.}, 2014. (b) Dependence of the spin structure of a nanoparticle on the particle size represented by selected magnetization configurations of Fe particles with diameters of $D = 20$ nm and $D = 40$ nm at an external field of $-30$ Oe. (c) Simulated upper part of the hysteresis loop (green line) and magnetization distribution (two-dimensional cuts out of three-dimensional distributions) at selected points on the hysteresis curve (approach-to-saturation, remanence, coercivity) obtained in the core-shell model of a Nd–Fe–B nanocomposite. Contributions to the hysteresis loops only from cores (blue dashed lines) and from shells (red dashed lines) are also shown as lower parts of these loops.

like a system of Stoner-Wohlfarth particles (each having a homogeneous magnetization), resulting in a high remanence and large coercivity of a nanocomposite. By contrast, the vortex state formed inside larger particles with sizes above the single-domain limit results in a very small remanence and in a reduced coercive field (the magnetization distributions of both types are presented in Fig. 18(b)).

Advancement of permanent-magnet materials requires a clear physical understanding of the relation between their microstructure and the macroscopic magnetic properties. To establish this relation for the core-shell model of a Nd–Fe–B nanocomposite, we show in Fig. 18(c) the magnetization distribution at selected points on the hysteresis curve (large positive field, remanence, coercivity). It can be clearly seen that in high fields the shells exhibit a larger magnetization projection in the field direction than the cores, since the shell anisotropy constants are reduced compared to the core regions. This situation prevails down to the remanent state, where a qualitatively similar spin distribution is observed. However, at negative fields the shells reverse their magnetization “easier” than the cores (compare, e.g., the dashed curves in Fig. 18(c)), which again can be attributed to the reduced anisotropy in the shell region. The most interesting feature of the presented hysteresis is the contradiction between a relatively low coercivity and a “hard” system behavior at large positive fields, where the saturation is not reached even at the applied field of 2 T (we call it the “hard-soft effect”). Erokhin 	extit{et al.}, 2018 were able to explain this effect by combining the core-shell model of nanograins with the influence of a considerable volume fraction of particles in a superparamagnetic state.

B. Simulation of magnetic neutron scattering: decrypting SANS cross sections

This section discusses simulation results for the magnetic SANS cross sections of soft magnetic nanocomposites. The most important advantage of numerical simulations is the possibility to decrypt the magnetic SANS cross sections, i.e., one can study the contribution of each individual Fourier component of the magnetization distribution separately. Also the twofold impact of the magnetodipolar interaction can be disentangled, which provides fundamental insights into magnetic SANS from polycrystalline materials (see below).

Since the focus is on magnetic spin-misalignment
scattering, the nuclear SANS contribution was ignored and the discussion was restricted to unpolarized neutrons. Note, however, that for polycrystalline texture-free magnetic nanocomposites the nuclear SANS signal is isotropic and independent of the applied magnetic field, and its magnitude is generally small compared to the spin-misalignment scattering discussed here (Michels and Weissmüller, 2008).

In the following we show selected results for the unpolarized magnetic SANS cross section \( d\Sigma_M / d\Omega \) in the perpendicular scattering geometry (compare Eq. (12) in Sec. II.B). The functions \( M_{x,y,z}(r) \) are obtained from the micromagnetic simulations; applying the Fourier transformation and combining them into Eq. (12) (or another one for a different scattering geometry), we derive the simulated magnetic SANS cross section. In order to compare the numerical results with experiment, it is necessary to obtain the scattering cross section in the plane of the detector. We apply three-dimensional Fourier transformation to \( M_{x,y,z}(r) \) and use data on the plane \((q_x,q_z)\) at \( q_x = 0 \), which corresponds to the standard SANS setup, where \( q_x \) is negligible in comparison to the other two components.

Returning to the simulations of NANOFORM discussed above, we display in Fig. 19 projections of the functions \(|\hat{M}_x|^2\), \(|\hat{M}_y|^2\), \(|\hat{M}_z|^2\), and of the cross term \( CT = -\left(\hat{M}_x\hat{M}_y^* + \hat{M}_y\hat{M}_x^*\right)\) present in Eq. (12) into the detector plane at selected external field values. Also, the field dependence of the magnetic SANS cross section \( d\Sigma_M / d\Omega \) is shown. This representation emphasizes the power of the approach: by employing numerical micromagnetics for the computation of magnetic SANS cross sections, it becomes possible to study the individual magnetization Fourier components and their contribution to \( d\Sigma_M / d\Omega \). In particular, the approach of combining micromagnetic and SANS simulations complements experiments, which provide a weighted sum of Fourier components, a fact which often hampers the straightforward interpretation of experimental SANS data. While it is in principle possible to determine some Fourier coefficients separately, e.g., through the application of a saturating magnetic field or by exploiting the neutron polarization degree of freedom via SANSPOL or POLARIS methods (e.g., Honecker et al., 2010 and Wiedenmann, 2010), it is difficult to unambiguously determine a particular scattering contribution without “contamination” by unwanted Fourier components. For instance, when the applied field is not large enough to completely saturate the sample, then the scattering of unpolarized neutrons along the field direction does not represent the pure nuclear SANS, but contains also the magnetic SANS due to the misaligned spins (Bischof et al., 2007).

Analyzing the individual contributions to the total magnetic SANS cross section presented in Fig. 19, we can see that \(|\hat{M}_z|^2\) is nearly isotropic (i.e., \(\theta\)-independent) over the whole ranges of the applied field and scattering vectors. By contrast, at the smallest \(q\) and largest fields, the Fourier coefficient \(|\hat{M}_y|^2\) reveals a pronounced angular anisotropy with maxima roughly along the diagonals of the detector (the so-called “clover-leaf” anisotropy), whereas at the smaller fields, the anisotropy of \(|\hat{M}_y|^2\) is rather of the \(\cos^2\theta\)-type (i.e., elongated parallel to \(\mathbf{H}\)). At saturation (\(\mu_0 H = 1.5\, T\)), both \(|\hat{M}_x|^2\) and \(|\hat{M}_y|^2\) are relatively small and the main contribution to \(d\Sigma_M / d\Omega\) is due to the term \(|\hat{M}_x|^2\), which originates from nanoscale jumps of the magnetization at phase boundaries. By decreasing the field, the magnitude of the transversal components increases as the spin misalignment on the scale of tens of nanometers develops.

The next important insight into the field-dependent relation between the different contributions in Eq. (12) can be deduced from the radially-averaged data shown in Fig. 20(c). For the particular applied field of \(\mu_0 H = 30\, mT\) used for this figure, the dominance of \(|\hat{M}_y|^2\) and \(|\hat{M}_x|^2\) in the small \(q\)-range becomes evident, especially when the logarithmic scale of the cross-section axis is considered. By contrast, at larger \(q\) the influence of the \(|\hat{M}_z|^2\) component on the resulting cross section clearly prevails.

Figure 20(d) is a vivid illustration of the strong dependence of magnetic SANS on the applied magnetic field. In the small \(q\)-range, the difference between the cross section at \(\mu_0 H = 10\, mT\) and at saturation (1500 nT) can
FIG. 19 Results of micromagnetic simulations for the Fourier components of the magnetization. The images represent projections of the respective functions into the detector plane $q_x = 0$ for $k_0 \perp H$. Columns from left to right: $|\tilde{M}_x|^2$, $|\tilde{M}_y|^2$, $|\tilde{M}_z|^2$, $CT = -(\tilde{M}_y\tilde{M}_z^* + \tilde{M}_z\tilde{M}_y^*)$ and $d\Sigma_M/d\Omega$. Pixels in the image corners correspond to $q \approx 1.8 \text{ nm}^{-1}$. Logarithmic color scale is used (color bars are in arbitrary units).

FIG. 20 (a) and (b) Two-dimensional cuts (shifted for a better visibility) of $|\tilde{M}_z|^2(q_x, q_y, q_z)$. (c) Radially-averaged individual scattering contributions to $d\Sigma_M/d\Omega$ as a function of the scattering vector $q$ ($k_0 \perp H$) (log-log scale): total $d\Sigma_M/d\Omega$ (black); $|\tilde{M}_z|^2\sin^2 \theta$ (blue); $CT$ (magenta); $|\tilde{M}_y|^2\cos^2 \theta$ (green); $|\tilde{M}_x|^2$ (red). (d) Radially-averaged total magnetic SANS cross section $d\Sigma_M/d\Omega$ as a function of the scattering vector $q$ and for several applied magnetic fields $H$ (see inset) ($k_0 \perp H$) (log-log scale). Materials parameters for NANOPERM were used.

be as large as two orders of magnitude. This observation demonstrates the tremendous sensitivity of SANS in the study of magnetic materials; particularly, it has a special importance for the investigation of magnetization distributions of such an extremely magnetically soft nanocomposite as NANOPERM, where the difference of total magnetizations at the above mentioned applied fields does not exceed 5%. The field-independent local maximum at $q \approx 0.78 \text{ nm}^{-1}$ corresponds to $\sim 8 \text{ nm}$ distance in the real space and agrees very well with the size of the hard magnetic crystallites used in the model.

The finding that $|\tilde{M}_z|^2$ is nearly isotropic and that $|\tilde{M}_y|^2 = |\tilde{M}_y|^2(q, \theta)$ strongly depends on the angle $\theta$, provides a straightforward explanation for the experimental observation of the clover-leaf anisotropy in the SANS cross section of the nanocrystalline two-phase alloy NANOPERM (Michels et al., 2006). The simulation results for the difference cross section, i.e., where the scattering at saturation ($\mu_0H = 1.5 \text{ T}$) has been subtracted, agree semiquantitatively (up to an unknown
scaling factor) with the experimental data, as shown in Fig. 21 (Erokhin et al., 2011, 2012b). Clover-leaf-type anisotropies in $d\Sigma_M/d\Omega$ have also been reported for a number of other materials, including precipitates in steels (Bischof et al., 2007), nanocrystalline Gd (Döbrich et al., 2012; Michels et al., 2008), and nanoporous Fe (Michels et al., 2009).

As a final point, the impact of the magnetodipolar interaction is discussed: the quantity of interest in an elastic magnetic neutron scattering experiment, the differential scattering cross section $d\Sigma_M/d\Omega$, depends in a twofold manner on this interaction. First, the interaction of the magnetic moment of the neutron with the sample’s magnetization results in dipolar selection rules which are embodied, e.g., by the appearance of trigonometric functions in $d\Sigma_M/d\Omega$ [via the Halpern-Johnson vector, Eq. (7)] (Squires, 1978). Second, the magnetodipolar interaction between the magnetic moments in the sample has a direct impact on its magnetization configuration and therefore on the Fourier components of the magnetization. The former determine the properties of $d\Sigma_M/d\Omega$ and the latter the total magnetization (as measured by magnetometry).

The corresponding difference can be clearly demonstrated by comparing the results for the correlation function of the spin-misalignment SANS cross section (Döbrich et al., 2012; Mettus and Michels, 2015; Michels et al., 2003; Weissmüller et al., 2004),

$$C_M(y,z) \sim \int \frac{d\Sigma_M}{d\Omega}(q_y,q_z) e^{iqr} d^2q,$$  \hspace{1cm} (55)

with the autocorrelation function $C_{SM}(r)$ of the magnetization component perpendicular to the applied field; the latter is defined as

$$C_{SM}(r) \sim \int M_\perp(x) M_\perp(x+r) d^3x \quad (56)$$

and is not decorated by the dipolar interaction between the neutrons and the sample magnetization. Using the convolution theorem, Eq. (56) can be rewritten as

$$C_{SM}(r) \sim \int \left( |\tilde{M}_x(q)|^2 + |\tilde{M}_y(q)|^2 \right) e^{iqr} d^2q. \quad (57)$$

Both correlation functions, simulated for porous Fe, are depicted at a field of $\mu_0 H = 0.6$ T in Fig. 22 along the horizontal ($z$) and vertical ($y$) direction; see Fig. 6, which depicts the spin structure around a spherical pore in Fe. One recognizes the existence of anisotropic correlations already for the autocorrelation function of the spin misalignment (not influenced by the interaction between neutrons and magnetic moments), which may be expected due to the long-range and anisotropic nature of the magnetodipolar interaction. The difference between both directions is significant (in particular for $r \approx 30 - 40$ nm) with $C_{SM}$ along the vertical direction being exclusively positive, while $C_{SM}$ along the horizontal direction intersects the $r$-axis at $r \approx 20$ nm and possesses a minimum at $r \approx 30$ nm.

The existence of “anticorrelations” in $C_{SM}$ around these particular $r$-values is a manifestation of the typical magnetization distribution $M_\perp(r)$ around a pore.
(see Fig. 6), which is due to the configuration of the magnetodipolar field in the vicinity of such an inclusion. Namely, the perpendicular magnetization component changes its sign along the direction of the applied field at a distance comparable with the pore diameter. Of course, the zeros and global minima of the correlation functions are dependent on the applied-field value.

Summarizing this section, we conclude that micromagnetic modeling perfectly complements magnetic SANS in decrypting the corresponding cross section by splitting it into different components or by providing the real-space magnetization distribution more appropriate for the physical analysis. As mentioned above, magnetic SANS and micromagnetic simulations allows the investigation of the properties of materials in bulk, therefore, their powerful combination will provide fundamental insights into the magnetism of nanocomposites.

VII. SANS AS A PROBE OF NANOSCALE MAGNETIC INHOMOGENEITY IN COMPLEX MAGNETIC SYSTEMS

A. Magnetic inhomogeneity and magnetic phase separation

Sections III to IV focused on systems with explicit chemical heterogeneity, or assemblies of nanoparticles. There are, however, nominally chemically uniform materials that spontaneously nanostructure magnetically; these are the focus of this section. A simple paradigm in condensed matter physics is that materials with spatial homogeneity in structure and chemistry can be expected to exhibit corresponding homogeneity in properties. The complex, chemically disordered materials in many contemporary research fields stretch this paradigm to its limits, however, in some cases violating it. One important example emerged from the study of perovskite manganites that began in the mid-1990’s, stimulated by the discovery of colossal magnetoresistance (CMR) (Coey et al., 1999; Dagotto, 2002; Dagotto et al., 2001; Tokura and Tomioka, 1999). Although these compounds were studied 40 years earlier (Jonker and Van Santen, 1950), their tendency to display large resistivity decreases in applied magnetic fields was overlooked. Originally identified in systems such as mixed-valence La$_{1-x}$Sr$_x$MnO$_3$, CMR was eventually demonstrated to reach MR ratios of 10$^12$ in just a few Tesla, justifying the term “colossal” (Coey et al., 1999; Dagotto, 2002; Dagotto et al., 2001; Tokura and Tomioka, 1999).

From the mid-90’s it was gradually understood that a feature of such materials, on both nano- and microscopic scales, is inhomogeneity in quantities such as conductance and magnetization, even in the highest structural/chemical quality (Dagotto, 2002; Dagotto et al., 2001). Evidence for this was accumulated from macroscopic measurements (e.g., magnetotransport, magnetometry, heat capacity), spectroscopic signatures of coexisting electronic/magnetic phases [e.g., from nuclear magnetic resonance (NMR)], imaging with scanning tunneling microscopy/spectroscopy (STM/STS), magnetic force microscopy and TEM, and reciprocal space methods such as neutron diffraction (ND) and inelastic neutron spectroscopy (INS) (Dagotto, 2002; Dagotto et al., 2001). Real space examples are provided in Fig. 23(a) and (b). Classic cases include nanoscale ferromagnetic metallic (FMM) clusters in a matrix of paramagnetic insulator around the Curie temperature ($T_C$) in La$_{0.7}$Ca$_{0.3}$MnO$_3$ [Fig. 23(b) (Fath et al., 1999)], and microscopic coexistence of FMM and charge- and orbitally-ordered antiferromagnetic insulator (COOAFI) phases in La$_{0.5}$Ca$_{0.5}$MnO$_3$ and (La$_{1-y}$Pr$_y$)$_{1-x}$Ca$_x$MnO$_3$ [Fig. 23(a) (Cheong et al., 2002; Uehara et al., 1999)]. These competing phases derive from the competition between spin, charge, orbital, and lattice degrees of freedom, metallic phases being favored by double-exchange, insulating ones by charge order (CO), orbital order (OO),
and superexchange AF (Coey et al., 1999; Dagotto, 2002; Dagotto et al., 2001; Tokura and Tomioka, 1999). This competition was eventually understood in detail, the global manganite phase diagram being mapped in the plane of electronic bandwidth (ionic radius) vs. quenched disorder (radius variance) (Tomioka and Tokura, 2004). The spatial variation in electronic and magnetic properties was termed electronic or magnetic inhomogeneity, or, as it often emerges from a uniform paramagnet on cooling, electronic or magnetic phase separation. These electronic inhomogeneities also occur in other oxides, including cobaltites and cuprates [Fig. 23(c) (Pasupathy et al., 2008)].

Great effort was expended on understanding the origin of this inhomogeneity, due to its role in CMR. Specifically, numerous studies concluded that CMR could be understood in terms of magnetic-field-induced coalescence of FMM regions in a non-FMM matrix [e.g., Fig. 23(b)] (Dagotto, 2002; Dagotto et al., 2001). Models based on finite temperature double-exchange, or double-exchange competing with a superexchange AF insulator, among others, were shown capable of reproducing phase separation [e.g., Fig. 23(d) (Mayr et al., 2001)], even semi-quantitatively reproducing the temperature ($T$) and field ($H$) dependence of the resistivity (Dagotto, 2002; Dagotto et al., 2001). While subtleties remain, including two forms of CMR (Tokuera et al., 1996), and two forms of phase separation [at nano- and microscales (compare Fig. 23(a) and (b)], this explanation for CMR has become widely accepted (Dagotto, 2002; Dagotto et al., 2001). The role of disorder vs. purely electronically-driven phase separation remains a challenge, however, as does the understanding of chemical/structural inhomogeneities accompanying and/or driving the magnetic/electronic inhomogeneity.

Importantly, SANS played, and continues to play, an important role in detecting and elucidating such magnetic inhomogeneity in complex oxides. The ability to probe nano- and mesoscopic scales, the high sensitivity to inhomogeneity, the magnetic sensitivity, and the penetration capability make SANS a powerful probe of this physics. Polarization, and measurement over a wide $T$-range, further enable separation of magnetic and structural/chemical contrast. Methods and approaches originally honed on systems such as manganites have also now been applied to other complex oxides, as well as nonoxidic systems, including alloys. This section provides a summary of the application of SANS to the study of short-range magnetic inhomogeneity in these complex systems.

B. Complex magnetic oxides

As discussed above, complex oxides are systems in which nano- and microscale magnetic inhomogeneities are especially important. We thus begin with a review of the use of SANS in the study of such materials, first in manganites (Sec. VII.B.1), then in cobaltites (Sec. VII.B.2).

1. Perovskite manganites

As discussed in Sec. II, the neutron scattering intensity near $q = 0$ is inherently sensitive to long-range FM order, often detected in elastic SANS via Porod scattering from magnetic domains. Such scattering follows Eq. (28), where $\Delta \eta$ arises due to magnetic contrast at domain walls and the power 4 can be generalized to an exponent $n$. This equation is valid for $q \gg 2\pi/D$, where $D$ is the size of the scattering object. For an FM these objects can be nonmagnetic (e.g., grains, or other extended defects), or magnetic (typically long-range FM-ordered domains), $q \gg 2\pi/D$ often being satisfied even at the lowest $q$ due to the large magnetic domains. The case $n = 4$ [i.e., $d\Sigma /d\Omega \propto q^{-4}$, the “Porod Law”, Eq. (28)] is particularly common, describing scattering from 3D objects with “smooth” surfaces (Willis and Carlile, 2009). “Rough” or “wrinkled” surfaces result in $n < 4$, $n = 3$ marking the transition between surface and volume fractals (Kreyssig et al., 2009). In FMs, nonmagnetic contributions to Eq. (28) are often $T$-independent while magnetic ones are strongly $T$-dependent, making separation of magnetic and nonmagnetic scattering facile, even without polarization. An example is shown in Fig. 24(a) from the manganite pyrochlore $\text{Tl}_2\text{Mn}_2\text{O}_7$, where the low-$q$ scattering (which follows the $n = 4$ Porod Law) clearly reflects the development of FM order, with $T_C \approx 120$ K (Lynn et al., 1998).

At a higher $q$ of 0.046 Å$^{-1}$ [Fig. 24(b)], $\text{Tl}_2\text{Mn}_2\text{O}_7$ reveals a different $T$-dependence of the SANS intensity, referred to as critical scattering (Lynn et al., 1998). This arises at a second-order FM transition due to quasi-elastic scattering from spin waves as $T \to T_C^-$, and from FM spin correlations as $T \to T_C^+$. The $q$-dependence can often be described by a Lorentzian:

$$
\frac{d\Sigma}{d\Omega}(q, T) = \frac{(\frac{d\Sigma_L}{d\Omega}) (T)}{q^2 + \xi(T)^2},
$$

where $(d\Sigma_L/d\Omega)$ parametrizes the strength of Lorentzian scattering, and $\xi$ is the magnetic correlation length (Furrer et al., 2009). This $q$-dependence results from an Ornstein-Zernike spin correlation function, i.e., $\langle S(0)S(r) \rangle \propto e^{-r/\xi}$ (Lovesey, 1984), which describes Heisenberg FMs for instance. $\xi(T)$ is thus readily obtained from SANS, Fig. 24(c) showing an example [again in $\text{Tl}_2\text{Mn}_2\text{O}_7$ (Lynn et al., 1998)] of the divergence of $\xi$ as $T \to T_C^-$. SANS data as a function of $T$ in $H = 0$ thus provide a detailed picture of FM ordering, and were widely used to study manganites.
From around 1997, SANS was used in several studies of the evolution of FM spin correlations across $T_C$ in compounds such as $La_{1-x}Ca_xMnO_3$. This was done at $x \approx 1/3$, where an insulator-metal transition occurs on cooling, accompanied by CMR. As shown in Fig. 24(d), De Teresa et al. found a critical-scattering-type peak at $q = 0.13 \AA^{-1}$ in $La_{2/3}Ca_{1/3}MnO_3$, with $T$ and $H$-dependence reminiscent of CMR (De Teresa et al., 1997). Magnetic intensity was detected from well above $T_C$, decreasing rapidly below it. Above $T_C$, Lorentzian fits [Eq. (58)] were used to extract a $\xi(T)$ exhibiting a long high $T$ tail, saturating at 12 Å (i.e., 3 unit cells), and increasing to 25 Å in $\mu_0H = 5$ T. Similar data were reported for $Sm_{1-x}Sr_xMnO_3$, (De Teresa et al., 2002). Along with resistivity, susceptibility, and thermal expansion, the SANS data on $La_{2/3}Ca_{1/3}MnO_3$ were interpreted as evidence for an entity known as a magnetic polaron. Such polarons, wherein doped carriers align the spins of magnetic ions within their Bohr radii, were originally developed for magnetic semiconductors (Coey et al., 1999; Nagao, 2001). The conclusion of polaron formation was criticized by Viret et al., however, who claimed that $d\Sigma/d\Omega$ in a $La_{0.75}Sr_{0.25}MnO_3$ single crystal was not well described by a Lorentzian, or by the form factor for discrete objects such as magnetic polarons (Viret et al., 1998). Viret et al. instead advanced a correlation function of longer range than Ornstein-Zernike, $\langle S(0), S(r) \rangle \propto e^{-r/\xi}$, yielding $d\Sigma/d\Omega \propto \xi^4/(1 + \xi^2q^2)^2$. This was found to describe data above $T_C$ over a substantial $q$-range, yielding $\xi(T)$ similar to simpler metallic FMs. Viret et al. thus argued for typical development of magnetic coherence as $T \to T_C^+$, albeit with quantitative differences due to double-exchange.

Further elucidation of magnetism near $T_C$ was achieved by adding energy resolution. As discussed in Sec. II.A, typical SANS measurements are not energy-resolved, instead integrating over some window centered on energy transfer $\Delta E = 0$. With an instrument such as a triple-axis spectrometer, however, energy resolution is added, and the low-$q$ intensity vs. $\Delta E$ spectra in FMs probe spin waves. Performing such measurements at multiple $q$ provides the dispersion relation, from which the spin-wave stiffness $D$ can be extracted vs. $T$. While somewhat different than typical INS, this approach proved expedient in early studies of manganites where it revealed an unconventional FM transition in $La_{0.67}Ca_{0.33}MnO_3$. Specifically, the FM order parameter and $D(T)$ were cut-off in a first-order fashion as $T \to T_C^-$, associated with the emergence of a “central peak” at $E = 0$ in the low-$q$ inelastic spectrum (Lynn et al., 1996). Measurements on single crystals confirmed the first-order transition at $x = 0.3$ (Adams et al., 2004). This situation was clarified by ND and INS measurements on $La_{1-x}Ca_xMnO_3$ crystals around $x = 0.3$ (Adams et al., 2000; Dai et al., 2000) [as well as layered manganites (Argyriou et al., 2002; Vasilii-Doloc et al., 1999)], which linked the central peak to diffuse Huang scattering from nanoscale lattice polarons, with short-range correlations. These polarons, dubbed “Jahn-Teller polarons” (due to Jahn-Teller-active $Mn^{3+}$ ions), were found to generate scattering with striking similarities to the $T$-dependence of the resistivity, elucidating a subtle competition between double-exchange and the lattice/charge/orbital degrees of freedom (Adams et al., 2000; Argyriou et al., 2002; Dai et al., 2000; Vasilii-Doloc et al., 1999).

The above focused on high-$x$ manganites, probing the
transition from insulating paramagnet to FMM on cooling. Further insight was obtained from studying lightly-doped manganites to understand how double-exchange FMM emerges from superexchange AF at \( x = 0 \). SANS, both conventional and when implemented with energy resolution on triple-axis systems, played a vital role here, revealing classic nanoscopic magnetic inhomogeneity. Well-studied systems include lightly hole-doped AF \( \text{LaMnO}_3 \) [i.e., \( \text{La}_{1-x}\text{Ca}_x\text{MnO}_3 \) (Biotteau et al., 2001; Hennion et al., 1998)] and lightly electron-doped AF \( \text{CaMnO}_3 \) [i.e., \( \text{Ca}_{1-x}\text{La}_x\text{MnO}_3 \) (Ling et al., 2003)]. Illustrative low-\( T \) elastic magnetic SANS cross section vs. \( q \) plots are shown in Fig. 25 for electron- and hole-doped polycrystalline \( \text{Ca}_{1-x}\text{La}_x\text{MnO}_3 \) (Ling et al., 2003). A broad peak is evident at \( 0.20 \text{ Å}^{-1} \), indicating nanoscale magnetic clustering in the AF matrix, with well-defined spatial extent. Single crystal measurements (Hennion et al., 1998) showed this scattering to be isotropic, leading to the term magnetic “droplets”. Quantitative analysis was performed by fitting to the general formula [see Sec. II.D, Eqs. (23)–(27)],

\[
\frac{d\Sigma}{d\Omega}(q, T) = n_p V_p^2 \Delta \eta^2 F^2(q) S(q),
\]

where \( n_p \) and \( V_p \) are the number density and volume of particles (in this case magnetic clusters), \( \Delta \eta \) is the (magnetic) contrast between clusters and matrix, \( F(q) \) is the cluster form factor, and \( S(q) \) is the (inter-cluster) structure factor. Several group’s data were well described by this (e.g., solid lines in Fig. 25), using a spherical \( F(q) \), and an \( S(q) \) for liquids (Biotteau et al., 2001; Hennion et al., 1998; Ling et al., 2003). A liquid-like distribution of magnetic “droplets” in an AF matrix was thus concluded, with diameters \( 10 \text{–} 17 \text{ Å} \) and center-to-center spacings \( 25 \text{–} 40 \text{ Å} \). Importantly, typical \( \Delta \eta \)’s correspond to magnetization contrast of only \( 0.7 \mu_B \), indicating not simple FM clusters, but rather differing AF canting angles in the clusters and matrix (Hennion et al., 1998). Both the magnetic SANS peak intensity and cluster diameter increased with doping, low-\( q \) structure and longer range FM eventually evolving, consistent with percolation (Biotteau et al., 2001; Hennion et al., 1998; Ling et al., 2003). Notably, the charge carrier to cluster ratios extracted from \( x \) and \( n_p \) were large, \( \approx 60 \), favoring clusters/droplets over magnetic polarons. Several theoretical works provided support for these magnetic cluster sizes. Building on the theory discussed in Sec. VII.A, Kugel et al., 2005 constructed a theory for manganites based on a Kondo lattice model capturing competition between Jahn-Teller and double-exchange. Electronic/magnetic phase separation was found at most dopings, and the energy was minimized to determine the FMM cluster size, accounting for long-range Coulomb interactions. The latter restrict phase separation to short length scales, \( \approx 2 \) unit cells, in reasonable agreement with experiment.

Nanoscale magnetic phase separation was also probed by SANS in other manganites, including \( \text{Ca}_{1-x}\text{Bi}_x\text{MnO}_3 \) (Qin et al., 2008) and \( \text{LaMnO}_{3-x} \) (Ritter et al., 1997). In the latter, the substantial (positive) \( \delta \) values enable a different approach to hole doping an AF manganite. FM order develops by \( \delta = 0.07 \) according to Ritter et al., but with unusual SANS features indicating \( \xi \approx 100 \text{ Å} \), even below \( T_C \). Yet another route to explore the homogeneity of the evolution of magnetism with composition in \( \text{La}_{1-x}\text{Ca}_x\text{MnO}_3 \) was realized via nonmagnetic doping on the Mn site, thus suppressing long-range FM. This was done in \( \text{La}_{2/3}\text{Ca}_{1/3}\text{Mn}_{1-y}\text{Ga}_y\text{O}_3 \), SANS revealing that a transition from long-range FM order to short-range FM correlations proceeds through an intermediate region where the two coexist (De Teresa et al., 2005). At \( y \approx 0.1 \text{–} 0.15 \), for example, long-range FM was found to coexist with short-range order with \( \xi \approx 50 \text{ Å} \).

Another example of the power of SANS in elucidating magnetic inhomogeneity in manganites comes from \( \text{Pr}_{1-x}\text{Ca}_x\text{MnO}_3 \) at \( x \approx 0.30 \). In this system the smaller
A-site cation radius leads to weakened double-exchange, typically viewed as resulting from reduced $e_g$-derived electronic bandwidth due to larger structural distortions (Tomioka and Tokura, 2004). Instabilities such as CO and AF thus compete with FM, a crossover occurring around $x = 0.27$ (Dagotto, 2002; Dagotto et al., 2001; Tokura and Tomioka, 1999). At $x = 0.30$, Pr$_{1-x}$Ca$_x$MnO$_3$ thus exhibits a CO transition on cooling (at $\approx 200$ K), followed by AF ordering (at $\approx 140$ K), the insulating ground state nevertheless exhibiting nonnegligible magnetization (Radaelli et al., 2001). This state is remarkably sensitive to field, pressure, illumination, etc., application of a few Tesla leading to a metamagnetic transition to FM [Fig. 26(a)], with a sharp insulator-metal transition and large CMR [Fig. 26(b)] (Saurel et al., 2010). In a key step, Radaelli et al. established that the magnetization in the zero-field-cooled, nominally COAF state arises due to magnetic phase separation (Radaelli et al., 2001). COAF and FMM domains were found to coexist in polycrystals, with $T$ and $H$-dependent volume fractions, the notable feature being microscopic length scales. Very different from La$_{2/3}$Ca$_{1/3}$MnO$_3$, ND revealed coexisting CO, AF, and FM, with peak widths indicative of $> 500 \AA$ length scales, and large variations in lattice parameters (Radaelli et al., 2001). This is therefore microscale phase separation, as in Fig. 23(a), as opposed to the nanoscale phenomenon in Fig. 23(b).

SANS was then applied extensively, primarily to single crystals (Mercene et al., 2003; Saurel et al., 2006, 2007, 2010; Simon et al., 2002; Yaicle et al., 2003; Yamada et al., 2001). Magnetic phase separation was established from $\approx 10$ nm to $\approx 1 \mu$m. One key observation in the $x = 0.30 - 0.33$ range was of a crossover from the Porod form at low-$q$ to a higher-$q$ regime with $d\Sigma/d\Omega \propto q^{-2}$. This was recognized as indicating reduced dimensionality, the first interpretation being rumpled 2D sheets of inter-penetrating FM and AF. This “red cabbage” model is illustrated in Fig. 26(c,d); sheet thicknesses in the nm-range were reported (Mercene et al., 2003; Simon et al., 2002; Yaicle et al., 2003). In 2004, however, this picture was reevaluated by Viret et al., who detected, in a Pr$_{0.67}$Ca$_{0.33}$MnO$_3$ single crystal, $q^6$-scattering, but with $n$ consistently 1.6 to 1.7, i.e., $\approx 5/3$ (Viret et al., 2004). Taking cues from the polymer literature, this was interpreted in terms of filamentary FM chains, as in Fig. 26(e). This figure shows a Monte Carlo simulation for a “hopping exchange” process, creating a random-walk network of FM chains (Viret et al., 2004). These competing models for Pr$_{1-x}$Ca$_x$MnO$_3$ were later compared (Saurel et al., 2006), considering Lorentzian fitting (De Teresa et al., 1997), “red cabbage” (Saurel et al., 2006), self-avoiding chains (Viret et al., 2004), and spherical FM regions described by a specific correlation function. While the data can be fit with multiple models, the extracted size scales were reassuringly similar.

SANS measurements have also been applied to the $H$-dependence in Pr$_{1-x}$Ca$_x$MnO$_3$. The $\sin^2\theta$ dependence in $q_xq_y$ maps taken in finite $H$ [see Sec. II.D, Eqs. (23)–(27)] was probed in several works, and the magnetic SANS tracked vs. $H$. A particularly clear view was provided by Saurel et al., who used wide $q$-range measurements to extract phase separation length scales and the specific area of FM/AF interfaces as a function of $H$ (Saurel et al., 2010). The latter can be extracted from Porod scattering (Eq. (28)). This analysis provided a simple picture for the manner in which the FM phase fraction grows with $H$ [see Fig. 26(c,d)], as well as the resulting percolation and CMR [Fig. 26(b)], the length scales spanning from nanometric to $> 1 \mu$m (Saurel et al., 2010). A special case of the $H$-dependent FM/AF phase separation, where ultra-sharp $H$- and time-dependent magnetization steps were observed (Mahendiran et al., 2002), was also studied by SANS (Woodward et al., 2004). Strain effects at FM/AF boundaries were invoked to explain these avalanche phenomena.

The magnetic phase separation effects described above in La$_{1-x}$Ca$_x$MnO$_3$ (at light doping and $x \approx 1/3$) and Pr$_{1-x}$Ca$_x$MnO$_3$ (at $x \approx 1/3$) are significantly different. The former, similar to La$_{1-x}$Sr$_x$MnO$_3$, involves nanooscopic FMM clusters, with the CMR peaking at the insulator-metal transition near $T_C$, accompanied by only subtle structural heterogeneity (Shibata et al., 2002). In Pr$_{1-x}$Ca$_x$MnO$_3$, however, phase separation occurs over longer scales, with clear structural differences between domains, often with sharp, percolative features in CMR. In contrast to the models discussed in Sec. VII.A, this longer-range magnetic/electronic phase separation is discussed in terms of quenched disorder near

FIG. 26 (a,b) Magnetic field dependence of the 30 K magnetization and resistance of a Pr$_{0.7}$Ca$_{0.3}$MnO$_3$ single crystal. Reproduced with permission from Saurel et al., 2010. (c,d) Schematic of the zero-field and in-field magnetic phase separation in the same compound. Gray signifies ferromagnetic metallic, white antiferromagnetic insulating. Adapted from Mercene et al., 2003. (e) Magnetic filaments obtained from Monte Carlo simulations for the same compound; arrows illustrate magnetization directions. Reproduced with permission from Viret et al., 2004.
a first-order phase transition between competing phases, such as FMM and AFI (Dagotto, 2002; Dagotto et al., 2001). This occurs near $x = 0.50$ in La$_{1-x}$Ca$_x$MnO$_3$ and $x = 0.30$ in Pr$_{1-x}$Ca$_x$MnO$_3$, and is tunable in (La$_{1-y}$Pr$_y$)$_{1-x}$Ca$_x$MnO$_3$ (Dagotto, 2002; Dagotto et al., 2001). The models predict emergence of clusters below some temperature $T^*$, with $H$-dependent competition giving CMR (Dagotto, 2002; Dagotto et al., 2001). Models emphasizing elastic driving forces for phase separation have also been advanced (Ahn et al., 2004). As a final comment on this issue of multiple types of magnetic phase separation, we note that two distinct forms of CMR are active in manganites, as highlighted in (Nd$_{1-y}$Sm$_y$)$_{0.5}$Sr$_{0.5}$MnO$_3$ (Tokura et al., 1996). In that work gradual $H$-dependence near the FM-to-paramagnet transition was contrasted with sharp $H$ effects at the AF/FM boundary, as discussed by Dagotto, 2002.

Finally, we briefly note that SANS has also proven useful in the study of multiferroic manganites. HoMnO$_3$ crystals, for example, exhibit AF order simultaneously with ferroelectric order below the Néel temperature of 72 K, in addition to a spin reorientation transition at 40 K (Ueland et al., 2010). Strong magnetic Porod scattering was found to develop below this reorientation temperature, with sensitivity to both magnetic and electric fields. The scattering was interpreted in terms of uncompensated magnetization at AF domain walls, the interaction between ferroelectric and AF domains resulting in electric field sensitivity (Ueland et al., 2010).

2. Perovskite cobaltites

The discovery of CMR in the manganites stimulated interest in other perovskites, one class being cobaltites. Much like manganites, cobaltites have a long history (Jonker and Van Santen, 1953), but were studied more extensively since the 1990’s. This work focused on two main issues: nanoscale magnetic phase separation, and the additional spin-state degree of freedom. The latter refers to the fact that the spin state of Co ions in cobaltites can vary with composition, pressure, $H$, $T$, etc. This is because the crystal field splitting between $t_{2g}$ and $e_g$ states is unusually close to the Hund’s rule exchange energy. In insulating undoped LaCoO$_3$ for example, the Co$^{3+}$ ions adopt a low-spin $t_{2g}^6e_g^0$ ($S = 0$) ground state, but with thermally-excited population of finite spin states (simplistically the $t_{2g}^5e_g^1$ intermediate spin state and $t_{2g}^4e_g^2$ high-spin state) by only 30 K; for a brief review see Imada et al., 1998. An insulator-metal transition then takes place at higher $T \approx 500$ K (Raccah and Goodenough, 1967). While much understanding of this spin-state transition/crossover has been achieved, the sequence of excited states has proven very difficult to pin down (Asai et al., 1994; Haverkort et al., 2006; Korotin et al., 1996; Kozenko et al., 2007; Lee and Harmon, 2013; Noguchi et al., 2002; Podlesnyak et al., 2006; Ropka and Radwanski, 2003). This is likely in part due to the desire to describe excited states in terms of low, intermediate, and high spin-states, i.e., with “atomic” language (Lee and Harmon, 2013), often also ignoring spin-orbit coupling (Ropka and Radwanski, 2003).

Remarkably, SANS was first applied to study the spin-state transition in LaCoO$_3$ in 2015 (El-Khatib et al., 2015). The $T$-dependence of $d\Sigma/d\Omega$ from a LaCoO$_3$ single crystal is reproduced in Fig. 27. At low-$q$, a steep $q$-dependence is observed, with the intensity increasing on cooling, while at high-$q$ the cross section is practically $q$-independent (indicating a local origin), increasing on warming. Considering the high-$q$ component first, this was shown to reflect the spin-state transition, being quasieleastic/inelastic scattering due to thermally-excited paramagnetism. Specifically, earlier INS data had shown thermal excitation of not only a broad inelastic paramagnetic continuum around $\Delta E = 0$, but also a distinct 0.8 meV excitation associated with the excited spin-state manifold (Phekan et al., 2006b). Taking the dynamic $S(q, \omega)$ from INS and performing an energy integral appropriate for SANS confirmed quantitative consistency with the high-$q$ ($\approx 0.1 \text{ Å}^{-1}$) SANS (El-Khatib et al., 2015), a rare example of quantification of inelastic magnetic SANS.

The low-$q$ scattering in Fig. 27, on the other hand, grows rapidly only below 60 K, and the magnetic component was thus isolated by subtracting high-$T$ data (see inset). The result is well fit by the Guinier form (see Eq. (29) in Sec. II.D) as expected for the $q \to 0$ behavior of an assembly of scattering centers with radius of gy-
ration $R_q$ (Furrer et al., 2009; Willis and Carlile, 2009). $R_q$ is 140 Å at low $T$, increasing with $T$ until this contribution vanishes above 60 K. Given the unusual $R_q(T)$, the onset of scattering at 60 K, and the observation of Guinier scattering from objects with well-defined extent, evidence for “spin excitons” or “spin-state polaron” was concluded. Such objects were theoretically proposed by multiple authors (Nagaev and Podel’shchikov, 1996; Podlesnyak et al., 2008), who envisioned that doped electrons or holes in LaCoO$_3$ would stabilize finite-spin Co ions in their vicinity, aligning those spins to form a magnetic polaron in a low spin matrix. Prior evidence for these polarons had been derived from magnetometry and muon spin spectroscopy (including collective ordering below 60 K) (Giblin et al., 2005), but this was the first neutron-based detection. The scattering intensity could be explained with oxygen deficiency in LaCoO$_{3-\delta}$ of only 0.0005, the increase in $R_q$ on warming being a classic signature of spin polarons (El-Khatib et al., 2015).

Returning to magnetic phase separation, doped cobaltites were intensively studied from the 1990’s, La$_{1-x}$Sr$_x$CoO$_3$ emerging as a classic example of a magnetically heterogeneous oxide. Pioneering work by Goodenough had earlier established a crossover from insulating behavior to FMM around $x_c = 0.18$, postulating formation of superparamagnetic clusters etc. (Señarís-Rodríguez and Goodenough, 1995). A series of studies then gathered direct evidence of this (Wu and Leighton, 2003). Caciuffo et al. concluded $x$-dependent phase separation into hole-rich FMM clusters in a hole-poor non-FM insulating matrix, from electron microscopy, susceptibility, ND, and thermal expansion (Caciuffo et al., 1999b). Kuhns et al. applied $^{59}$Co zero-field and in-field NMR to polycrystalline La$_{1-x}$Sr$_x$CoO$_3$, separately detecting the $x$-dependent FM and non-FM components (Kuhns et al., 2003). In 2005, Wu et al. then applied SANS to polycrystals, coarsely spanning $x_c$. Low-$q$ Porod scattering from long-range FM domains was separated from high-$q$ Lorentzian scattering from nanoscale FM correlations as a function of $x$ and $T$ (Wu et al., 2005). This solidified the picture of short-range (10 – 30 Å) FM clusters at low $x$, which increase in size and density as $x$ is increased, percolating at $x_c$. In the same work, a large, negative, hysteretic MR was discovered in low-$x$ single crystals, arising due to spin-dependent intercluster transport (Wu et al., 2005), analogous to artificial systems such as Co-SiO$_2$ granular films (Sankar et al., 2000). Using INS on single crystals, Phelan et al. also detected these nanoscopic FM “droplets”, which were confirmed isotropic (Phelan et al., 2006b).

More detailed La$_{1-x}$Sr$_x$CoO$_3$ SANS studies of single crystals at multiple $x$ provided a complete picture (He et al., 2009). As shown in Fig. 28(a-c), $T$-dependent measurements provided a similar view to earlier work, the low-$q$ scattering [Fig. 28(a)] being of Porod type from FM domains, with order-parameter $T$-dependence.

The higher-$q$ Lorentzian part [Fig. 28(b)] exhibits critical scattering, which weakens and broadens with decreasing $x$, resulting, below $x_c$, in intensity growing monotonically on cooling. The extracted $\xi(T)$ [Fig. 28(c)] evolves from divergence as $T \to T_C^-$ at high $x$, to saturation (at 10 – 30 Å) at low $x$. Plotting the low-$q$ (0.007 Å$^{-1}$) and high-$q$ (0.049 Å$^{-1}$) cross sections vs. $x$ at 10 K produces Fig. 28(d,e). Considering high-$q$ first [Fig. 28(d)], the data show negligible long-range FM for $x \leq 0.15$, as expected. This intensity then turns on around percolation (at $x_c = 0.17 – 0.18$) peaking just above this, before leveling off above $x = 0.22$. Insight into the latter observation came from the high-$q$ intensity [Fig. 28(e)], which reveals two surprises. First, this signature of short-range FM abruptly vanishes at $x = 0.22$, suggesting phase-pure long-range FM above this doping. Magnetic phase separation in La$_{1-x}$Sr$_x$CoO$_3$ crystals, unlike more disordered polycrystals (Kuhns et al., 2003), thus does not pervade the entire phase diagram, but ends abruptly at $x = 0.22$, the constant low-$q$ scattering above this coming from conventional FM domains. The second surprise from the low-$q$ SANS is that the scattering, which peaks near percolation, also vanishes at low $x$, around 0.04. This observation of a finite window for magnetic phase separation in the highest quality La$_{1-x}$Sr$_x$CoO$_3$ was corroborated by multiple techniques, including La NMR and heat capacity, each in precise agreement on the lower and upper doping limits of 0.04 and 0.22 (He et al., 2009; Smith et al., 2008).

These observations were used by He et al. as the starting point for statistical modeling showing that on the scales of the observed $\xi$-values [Fig. 28(e)], local compo-
tional fluctuations are sufficient to explain the experiments (He et al., 2009). With no adjustable parameters, simulations were shown to predict start and end values for the phase-separated regime of 0.04 and 0.21, in remarkable agreement with experiment. The model also reproduces key features of the $x$-dependent SANS cross sections, as well as volume fractions of FMM and non-FMM phases. This demonstrated that nanoscale magnetic inhomogeneity in La$_{1-x}$Sr$_x$CoO$_3$ can be explained solely via local compositional fluctuations that must always be present in randomly doped solids on such short scales, with no need to invoke electronic mechanisms. Understanding to what extent similar local compositional /structural inhomogeneities occur in other systems with nanoscale magnetic inhomogeneity remains an open challenge.

One unresolved issue in La$_{1-x}$Sr$_x$CoO$_3$ arose from the work of Phelan et al., who detected unanticipated incommensurate scattering around the (001) FM Bragg peak (Phelan et al., 2006a). The deduced spin superstructure was found to have $x$- and $T$-dependent intensity and incommensurability, speculatively linked to spin-state polarons. As discussed above, these spin-state polarons form around doped carriers in LaCoO$_{3-\delta}$. In the case of La$_{1-x}$Sr$_x$CoO$_3$ they were shown to be octahedrally-coordinated complexes from the $q$-dependence of a characteristic 0.75 meV excitation (Podlesnyak et al., 2008). While the exact link between these polarons and incommensurate magnetism remains unclear, Fig. 28(e) provides some insight by overlaying the incommensurate intensity (Phelan et al., 2006a) with the high-$q$ SANS cross section (He et al., 2009). The comparison is striking, indicating that the two forms of scattering probe the same physics. Another open issue in La$_{1-x}$Sr$_x$CoO$_3$ is the relationship between the spin-state polarons at light doping, and the magnetic clusters at higher $x$. While much remains to be learned about the distinguishing features of these entities, as well as how one evolves into the other, recent work took a first step. Specifically, INS and magnetometry data were acquired vs. $x$, indicating “collapse” of the spin-state polarons into magnetic clusters, evidenced by a rapid drop in magnetization per doped hole (Podlesnyak et al., 2011).

We note as an aside here that magnetic phase separation in La$_{1-x}$Sr$_x$CoO$_3$ also plays a role in heterostructures. A central problem in such systems is the suppression of the FMM phase in ultrathin films. This “dead layer” effect has been extensively studied in La$_{0.5}$Sr$_{0.5}$MnO$_3$ and La$_{0.3}$Sr$_{0.7}$CoO$_3$ films, including in the latter case with SANS. The crossover from FMM to non-FM insulating phases below 70 Å thickness in SrTiO$_3$(001)/La$_{0.5}$Sr$_{0.5}$CoO$_{3-\delta}$, for example, was shown to be driven by a decrease in effective doping, due to accumulation of interfacial oxygen vacancies (Torija et al., 2011). In a novel mechanism, these vacancies play the key role in strain relaxation, ordering into alternating oxygen deficient/sufficient planes, similar to brownmillerite Sr$_2$Co$_2$O$_5$ (SrCo$_2$O$_5$)$_3$, to better lattice match the substrate (Gazeque et al., 2013). TEM with electron energy loss spectroscopy showed that the effective doping thereby falls to $\approx 0.22$ at the interface with SrTiO$_3$(001), triggering magnetic phase separation. Exactly this was detected by SANS in 650 Å-thick films. Scattering of the form $q^n$ was observed at low-$q$, with $n \approx 2.7$ and order-parameter shape (Torija et al., 2011). This derives from long-range FM domains, the $n$ indicating domain wall pinning at defects. At higher $q$, a weaker $q$-dependence was found, the $T$-dependence revealing critical scattering, as expected, but on a background of additional intensity not present in bulk. This short-range FM was interpreted as arising from the interface, directly evidencing interface-induced magnetic phase separation as the origin of the dead layer in La$_{1-x}$Sr$_x$CoO$_3$ films (Torija et al., 2011). This represents one of the few applications of SANS to thin film oxides. While intensity-limited (multiple films were stacked), this work demonstrates the feasibility of SANS on complex oxide heterostructures, which could expand in the future.

The majority of the above discussion of La$_{1-x}$Sr$_x$CoO$_3$ focused on the evolution in magnetism as $x$ is varied at low $T$. A complementary approach is to understand how the FMM state evolves with $T$, as first studied with SANS in Caciuffo et al., 1999a. In that work, the behavior around $T_C$ in polycrystalline La$_{0.7}$Sr$_{0.3}$CoO$_3$ revealed $T$-dependent magnetic inhomogeneity. The SANS included a strongly $T$-dependent Guinier component, indicating an asymmetric peak in magnetic intensity and cluster size (peaking at 15 Å) around $T_C$. This was interpreted as preformation of magnetic clusters that are eventually subsumed by longer-range FM at low $T$. A somewhat similar picture was concluded from SANS on polycrystalline La$_{1-x}$Sr$_x$CoO$_3$ at $x \approx 0.20-0.50$ (He et al., 2007). As shown in Fig. 29(a-c), that work found critical scattering, well described by Eq. (58). Interestingly, the deduced $\xi(T)$ exhibits well-defined onset temperatures for FM spin correlations (Fig. 29(d-f) and insets), unlike the typical power-law growth as $T \to T_C^\ast$. The susceptibility also exhibits deviations from Curie-Weiss behavior at this temperature. $T^\ast \approx 360$ K. These findings are reminiscent of the Griffiths model, referring to a classic theoretical treatment of randomly-diluted Ising FM (Bray, 1987; Griffiths, 1969). In that model, cooling below the undiluted $T_C^\ast$ results in a spatially-inhomogeneous clustered phase, with nonanalytical behavior of thermodynamic quantities. The applicability of this model to manganites was quite intensively studied, several authors concluding preformation of magnetic clusters in a manner consistent with a Griffiths phase (Deisenhofer et al., 2005; Salamon et al., 2002). In La$_{1-x}$Sr$_x$CoO$_3$, however, the specifics were found quite different from Griffiths, including the sign of the deviations from Curie-Weiss (He et al., 2007). A well-defined onset temperature for FM clustering nev-
et al., 2006). The insets in (d-f) are expanded views of the low correlation length region, the arrows marking $T^* \pm 5 \, \text{K}$, where correlations turn on. Reproduced with permission from He et al., 2007. (g) Temperature dependence of the ferromagnetic Bragg intensity (left axis), and $q = 0.1 \, \text{Å}^{-1}$ magnetic SANS intensity (right axis) of polycrystalline Tb$_5$Si$_2$Ge$_2$ at various magnetic fields. Curie and Griffiths temperatures are marked with vertical dashed lines. Reproduced with permission from Magen et al., 2006.

Nevertheless occurs, and it remains to be clarified why various perovskite oxides behave differently with regard to comparisons to the Griffiths model.

Finally, we briefly mention that SANS has also been used as a probe of nanoscale magnetic order in other cobaltites. Pr$_{1-x}$Sr$_x$CoO$_3$ is one example, primarily because compositions around Pr$_{0.5}$Sr$_{0.5}$CoO$_3$ display a second magnetic transition below $T_C$ (Mahendiran and Schiffer, 2003). This was eventually shown, with the assistance of SANS, to signify a change in magnetocrystalline anisotropy, driven by an unusual isomorphic structural transition (Leighton et al., 2009). Further decreasing the average cation radius leads to Pr$_{1-x}$Ca$_x$CoO$_3$. In this material the suppression of the FMM phase leads to acute competition with other instabilities, inducing complex phase separation. Based in large part on SANS, polycrystalline Pr$_{0.7}$Ca$_{0.3}$CoO$_3$ was shown to separate into short-range and long-range-ordered FM phases, with different coercivities (El-Khatib et al., 2010). The coupling between the two was then found to create a hard/soft FM composite displaying exchange spring behavior. At higher $x$, Pr$_{1-x}$Ca$_x$CoO$_3$ was discovered to exhibit an extraordinary first-order metal-insulator transition driven by a Pr valence transition. This effect was first detected in Pr$_{0.5}$Ca$_{0.5}$CoO$_3$ (Tsubouchi et al., 2002) and later stabilized via Y substitution (Hejtmanek et al., 2010). SANS has again been applied to these systems, not only to probe the FM order in related compounds [e.g., Nd$_{1-x}$Ca$_x$CoO$_3$ (Phelan et al., 2013)], but also to understand the evolution of magnetism across the valence transition (Phelan et al., 2014). The latter measurements reveal striking inhomogeneity, complementary to chemical/structural studies by TEM (Gulec et al., 2016). These valence transitions remain an active research area in cobaltites.

**C. Complex magnetic alloys**

As illustrated in other sections of this review (e.g., Sec. III on Nd–Fe–B and Sec. IV on steels), SANS has a long history of applicability to magnetic alloy systems, in addition to oxides. Along with the study of micromagnetics in alloys, phenomena such as spin-glass freezing have been extensively studied with SANS. In terms of nanoscale magnetic inhomogeneity, alloys exhibiting reentrant spin-glass behavior are of particular interest. In these materials, such as Pd–Fe–Mn, Cr–Fe, Au–Fe, Ni–Mn, Fe$_{3-x}$Al$_x$, a-(Fe$_{1-x}$Mn$_x$)$_7$P$_{16}$B$_6$Al$_3$, and a–Fe–Zr, a paramagnet-to-FM transition on cooling is followed by a transition to a spin-glass, i.e., from an ordered to a glassy state (Aeppli et al., 1983; Garcia-Calderón et al., 2005; Rhyne and Fish, 1985; Shapiro et al., 1980). SANS has been used for decades to probe magnetic ordering and correlation lengths in such alloys (Mettus et al., 2017), which have proven challenging to understand. As discussed in a-Fe$_{1-x}$Zr$_x$, nanoscale magnetic inhomogeneity is common to essentially all models, where FM order coexists with spin-glass regions (Garcia-Calderón et al., 2005). An important advance was made in 1983, when SANS studies on a–(Fe$_{1-x}$Mn$_x$)$_7$P$_{16}$B$_6$Al$_3$, were used to develop a heuristic model for reentrant spin-glasses, based on random-field effects in systems with coupled FM and spin-glass order parameters (Aeppli et al., 1983). The model emphasizes coupling between spin-glass and FM networks, introducing a scattering function based on...
a sum of Lorentzian and Lorentzian-squared terms:

$$\frac{d\Sigma}{d\Omega}(q,T) = \frac{A(T)}{q^2 + \xi_A^2(T) + (q^2 + \xi_B^{-2}(T))^2}.$$  

(60)

where $A$ and $\xi_A$ are defined as in Eq. (58), and $B$ and $\xi_B$ result from the random-field, which creates finite FM regions. This function has been widely employed, although its general applicability is not entirely clear (Garcia-Calderón et al., 2005). The similarity of Eq. (60) to a power law ($d\Sigma/d\Omega \propto q^{-n}$) in the experimental $q$-range likely complicates this situation (Aeppli et al., 1983). While there remain competing models in materials such as Fe$_{1-x}$Zr$_x$, involving finite AF and FM clusters, long-range FM networks, random anisotropy fields, noncollinear spin structures, and separately responding longitudinal and transverse spins, nanoscopic magnetic inhomogeneity features prominently (Garcia-Calderón et al., 2005).

Other examples of SANS elucidation of magnetic inhomogeneity in complex alloys arise in magnetocaloric intermetallics and transition metal/rare-earth alloys. One example of the former is highlighted in Fig. 29(g), where the $T$-dependence of the FM Bragg intensity (left axis) and $q = 0.1$ Å$^{-1}$ SANS (right axis) are plotted for Tb$_5$Si$_2$Ge$_2$ (Magen et al., 2006). This compound is a member of the series RE$_5$(Si$_{1-x}$Ge$_x$)$_4$ (RE = rare earth), in which giant magnetocaloric effects occur. Interestingly, as shown in Fig. 29(g), in Tb$_5$Si$_2$Ge$_2$ the FM ordering at $T_C = 110$ K is preceded by the onset of magnetic SANS at 200 K. In the interval 110 K $T < 200$ K, the behavior of the susceptibility, $d\Sigma/d\Omega(q,T)$, and $\xi(T,H)$ (from Lorentzian fitting) were shown consistent with the Griffiths model (see Sec. VII.B.2 above). The 200 K temperature scale was identified with the $T_C$ of the Si-rich Tb$_5$(Si$_{1-x}$Ge$_x$)$_4$ end-member, leading to an interpretation where the disorder results from structural/chemical fluctuations (Magen et al., 2006). An example of the use of SANS to elucidate magnetic inhomogeneity in correlated electron systems is found in CeNi$_{1-x}$Cu$_x$, where percolation from a cluster-glass to long-range FM order was proposed (Marcano et al., 2007).

The remainder of this section focuses on a different class of complex alloys that have recently been shown to display magnetic phase separation. We refer here to off-stoichiometric Heusler alloys. Full Heusler and half Heusler alloys are simply ordered cubic alloys of the form X$_2$YZ and XYZ (Felser and Hirohata, 2015). Many such alloys are magnetic, including numerous FMs, such as Ni$_2$MnSn. The latter has $T_C = 340$ K, the moment residing primarily on Mn (Krenke et al., 2005). One of the many contemporary research areas with these Heuslers involves deliberate generation of nonstoichiometry by substituting excess Mn for Sn, resulting in disordered alloys such as Ni$_{50}$Mn$_{25+y}$Sn$_{25-y}$ (Krenke et al., 2005). The excess Mn creates Mn-Mn bonds, with strongly AF character, thus generating tunable FM/AF phase competition. These off-stoichiometric Heuslers often also exhibit first-order martensitic phase transformations from cubic to lower symmetry on cooling, with substantial impact on magnetism (Bhatti et al., 2016, 2012; Krenke et al., 2005). This results in ferroelasticity, multifiorioicity, shape memory behavior, magnetocaloric effects, etc., with potential applications in sensors, actuators, refrigeration, and energy conversion (Bhatti et al., 2016, 2012; Krenke et al., 2005).

A central issue in such alloys is the magnitude of the thermal hysteresis at the martensitic transformation. Recent theory has had a dramatic impact in this respect, identifying geometrical compatibility criteria between the austenite and martensite phases that minimize hysteresis (Cui et al., 2006; Zhang et al., 2009). Complex alloys such as Ni$_{50}$Mn$_{25+y}$Sn$_{25-y}$ (Bhatti et al., 2016, 2012) and Ni$_{50}$Mn$_{25+y}$In$_{25-y}$ (Karaca et al., 2009) thus emerged, possessing close-to-300 K phase transformations, with sub-10 K hysteresis. The phase diagram of Ni$_{50-x}$Co$_x$Mn$_{40}$Sn$_{10}$ is shown in Fig. 30(a) (Bhatti et al., 2016). The martensitic phase transformation temperature decreases gradually from 430 to 310 K from $x = 0$ to 10, before dropping by over 300 K with 1% additional Co. For $x > 4$, FM emerges in the austenite, the martensitic transformation inducing a first-order transition from FM to non-FM on cooling. Compositions around Ni$_{44}$Co$_6$Mn$_{40}$Sn$_{10}$ thus exhibit a soft FM austenite phase with high $T_C$ (430 K), transforming at just above ambient to non-FM martensite with $\approx 1000$ emu/cm$^3$ magnetization change across only a 6 K hysteresis region (Bhatti et al., 2016).

Detailed magnetic studies of this and related alloys reveal a number of unanticipated low-$T$ features, including substantial magnetization in the nominally non-FM state, Langevin-like $T$ and $H$-dependence, magnetic freezing transitions, and “intrinsic” exchange bias (Cong et al., 2010, 2012; Wang et al., 2011). The superparamagnet-like behavior led to various hypotheses of magnetic clustering, recently verified by SANS (Bhatti et al., 2012). Specifically, a study of Ni$_{44}$Co$_6$Mn$_{40}$Sn$_{10}$ revealed not only the typical low-$q$ Porod and high-$q$ Lorentzian components associated with long-range FM domains and short-range FM spin correlations, but also a distinct peak at intermediate $q$. A $T$-dependent summary is provided in Fig. 30(b,c), where $T_C$ is marked by the onset of FM domain scattering at $q = 0.005$ Å$^{-1}$ [Fig. 30(b)], as well as a critical scattering peak at $q = 0.1$ Å$^{-1}$ [Fig. 30(c)]. The weakly hysteretic FM to non-FM transition at the martensitic transformation is also clear below 400 K, but with nonnegligible high-$q$ scattering in the low-$T$, nominally non-FM phase. As shown in the inset, this weak scattering reveals a clear phase transition, associated with the intermediate $q$-peak. The peak was interpreted as an $S(q)$-derived feature reflect-
FIG. 30  (a) Phase diagram of Ni$_{50-x}$Co$_x$Mn$_{40}$Sn$_{10}$, reproduced with permission from (Bhatti et al., 2016). $T_M$, $T_C$, $T_{SP}$, and $T_{EB}$ denote the martensitic phase transformation temperature, Curie temperature, and superparamagnetic and exchange bias blocking temperatures, respectively. P, F, AF, and SP denote paramagnetic, ferromagnetic, antiferromagnetic and superparamagnetic; “Aust.” and “Mart.” refer to austenite and martensite. The top axis shows the valence electron per atom ratio.  (b) and (c) show the temperature dependence of the SANS cross section from Ni$_{50}$Co$_6$Mn$_{40}$Sn$_{10}$ at $q = 0.005 \text{ Å}^{-1}$ and $q = 0.1 \text{ Å}^{-1}$, taken on cooling and heating. The inset shows a close-up at low temperature. Reproduced with permission from Bhatti et al., 2012.

VIII. SKYRMION LATTICES AND NONCOLLINEAR SPIN STRUCTURES

The recent discovery of a magnetic skyrmion lattice (SkL) in MnSi (Mühlbauer et al., 2009a) provided a showcase for a new type of magnetic order, where magnetic whirls exhibit particle like properties due to their non-trivial topology. With SANS in a leading role, magnetic skyrmions are observed in increasing number in a wide range of material classes rendering their emergence as generic phenomenon of materials which promote noncollinear chiral magnetic interactions. Together with the ease of manipulations SkLs with ultra-low current densities, their topological (and sometimes multiferroic) properties ideally suit future data storage and logical devices.

This section provides an introduction into the variety of competing or oscillatory interactions (cf. Sec. VIII.A.1) including the Dzyaloshinskii-Moriya interaction (DMI) (cf. Sec. VIII.A.2) that lead to the large zoo of noncollinear spin structures. We then review the intrinsic multiferroic properties of noncollinear spin structures (cf. Sec. VIII.A.3) before we concentrate on their properties accessible by means of SANS (cf. Sec. VIII.A.5) and the peculiarities of diffraction in SANS geometry (cf. Sec. VIII.A.4). We finally summarize recent developments in the area of magnetic skyrmions (cf. Sec. VIII.B.1 and VIII.B.2).

A. Noncollinear magnetic structures

Spiral or noncollinear spin structures describe a large variety of spin modulation which share rotation of the magnetic moment or spin, whether localized or itinerant, from one unit cell to the next by canting angle $\alpha_r$. Such modulations can be described by a single propagation vector $k$, like, e.g., in helical, cycloidal, longitudinal-conical or transverse-conical spin spirals. The superposition or mixture of multiple propagation vectors—even with different moduli of $k$—leads to more complex textures like, e.g., helifan or skyrmionic structures with potentially nontrivial topology. Note that the different propagation vectors of multi-$k$ structures exhibit a fixed...
phase relation with respect to each other while multido-
main single-k structures have no fixed phase relation.
Noncollinear spin structures exhibit a wide range of wave-
lenghts from a few atomic layers to smooth rotations on
a scale of several thousand Å.

1. Frustrated and oscillating interactions

A large variety of noncollinear spin structures can be
cased by frustrated and oscillating interactions. Here,
we provide some typical examples, illustrating the differ-
ent coupling mechanisms. For further details we refer to
the book by Chatterji, 2006 and the review articles by
Coey, 1987; Freeman and Nakamura, 2004; and Kawa-

The interplay of FM or AF nearest neighbor interac-
tions and AF next-nearest neighbor coupling can lead to
a spin spiral ground state (Enz, 1961; Nagamiya et al.,
1962; Yoshimori, 1959a,b). A typical example for a Mott
insulator showing spiral spin structures is perovskite
TbMnO$_3$ (Kimura et al., 2003). While this mechanism
is relevant for localized and itinerant electron materials,
long-range interactions of the electron gas are expected
to play an additional role for the latter. For itinerant
systems, the role of these interactions is reflected in the
structure of the electronic response. In the case the spin
susceptibility exhibits a maximum at a finite wavevector
k, this may result in the formation of a spin density wave
(SDW) or spin spiral with propagation vector k, respec-
tively. Moreover, a real space oscillatory RKKY interac-
tion (Kasuya, 1956; Ruderman and Kittel, 1954; Yosida,
1957) can be induced by a kink or a derivative singularity
(for a 2D/3D electron gas, respectively), leading to the
formation of noncollinear spin structures. Examples in-
clude the good metal FeAs (Segawa and Ando, 2009; Selte
and Kjekshus, 1973) and the rare-earth metal Ho, where
RKKY exchange between the localized 4f spins through
the conduction electrons leads to a variety of spiral phases
(Coqblin, 1977; Szary et al., 2016). In the strongly cor-
related f electron material CeRhIn$_5$ the RKKY spiral
state is strongly coupled to the conduction electrons via
the Kondo effect (Fobes et al., 2018) generating emergent
electronic heterostructures. The spin fluctuations related
to the spiral state are believed to be closely tied to the
formation of textured superconductivity and electronic
nematic textures (Ronning et al., 2017).

Noncollinear spin structures can be also caused by
geometric frustration as e.g in the spinel Chromites
ZnCr$_2$Se$_4$ and CdCr$_2$O$_4$ (Chung et al., 2005; Murakawa
et al., 2008; Plunier, 1966; Siratori et al., 1980). Re-
cently, it was pointed out by Azhlar and Mostovoy, 2017
that the FM state of systems showing double exchange is
generally unstable against the formation of noncollinear
spin structures. Typical examples are the cubic per-
ovskite compound SrFeO$_3$ (Ishiwata et al., 2011) and
MnAu$_2$, actually one of the oldest spin spiral materi-
als known (Herpin and Meriel, 1961; Meyer and Taglang,
1956). It is argued that the spiral state is induced by
a competition of short-range AF and long-range interac-
tions induced by the polarization of the Au bands (Glas-
brenner et al., 2014).

In general, a large variety of spiral wavelengths are
found for frustrated and oscillatory interactions (Glas-
brenner et al., 2014). Furthermore, if chirality is present,
the samples show a mixture of left and right chiral do-

2. Dzyaloshinskii-Moriya interaction

For crystalline space groups lacking inversion symme-
try, the largest term of the anisotropic superexchange
that is linear in the spin-orbit coupling has the anti-
symmetric form $H_{DM} = \mathbf{D} \cdot (\mathbf{S}_i \times \mathbf{S}_j)$ denoted as DMI
(Dzyaloshinskii, 1958; Moriya, 1960). The DMI favors a
perpendicular rather than a FM or AF arrangement of
spins. $\mathbf{D}$ is a vector, which depends on the symmetry of
the magnetic exchange-path of the two involved spins. In
contrast to frustrated and oscillatory interactions, mag-
nentic structures involving the DMI can be conveniently seen as the result of a hierarchy of energy scales with symmetric exchange (FM or AF) $J_1$ as leading contribution. Due to weaker spin-orbit coupling, the DMI term leads to a small canting of the AF parent structure, the local arrangement of spins is hence almost FM or AF, as schematically indicated in panels (A) of Fig. 31. The associated propagation vector $k$ is given by $D/J_1$, leading to modulations that are long compared to typical interatomic distances, 100 Å to a few 1000 Å. Note, that weak FM behavior induced by canting of spins due to DMI should not be confused with weak itinerant FM, which is characteristic of a small localized but a large fluctuating moment (Lonzarich and Taillefer, 1985).

The spin structure (e.g., helix or cycloid) is fixed by the interplay of magnetocrystalline anisotropies and the sign and orientation of $D$, hence the crystallographic space group. Zeeman coupling to the external magnetic field can lead to a wealth of different spiral structures and associated phase transitions, respectively. As the orientation and sign of $D$ is solely derived from the noncentrosymmetric crystallographic space group, chiral magnetic structures based on DMI show a single chirality or handedness (Grigoriev et al., 2014) rather than statistical mixtures of left and right-handed structures. However, a noncentrosymmetric crystallographic space group not necessarily implies a chiral magnetic structure. Note that also surfaces and interfaces locally break inversion symmetry and hence allow for DMI terms.

A large body of work exits on the wealth of DMI helimagnets. To name a few prominent examples, Cr$_{1/3}$NbS$_2$ (Togawa et al., 2012), NdFe$_3$(BO$_4$)$_4$ (Janoschek et al., 2010) and CuB$_2$O$_4$ (Roessli et al., 2001) show helimagnetic order which distorts to a soliton lattice. CsCuCl$_2$ (Adachi et al., 1980) is a further classic helimagnetic material that exhibits a DMI based spin helix. Noncentrosymmetric MnSi ($P2_13$), representative of the cubic B20 family (an overview on the B20 series is given in Table I in the review by Nagaosa and Tokura, 2013), and tetragonal Ba$_2$CuGe$_2$O$_7$ ($P\overline{4}2_1m$) serve as two further examples where SANS was involved in key experiments. The weak itinerant FM MnSi exhibits a spin helix, (Bak and Jensen, 1980). Ba$_2$CuGe$_2$O$_7$ (Zheludev et al., 1997a, 1996) shows an AF spin cycloid. A compact survey from the neutron scattering view on noncollinear spin structures is given in the book by Chatterji, 2006; see also the reviews by Freeman and Nakamura, 2004; Sandratskii, 1998; and Sandratskii and J., 1996 for further reading. The recent impact of SANS on helical magnets and noncollinear spin structures is reviewed in Sec. VIII.A.5.

### 3. Multiferroic properties of spiral magnets

The multiferroic/magnetoelectric properties of noncollinear magnetic structures have lately gained remark-

4. Noncollinear magnetism; diffraction in SANS geometry

Magnetic SANS ideally suits the wavelengths of noncollinear spin structures of typically 30 − 1000 Å. However, in contrast to diffuse SANS, based on the definition of a SLD which varies smoothly on an atomic scale (see Sec. II.B), an approach based on magnetic single-crystal diffraction is typically used for noncollinear spin structures (a detailed derivation is given in the textbooks by Squires, 1978, Lovesey, 1984, and Furrer et al., 2009). We start with the general cross section for elastic magnetic neutron diffraction of unpolarized neutrons on a single-crystalline sample with negligible mosaicity. Considering only magnetic scattering, the cross section reads:

$$\frac{d\Sigma}{d\Omega} = (\gamma_n r_0)^2 e^{-W(q)} f^2(q) \sum_{\alpha,\beta} \left( \delta_{\alpha,\beta} - \frac{q_\alpha q_\beta}{q^2} \right) e^{i\alpha l} \langle \hat{S}_0^{\alpha} \rangle \langle \hat{S}_l^{\beta} \rangle$$

with the spin operators $\langle \hat{S}_0^{\alpha} \rangle$, and $\alpha,\beta = x,y,z$ ($\gamma_n = 1.913$ and $r_0 = 2.818$ fm). The dimensionless magnetic form factor $f(q)$ is the Fourier transform of the normalized spin density associated with the magnetic ions (not to be confused with the form factor used in the classical, diffuse SANS theory), and $e^{-W(q)}$ denotes the temperature-dependent Debye-Waller factor. The sum over $l$ embodies the integration over all unit cells of a sample with $l = R_j - R_{j'}$ and atomic positions $R_j$. 

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with the spin operators $\langle \hat{S}_0^{\alpha} \rangle$, and $\alpha,\beta = x,y,z$ ($\gamma_n = 1.913$ and $r_0 = 2.818$ fm). The dimensionless magnetic form factor $f(q)$ is the Fourier transform of the normalized spin density associated with the magnetic ions (not to be confused with the form factor used in the classical, diffuse SANS theory), and $e^{-W(q)}$ denotes the temperature-dependent Debye-Waller factor. The sum over $l$ embodies the integration over all unit cells of a sample with $l = R_j - R_{j'}$ and atomic positions $R_j$. 

### 4. Noncollinear magnetism; diffraction in SANS geometry

Magnetic SANS ideally suits the wavelengths of noncollinear spin structures of typically 30 − 1000 Å. However, in contrast to diffuse SANS, based on the definition of a SLD which varies smoothly on an atomic scale (see Sec. II.B), an approach based on magnetic single-crystal diffraction is typically used for noncollinear spin structures (a detailed derivation is given in the textbooks by Squires, 1978, Lovesey, 1984, and Furrer et al., 2009). We start with the general cross section for elastic magnetic neutron diffraction of unpolarized neutrons on a single-crystalline sample with negligible mosaicity. Considering only magnetic scattering, the cross section reads:

$$\frac{d\Sigma}{d\Omega} = (\gamma_n r_0)^2 e^{-W(q)} f^2(q) \sum_{\alpha,\beta} \left( \delta_{\alpha,\beta} - \frac{q_\alpha q_\beta}{q^2} \right) e^{i\alpha l} \langle \hat{S}_0^{\alpha} \rangle \langle \hat{S}_l^{\beta} \rangle$$

with the spin operators $\langle \hat{S}_0^{\alpha} \rangle$, and $\alpha,\beta = x,y,z$ ($\gamma_n = 1.913$ and $r_0 = 2.818$ fm). The dimensionless magnetic form factor $f(q)$ is the Fourier transform of the normalized spin density associated with the magnetic ions (not to be confused with the form factor used in the classical, diffuse SANS theory), and $e^{-W(q)}$ denotes the temperature-dependent Debye-Waller factor. The sum over $l$ embodies the integration over all unit cells of a sample with $l = R_j - R_{j'}$ and atomic positions $R_j$.
We assume the general form of a spin helix with propagation vector $k$ (corresponding to a wavelength $k = 2\pi/\lambda_h$) along the $z$-axis and spins lying in the $x$-$y$-plane. The expectation values of the spin operators are given by

$$\langle \hat{S}_i^\tau \rangle = \langle \hat{S} \rangle \cos(k \cdot l), \langle \hat{S}_i^\rho \rangle = \langle \hat{S} \rangle \sin(k \cdot l), \langle \hat{S}_i^\sigma \rangle = 0. \quad (62)$$

Different choices of $S$ and $k$ may reproduce alternative magnetic structures with different periodicities, including multiple $k$ and higher harmonics of $k$. Inserting Eq. (62) into Eq. (61) finally leads to (Squires, 1978):

$$\frac{d\Sigma}{d\Omega} = \frac{N}{4} \frac{(2\pi)^3}{v_0} (\gamma r_0)^2 \exp(-W(q)\langle f^2(q)\rangle \langle \hat{S} \rangle^2 (1 + q^2/\lambda^2)) \times \sum_{\tau} (\delta(q + k - \tau) + \delta(q - k - \tau)) \quad (63)$$

Bragg scattering occurs for $q = \tau \pm k$, where $\tau$ is a reciprocal lattice vector of the crystal. Each nuclear Bragg reflection is accompanied by a pair of two incommensurate satellite reflections at $k = 2\pi/\lambda_h$. As SANS is capable of mapping reciprocal space in a limited region around the forward direction $\tau = (0,0,0)$, incommensurate reflections are seen directly at $\pm k$. The transformation from real to reciprocal space of a smooth modulation on top of a Bravais lattice is illustrated in panels B1–B3 of Fig. 31.

In contrast to diffuse SANS, incommensurate magnetic satellites represent Bragg reflections that are seen in the SANS geometry, leading to important consequences: Noncollinear magnetic structures may be generally seen as a modulation on top of a FM or AF, depending on the leading terms $J_1$ in their Hamiltonian (see panel (A) of Fig. 31). In the latter case, the rotation angle $\alpha_r$ can be rewritten as $\alpha_r + \pi/2$, the staggered AF magnetization is used as order parameter. For AF spin spirals, incommensurate magnetic satellites appear in pairs centered around the AF-Néel point (typically $(\pi, \pi)$). As the direct beam ($\tau = (0,0,0)$) corresponds to a FM zone center, only the FM components of any noncollinear magnetic structure are visible by SANS. A typical case is illustrated in panels (C) and (D) of Fig. 31: A small canting of an otherwise purely AF cycloidal spin structure of only $1^\circ$ leads to small FM correlations (corresponding to an amplitude of the FM SDW of only 0.09 $\mu_B$/Fe). This causes a SANS signal in ferroelectric BiFeO$_3$ (Ramazanoglu et al., 2011). As the magnetic form factor of the corresponding ions $f(q) \approx 1$ for small $q$ around (0,0,0), SANS is very sensitive to small FM modulations.

Typically dealing with single-crystal samples, the magnetic selection rules apply as for every other magnetic neutron diffraction experiment: Only components of $S$ perpendicular to $q$ lead to a scattering signal [see Eq.(63)].

Further consequences of diffraction in the SANS geometry concern the coherence volume, which corresponds to a flat ellipsoid with the shortest half-axis along the beam direction for diffuse SANS, caused by the coarse wavelength resolution $\Delta \lambda/\lambda$ (typically $10\%$) (Felber et al., 1998). In perpendicular direction the coherence length is mainly governed by the divergence of the neutron beam. By contrast, Grigoriev et al., 2010 showed that the coherence volume of diffraction in the SANS geometry on objects with a two-dimensionally ordered nanostructure and a third nonperiodic dimension can go beyond the Born approximation: The Bragg reflection at the sample effectively acts as a neutron monochromator and leads to exceptionally elongated coherence lengths along the beam direction up to the $\mu$m regime. This is particularly relevant for well ordered periodic structures, e.g., vortex lattices (Yaron et al., 1994) or SkLs (Adams et al., 2011). Vice versa, the coherence lengths of the magnetically ordered state can be deduced from the inverse peak widths of such diffraction peaks, $\xi = 2\pi/\Delta q$ corrected for instrumental resolution.

Similar to single-crystal diffraction, diffraction in the SANS geometry requires the use of goniometers/rotation stages to rotate the reciprocal lattice of single-crystal samples through the Ewald sphere. Despite the Bragg angles and associated rocking angles being small, helical magnets, skyrmion lattices and also superconducting vortex lattices can yield exceptionally sharp rocking curves similar to the resolution limit of a SANS instrument at full collimation, typically 0.1° (Adams et al., 2011). To capture the full intensity of a Bragg peak, it is always necessary to record the integrated intensity of a rocking curve. As a further complication, the existence of strong Bragg peaks may sometimes lead to a significant amount of double scattering, which can be easily confused with true higher harmonic terms at multiples of $k$ (Adams et al., 2011). By means of Renninger scans (the sample is rotated around the scattering vector $Q$ (Shirane et al., 2002)) it is possible to discriminate true higher-order intensity from double scattering.

As spiral magnets often exhibit intrinsic magnetic chirality, the use of polarized neutrons, polarization analysis and even 3D polarimetry can be beneficial for the unambiguous identification of magnetic structures and the separation of magnetic and nuclear scattering. The cross-sections for polarized neutron scattering on chiral magnetic structures have been derived by Izyumov and Maleev, 1962; Maleev, 1961; and Maleev et al., 1963 and Blume, 1963. An introduction to 3D polarimetry is given by Brown, 2006. The papers by Kindervater et al., 2014 and Janoschek et al., 2007 provide typical examples.

5. Properties of B20 spiral magnets as inferred from SANS

A large body of SANS studies has been performed on the B20 family with MnSi ($P2_13$) as a drosophila in many regards (the skyrmionic spin textures found in these systems will be covered in section VIII.B.1
and VIII.B.2). While all B20 compounds share a generic hierarchy of energy scales and the diagram as introduced in Sec.VIII.A.2, their electronic properties, pitch lengths and transition temperatures vary considerably: a compact summary over the characteristic properties of the B20 series is given by Nagaosa and Tokura, 2013

The generic phase diagram representative of the B20 series is shown in Fig. 32, panels (B). Its archetypal example, MnSi is a weak itinerant FM, with a magnetic moment of 0.4 $\mu_B$ per Mn atom and a fluctuating moment of 2.2 $\mu_B$ (Lonzarich and Taillefer, 1985). In the helical phase below $T_C = 29.5$ K and below $H_{cl} \approx 0.1$ T, a large body of SANS studies established a mono-chiral helical order, e.g., by Grigoriev et al., 2007, 2006a,b, 2005; Hansen, 1977; Ishikawa and Arai, 1984; Ishikawa et al., 1976; and Lebech et al., 1995 with a wavelength of 180 Å caused by DMI. The propagation vector $k$ aligns along the (111) directions due to second-order anisotropy terms (Bak and Jensen, 1980). For increasing magnetic fields $k$ realigns in the direction of $H$, characteristic of a spin flop transition and the helix deforms to a cone phase (Ishikawa et al., 1976), until a fully field polarized state is reached at the upper critical field $H_{c2}$. The length of the pitch only weakly depends on temperature or magnetic field. Bulk measurements, (e.g. by Bauer and Pfleiderer, 2012) showed that due to demagnetizing effects, phase coexistence and extended crossover regions can be observed.

An extended region dominated by fluctuations is observed around the transition to the PM phase below a tricritical point (Bauer et al., 2013), which has been studied by means of SANS (Janoschek et al., 2013): As a consequence of the small propagation vector $k$, magnetocrystalline anisotropies are less effective in fixing the orientation of the magnetic order. Depending on the crystalline symmetry of the material, the energies of spiral magnetic textures might then be almost degenerate for orientations of $k$ that belong to manifolds in momentum space, e.g., a sphere or a ring. It was proposed by Brazovskii, 1975 that such a manifold may qualitatively alter the nature of a phase transition: Upon approaching the phase transition from high $T$, magnons soften on this manifold at finite $k$, giving rise to a large phase space. The abundance of fluctuations results a substantial magnetic entropy. To avoid this entropy the phase transition is driven to first-order. Based entirely on symmetry arguments, this scenario was discussed in various contexts, e.g., for weak crystallization (Brazovskii et al., 1987), liquid crystals (Brazovskii and Dmitriev, 1975; Swift, 1976), diblock copolymers (Bates et al., 1990; Leibler, 1980), and Bose-Einstein condensates (Gopalakrishnan et al., 2009).

Recent SANS experiments by Janoschek et al., 2013 showed that such a scenario is realized in MnSi with magnons softening on a sphere in $k$-space. Typical data is shown in Fig. 33. Panel (A) shows the evolution from sharp diffraction peaks associated to the helical domains ($T < T_C$), to a diffuse ring, representing a cut through a sphere ($T > T_C$). Panel (B) shows the temperature dependence of the inverse correlation length $\kappa$ as inferred from SANS and measurements of the susceptibility. Subsequently, it was demonstrated by means of susceptibility and SANS that other members of the B20 series like Cu$_2$OSeO$_3$ (Živković et al., 2014) show similar behavior. However, SANS and spin-echo studies by Bannenberg et al., 2017 on FeCo$_{1-x}$Si$_x$ revealed a more complex behavior with a coexistence of fluctuations and helimagnetic order over a broad temperature interval and long relaxation times with a stretched exponential that persists even under magnetic field.

In summary, these studies and related spin-echo mea-
measurements on MnSi (Pappas et al., 2009) essentially demonstrate an important feature of noncollinear magnets: The length scale of $k$, is still present in the characteristic fluctuations of incommensurate spiral magnets well above $T_C$ in the PM regime.

Similarly in the field-polarized FM state of a chiral spin structure like MnSi, the spin wave dispersion is distinct from a conventional FM. Due to the DMI, the spin wave dispersion is centered at $\pm k$ instead of $q = 0$ (Kataoka, 1987). The antisymmetric interaction of polarized neutrons with the chiral spin structure of MnSi and related materials has been used in a series of SANS experiments by Grigoriev et al., 2015 and Siegfried et al., 2017 to measure the spin-wave stiffness in the spirit of a similar approach made earlier for FM spin waves (see, e.g., Toperverg et al., 1993). This method is particularly helpful for materials like FeGe, were high-pressure synthesis only allows the growth of tiny crystallites or powders which do not allow the use of typical inelastic neutron instruments like triple-axis spectrometers (see, e.g. Kugler et al., 2015). Moreover, it benefits from the superior resolution of SANS instruments at small $k$, which are otherwise challenging to resolve. In the field-polarized states of MnSi (Grigoriev et al., 2015) and FeGe (Siegfried et al., 2017), SANS shows intensity in a circle, centered at $\pm k$. The radius or critical angle $\theta_c$ of the circle allows deducing the spin-wave stiffness. For this method, the magnetic field is aligned perpendicular to $k_0$. Note that for FM spin waves the magnetic field needs to be tilted with respect to $k_0$ as these do not show the inherent chirality of helical magnets.

A further point addressed in the B20 series concerns the coupling of crystalline and magnetic chirality. While the absolute chirality of the crystalline structure of the B20 monosilicides of Fe, Co and Mn is randomly governed by crystal growth, the relative chirality of the helices with respect to the crystal is given by the orientation and the sign of $D$. The handedness of both the magnetic helix and crystal can be addressed by means of polarized SANS and single-crystal X-ray diffraction, respectively. Measurements on concentration series of $\text{Mn}_{1-x}\text{Fe}_x\text{Ge}$ (Grigoriev et al., 2013), $\text{Fe}_{1-x}\text{Co}_x\text{Ge}$ (Grigoriev et al., 2014) and $\text{Fe}_{1-x}\text{Co}_x\text{Si}$ (Siegfried et al., 2015) reveal a breakdown of the helical order with vanishing $k$ associated with a flip of relative chirality and reentrant FM behavior at the critical concentration $x_c$. Consequently, opposite relative chirality is observed for opposite ends of each concentration row. The microscopic reason for the associated change of sign of the DMI is not resolved unambiguously, although is has been discussed that a competition of DMI and cubic anisotropy might play a role. Recent ab-initio theoretical calculations by Koretsune et al., 2015 showed that the behavior of DMI can be systematically understood in terms of the details of the electronic band structure.

Besides a systematic tuning of pitch length, relative chirality, $T_C$ and electronic properties, concentration series of the B20 members allow for systematic studies of quantum chirality and the effects of disorder. Bulk measurements by Bauer et al., 2010 revealed a critical concentration of $x_{c\text{Fe}} = 0.192$ and $x_{c\text{Co}} = 0.084$ where the transition temperature vanishes, for single crystals of $\text{Mn}_{1-x}\text{Fe}_x\text{Si}$ and $\text{Mn}_{1-x}\text{Co}_x\text{Si}$, respectively. The behavior of both concentration series agrees on a normalized doping scale $x/x_c$, suggesting the existence of a putative quantum-phase transition, dominated by FM quantum fluctuations. Recent SANS measurements on polycrystalline $\text{Mn}_{1-x}\text{Fe}_x\text{Ge}$ by Altybaev et al., 2016, and on powders of $\text{Mn}_{1-x}(\text{Co, Rh})_x\text{Ge}$ by Martin et al., 2017, show a far more complex behavior, although the influence of crystalline quality remains unclear. Alternative to doping, pressure serves as another tuning parameter. The transition temperature of MnSi is
suppressed until it reaches \( T_C = 0 \) at \( p_c = 15 \text{kbar} \) and the phase transition thereby changes from second order to weak first order at \( p^* = 12 \text{kbar} \) (Pfleiderer et al., 1995, 2007a). An extended region of non-Fermi-liquid behavior emerges above \( p_c \) (Doiron-Leyraud et al., 2003; Pfleiderer et al., 2001; Schofield, 1999), characterized by an unusual exponent of the electrical resistivity \( \rho(T) \propto T^\alpha \) with \( \alpha = 3/2 \). SANS and neutron diffraction observed a weak partial magnetic order (PO) above \( p^* \) (Pfleiderer et al., 2007a,b, 2004), where the spontaneous formation of a skyrmionic ground state at zero field was discussed by Rößler et al., 2006. Recent measurements by Ritz et al., 2013 connect the topology of the Hall signal and the non-Fermi-liquid regime, suggesting the presence of disorder structures with nontrivial topology.

6. Properties of non-B20 spiral magnets as inferred from SANS

The noncentrosymmetric tetragonal AF \( \text{Ba}_2\text{CuGe}_2\text{O}_7 \) (\( P\overline{4}2_1m \)) shows a long-range incommensurate, AF cycloidal spin spiral due to DMI below \( T_N \approx 3.05 \text{K} \) of which a number of neutron scattering studies (see, e.g., Mühlbauer et al., 2012 and Zheludev et al., 1997b and Refs. therein) and theoretical work (Bogdanov et al., 2002; Chovan et al., 2013, 2002) have established a quite complete understanding. However, it was predicted that a second, sign alternating component of the DMI vector pointing along the \( c \)-axis should give rise to incommensurate weak FM behavior (Chovan et al., 2013, 2002) with the same \( k \). This prediction has been confirmed by SANS (Mühlbauer et al., 2017). A similar situation, observed in the ferroelectric AF \( \text{BiFeO}_3 \) has already been illustrated in Sec. VIII.A.4. Here, polarization analysis was used in addition to separate nuclear from magnetic scattering.

\( \text{Ba}_2\text{CuGe}_2\text{O}_7 \) is also relevant in terms of a Brazovskii scenario (Brazovskii, 1975), however, with reduced 2D symmetry caused by the weak inter-plane coupling of \( \text{Ba}_2\text{CuGe}_2\text{O}_7 \): Due to crystallographic anisotropy, the manifold of fluctuations is reduced to a ring in reciprocal space. SANS and neutron diffraction have been used to shed light on the interplay of Brazovskii and 2D physics (Mühlbauer et al., 2017).

The ternary compound \( \text{Pr}_5\text{Ru}_3\text{Al}_2 \) crystallizes in the cubic noncentrosymmetric/nonmirror symmetric space group \( I2_13 \), hence allowing for DMI. Indeed, a study by Makino et al., 2016 on high quality powder samples have indicated the existence of incommensurate peaks at \( \mathbf{k} \approx (0.066, 0.066, 0.066) \) (r.l.u.) below \( T_N = 3.8 \text{K} \), while the FM transition found earlier was attributed to impurity phases. These incommensurate peaks have been confirmed by means of SANS measurements (Okuyama et al., 2017). Representation analysis was used to propose a helical spin structure of a composed moment of the different \( \text{Pr} \) layers.

The rare-earth metal \( \text{Ho} \) shows various spiral magnetic phases, generated by an oscillatory RKKY interaction of the localized \( 4f \)-spins through the conduction electrons. Between the Néel temperature \( T_N = 132 \text{K} \) and the Curie point \( T_C = 20 \text{K} \), an AF helical phase is found by neutron diffraction for single-crystal samples. Below \( T_C \), an additional FM component gives rise to a conical phase. Upon increasing magnetic field, the spin structure shows a series of transitions to a helifan(3/2), a helifan(2) and finally a fan structure for field along the \( b \)-axis (Jensen and Mackintosh, 1990; Kosugi et al., 2003). Helifan denotes a mixture of helix and fan each with different periodicitics. SANS experiments have been used to study the influence of grain size on the spin structures of \( \text{Ho} \) and related \( \text{Tb} \) (Michels et al., 2011; Szary et al., 2016). In contrast to single-crystalline samples, Szary et al., 2016 find no evidence of the helifan structures in nanocrystalline \( \text{Ho} \) by means of field-dependent SANS. For coarse grained samples, a streak pattern indicative of a long-period magnetic structure is observed, correlated with the vanishing of the helix for increasing field.

The members of the series \( \text{RMn}_6\text{X}_6 \), where \( R \) is a rare-earth element and \( X \) is a metal with an unfilled \( p \)-shell exhibit a hexagonal, layered structure and show a large diversity of magnetic phases, generated by competition of FM exchange between the \( X-X-X \) planes, AF coupling via the \( R-X \) planes and finally RKKY exchange between the next-nearest neighbor \( \text{Mn} \) planes. For \( \text{Y Mn}_6\text{Sn}_6 \), SANS studies by Bykov et al., 2015 have found a helical ordering with \( \mathbf{k} = 1.7 \text{nm}^{-1} \), that gives way to a state with spatial FM fluctuations at zero wavevector in the interval of \( 250-360 \text{K} \) and finally leads to a PM regime. Since the system is strongly anisotropic with the moments confined in the \( (001) \) plane, the phase transition is considered to be quasi two-dimensional. A qualitatively similar behavior was observed by Altynbaev et al., 2016 in powder samples of \( \text{B}_20 \text{MnGe} \) (crystallite size 1 \( \mu \text{m} \)). Here, a Gaussian peak at incommensurate 2 \( \text{nm}^{-1} \), associated with static helical order, is continuously replaced by a Lorentzian peak at the same \( k \) indicative of fluctuations, for increasing temperature. At \( T_N = 130 \text{K} \), the Lorentzian contribution dominates. Both contributions exist above \( T_N \) and show exponentially activated behavior. Well above \( T_N \), additional Gaussian correlations are observed at lower \( q \), attributed to static short-range FM inhomogeneities. The phase transition observed for \( \text{MnGe} \) is interpreted as a complex order-to-disorder transition, although the influence of the crystallite size remains unclear.

The bond frustrated compound \( \text{ZnCr}_2\text{Se}_4 \) (cubic spinel (\( Fd\overline{3}m \)) structure) is magnetoelectric and shows a strong spin-lattice coupling. An AF spiral with a screw angle of 42° is observed below \( T_N = 20 \text{K} \) that goes along with a structural transition to tetragonal symmetry (Felea et al., 2012 and Refs. therein). SANS (Cameron et al., 2016) has been used to study the field dependence of the multidomain spin spiral phase with particular focus on
FIG. 34 Panels (A) and (B): Schematic depiction of a Néel-type and a Bloch-type single skyrmion. A cut through a single skyrmion yields the spin structure of a Néel and a Bloch domain wall, respectively. Reproduced with permission from Markus Garst. Panel (C): Schematic depiction of a skyrmion lattice with the hexagonal arrangement of skyrmion lines through the sample (Milde et al., 2013).

1. The concept of skyrmions

The concept of skyrmions goes back to British physicist Tony Skyrme (Skyrme, 1961a,b, 1962) and later contributions by Adkins et al., 1983, who managed to show that in the presence of nonlinear coupling, excitations of continuous fields can assume particle-like character. Skyrmionic states are characterized by a nontrivial topology, which can be conveniently classified by means of winding numbers (Manton and Sutcliffe, 2004). A nonzero, integer winding number describes a topology that cannot be smoothly distorted to a trivial state and, as a consequence, leads to countable entities (e.g., a vortex state in 2D). In turn, the topological properties are intimately connected to particle stability. Originally developed to explain the constituents of the nucleus, protons and neutrons, as topological excitations of a spinless pion field, the concept of skyrmions has been used in different physical contexts. In condensed matter physics, skyrmionic states have been described in quantum Hall magnets (Sondhi et al., 1993) at finite magnetic field and in topological insulators at zero field (Hsieh et al., 2008; König et al., 2007). Recent reviews of the general concept of skyrmions have been given by Brown and Rho, 2010 and Manton and Sutcliffe, 2004.

In their seminal studies Bogdanov and Hubert, 1994; Bogdanov, 1995; Bogdanov and Rößler, 2001; and Bogdanov and Yablonskii, 1989 predicted the stabilization of magnetic vortices in FMs by Bloch domain walls and in anisotropic noncentrosymmetric magnetic materials, similar to superconducting vortices which are stabilized by the negative energy of a normal to superconducting interface. If vortex states in bulk magnets have a non-trivial winding density, they can be regarded as skyrmions. Smectic, nematic, and hexatic forms of order, akin to liquid crystals and multi-k structures have been discussed for spin liquids or spin glasses found in frustrated systems like e.g. Kagomé lattices or 3D Pyrochlore AFs (Forgan et al., 1989; Lawler et al., 2008), however, their topological properties have not been discussed in detail. A schematic depiction of the spin structure of a Néel-type and a Bloch-type single skyrmion are shown in panels (A) and (B) of Fig. 34. Note that both exhibit similar topological properties/winding numbers. Panel (C) shows the arrangement of skyrmions to a SkL with hexagonal symmetry.

The first experimental indication of magnetic skyrmions in helimagnetic MnSi revealed magnetic whirls, that arrange in a SkL (Mühlbauer et al., 2009a). SANS established a well-ordered Bloch-type SkL phase pocket (formerly denoted A-phase) at ≈ 0.2 T, close to $T_c$ in MnSi. A resolution limited hexagonal scattering pattern akin to a superconducting vortex lattice, aligned perpendicular to the magnetic field irrespective of the crystal orientation (Mühlbauer et al., 2009a) is observed. The spin structure is translationally invariant in the direction of the magnetic field, leading to skyrmion lines. A Ginzburg-Landau Ansatz based on a triple-k state including Gaussian fluctuations in the spirit of an order-by disorder mechanism qualitatively reproduced the stable region of the SkL phase in the $B$-$T$-phase diagram (panel (A) of Fig.35) and showed that the SkL state corresponds to the thermodynamic minimum. Typical SANS data from MnSi are shown in panels (B) and (C).

Characterized by a nonzero winding number of −1, these whirls have skyrmionic character. Their topology is distinct from the surrounding conical phase, leading to particle-like characteristics of single skyrmions. Since SANS does not couple directly to the topology of the SkL state (single-k multi-domain vs. multi-k single-domain), its topological properties have been probed by Hall measurements by Neubauer et al., 2009. Besides the normal
and anomalous Hall effect, an additional signal appears, caused by the Berry phase which the moving conduction electrons acquire as their spin follows the nontrivial topology of the SkL. Since then the topological Hall effect has been established as a hallmark of nontrivial skyrmionic structures (Ritz et al., 2013).

Further high resolution SANS experiments showed an exceptionally long-range ordered SkL and were able to prove the existence of weak, higher-order reflections (i) indicative of the particle-like character of the SkL (Adams et al., 2011) and (ii) the fixed phase relation of the triple-$k$ state. The pinning of the SkL to the crystal lattice in the plane perpendicular to the field is given by weak higher-order anisotropy terms, which have been examined in a series of SANS experiments (Adams, 2015) including the use of uniaxial pressure (Chacon et al., 2015). As for the helix state, the SkL state in MnSi is only weakly affected by crystal anisotropies, which reflects in a essentially isotropic temperature-field phase diagram and the particular coupling of the SkL spin structure to the magnetic field.

Inspired by the observation of magnetic skyrmions in MnSi and later in FeCo$_{1-x}$Si$_x$ (Mühlbauer et al., 2009a; Münnzer et al., 2010) by means of SANS, similar textures with topological properties have been observed in increasing number in various materials, including metals (Nagaosa and Tokura, 2013), semiconductors (Münnzer et al., 2010) and insulators (Adams et al., 2012), again with SANS playing a pivotal role. The finding of skyrmions beyond the B20 family (Bordacs et al., 2017; Kézsmárki et al., 2015; Kurumaji et al., 2017; Nayak et al., 2017; Tokunaga et al., 2015) and in particular their observation in thinned bulk samples (e.g., Yu et al., 2011), thin magnetic films (Heinze et al., 2011) and nanostructured patterns (Boule et al., 2016) using real space methods like MFM, spin polarized STM or LTEM have established magnetic skyrmions as a generic phenomenon of materials that support chiral interactions due to broken inversion symmetry. A general review of the properties of magnetic skyrmions has recently been given by Nagaosa and Tokura, 2013 and Bauer and Pfleiderer, 2010. In analogy to superconducting vortex matter (Sec. IX), a similarly large variety of skyrmion matter is expected.

Note that SANS is only able to access skyrmionic matter of bulk samples that shows at least short-range order, e.g., SkLs, glasses or liquids. The observation of individual skyrmions and their particle-like character (in particular for surfaces and thin film samples) is possible by the real space methods mentioned in the previous paragraph.

2. Skyrmions in spiral magnets seen by SANS

SANS studies by Jonietz et al., 2010 showed that spin transfer torque effects allow for manipulation of the SkL at ultra-low current densities of $\approx 10^6$A/m$^2$, facilitated by (i) the efficient decoupling of SkL and crystal lattice due to the smooth texture and (ii) the efficient coupling of the transport currents due to the topological properties of the SkL (Everschor et al., 2012). Complementary to measurements of spin transfer torque, recent time-resolved TISANE measurements with periodically oscillating magnetic fields on MnSi (Mühlbauer et al., 2016) consistently show a depinning transition of the SkL at a critical oscillation amplitude. These measurements allow to directly track the rigidity and pinning of the SkL also for non-conducting samples.

The stabilization of SkL phases is a central point of an increasing effort in experiment and theory: (i) Besides stabilization by fluctuations as found for MnSi (Mühlbauer et al., 2009a), Fe$_{1-x}$Co$_x$Si (Münnzer et al., 2010), FeGe (Yu et al., 2011), Cu$_2$OSeO$_3$ (Seki et al., 2012b) and the $\beta$-Mn series (Tokunaga et al., 2015), several studies using real space techniques revealed (ii) an

![FIG. 35 Panel (A) shows the phase diagram of MnSi, calculated by means of a Ginzburg-Landau Ansatz based on a triple-$k$ state. The inset shows the effect of Gaussian fluctuations on lowering the free energy of the SkL phase. Panel (B) and (C) show SANS data of the SkL in MnSi, for magnetic field parallel to (110) (B) and for a random direction of the magnetic field (C) (Mühlbauer et al., 2009a). Panel (D) shows a SANS pattern of the SkL in ferroelectric Cu$_2$OSeO$_3$ with magnetic field along the (100) axis (White et al., 2014). Note the multiple domain state. Panel (E) shows SANS data of the SkL in a single crystalline sample of Co$_3$Zn$_8$Mn$_4$ with magnetic field along (110) (Tokunaga et al., 2015).](image-url)
increasing skyrmion phase with decreasing sample thickness (Seki et al., 2012c; Yu et al., 2011, 2010). (iii) The influence of crystalline anisotropy have been investigated in several studies (Adams, 2015; Chacon et al., 2018). β-Mn-type Co$_2$Zn$_y$Mn$_z$ (Karube et al., 2016) exhibits a transition from a conventional triangular lattice to a metastable square SkL for low temperatures due to magneto-crystalline anisotropy. (iv) The destabilization of competing phases (Seki et al., 2012c; Yu et al., 2011, 2010), (v) strain (Fobes et al., 2017; Nii et al., 2015) and (vi) terms induced by free surfaces (Rybakov et al., 2015) and interface spin orbit effects (Heinze et al., 2011; Römmling et al., 2013) play an important role, in particular with decreasing sample thickness. Here, SANS, GISANS and polarized neutron reflectometry (PNR) have been used to study the possible formation of skyrmionic structures in B20 thin films: While such textures have been claimed to exist in MnSi thin films (Karhu et al., 2012; Meynell et al., 2017; Wilson et al., 2013), based on magnetization and PNR and SANS measurements, recent GISANS studies (Wiedemann et al., 2017) did not reveal any hints for SKL spin textures in thin epitaxial films of MnSi.

As the SkL is topologically distinct from the surrounding conical phase, a continuous transformation of a conical state to a SkL is impossible, or is at least associated with a large energy barrier. In combination with pinning induced by defects, this offers the study of considerable metastable, out-of-equilibrium and hysteretic effects, which have been demonstrated by quenching the SkL into long-lived metastable phases in various compounds (Karube et al., 2016; Münzer et al., 2010; Nakajima et al., 2017; Oike et al., 2016). The topological difference raises the generic question of the process of skyrmion nucleation/decay at the borders of the SkL phase pocket, where a first-order phase transition is observed by bulk measurements (Bauer and Pfleiderer, 2012). SANS combined with real space MFM measurements on FeCo$_{1-x}$Si$_x$ (Milde et al., 2013) showed that skyrmions decay by means of a topological defect that zips two neighboring skyrmions and propagates along the skyrmion line. Interestingly, the topological defect has the mathematical form of a monopole of emergent flux. In line with bulk and neutron imaging measurements (Bauer and Pfleiderer, 2012), geometry, demagnetization and edge effects play an important role (Müller et al., 2016), suggesting that macroscopic phase coexistence is present and might have been underestimated (Reimann et al., 2018).

SANS experiments by Seki et al., 2012b, Adams et al., 2012 and White et al., 2012, 2014 and LTEM studies by Seki et al., 2012c have established the existence of a SkL in the multiferroic B20 insulator Cu$_2$OSeO$_3$. The phase diagram shares some qualitative features of all B20 compounds, although a (100) pinning of the helical order at zero field indicates that (i) the magnetocrystalline anisotropy terms are different and (ii) the difference of free energy for pinning along (100) and (110) is very small. The different anisotropy terms reflect in the orientation of the SkL, multi-domain states are typically seen for Cu$_2$OSeO$_3$ (Adams et al., 2012; White et al., 2012) (see panel (D) of Fig. 35 for typical SANS data). Moreover, a second independent skyrmion phase for H // (100) at low temperature observed by means of SANS in Cu$_2$OSeO$_3$ is attributed to a stabilization mechanism invoking cubic anisotropy terms (Chacon et al., 2018).

The influence of the SkL to an electric field has been investigated by White et al., 2012: For magnetic fields along a (1, 1, 0) axis and electric field along (111) it is possible to induce small rotations of the SkL around the magnetic field axis. Consistent with symmetry arguments (Seki et al., 2012a) the SkL state permits a polar state for field along (1, 1, 0), namely P // (001). Each electric dipole associated to a single skyrmion hence is inclined with respect to the field along E // (111). The coupling of P and E is made responsible for the rotation of the SkL. It was found in a follow-up study (White et al., 2014) that electrical fields in combination with slight oscillations of the magnetic field amplitude helps overcoming SkL pinning and leads to significantly larger rotation angles.

Recently, SANS studies have revealed the existence of SkL phases in an increasing number of non-B20 compounds. β-Mn-type Co$_2$Zn$_y$Mn$_z$, (x + y + z = 20), crystallizes in the cubic, chiral P4$_1$32 space group. Accordingly, the ground state is a helical structure provided by DMI. SANS, magnetometry and LTEM measurements by Tokunaga et al., 2015 in thin platelets and small bulk samples have identified a Bloch-type SkL with a phase diagram that shares great similarities to cubic MnSi, although the transition temperature is much higher ≈ 320 K. Similar to MnSi, the SkL is stabilized by an order-by-disorder mechanism.

A qualitatively different situation if found in polar VOSe$_2$O$_5$ (space group C$_{4v}$), GaV$_4$S$_8$ and GaV$_4$Se$_8$ (space group C$_{3v}$) where the ground state is a cycloidal spin arrangement. Correspondingly, SANS, atomic force microscopy (AFM), electron spin resonance (ESR) and magnetization measurements have identified a Néel-type SkL in these materials (Bordacs et al., 2017; Kézmárski et al., 2015; Kurumaji et al., 2017). In stark contrast to the cubic compounds, where weak anisotropies govern the orientation of the helical order and the SkL with respect to the crystal lattice and the SkL is essentially tied to the magnetic field as leading contribution, the SkL state is confined to certain crystallographic planes by crystal anisotropy for polar GaV$_4$S$_8$ (Kézmárski et al., 2015) and GaV$_4$Se$_8$ (Bordacs et al., 2017). Moreover, the competing conical state is forbidden by the DMI pattern in the polar C$_{4v}$ class. Accordingly, the phase diagram is qualitatively different, showing an extended region of
stability of the SkL phase, although it is strongly dependent on the crystallographic direction (Bordacs et al., 2017; Kézsmárki et al., 2015). Antiskyrmion structures, characterized by alternating Bloch and Néel spin rotation following the boundary of the single skyrmion, have been identified in the acentric tetragonal Heusler compound Mn$_{1.4}$Pt$_{0.9}$Pd$_{0.1}$Sn with D$_{2d}$ symmetry by means of LTEM (Nayak et al., 2017). The emergence of skyrmionic textures in magnetically frustrated centrosymmetric materials with high lattice symmetry has recently been in the focus of intense theoretical effort (Hayami et al., 2017; Leonov and Motovoy, 2015; Okubo et al., 2012; Wang et al., 2015). Without the restrictions on the symmetry imposed by the DMI interaction, a large body of different helimagnetic structures and multiple-$q$ modulations, based on frustration effects, remain to be investigated for their topological properties. Due to lesser restrictions on symmetry, skyrmions with topological charge 1 and 2 (Leonov and Motovoy, 2015), skyrmion-antiskyrmion lattices (Okubo et al., 2012) and 3-D modulated structures (Wang et al., 2015) are proposed. It is expected that neutron diffraction and SANS will play a key role in their identification.

IX. VORTEX LATTICES IN SUPERCONDUCTORS

Superconductors are classified as type-I or type-II depending on their response to an applied magnetic field (Tinkham, 1996). Most superconducting materials, and almost all that are technologically relevant, belong to the second category. Subjecting a type-II superconductor to a magnetic field will introduce vortices, each carrying one quantum of magnetic flux $\Phi_0 = \hbar/2e = 2068$ Tm$^2$ (Abrikosov, 1957; Huebener, 2001). The vortices introduce singularities in the order parameter and may be used as probes of the superconducting state in the host material. Moreover, moving vortices give rise to dissipation within the superconducting state, and the dynamical properties of vortex matter constitutes its own important area of research (Blatter et al., 1994; Brandt, 1995).

A. Imaging the vortex lattice by neutron diffraction

Due to their mutual repulsion, vortices arrange themselves in an ordered vortex lattice (VL), as long as the vortex-vortex interactions dominate external influences such as pinning to impurities or thermal disordering. The VL gives rise to a periodic magnetic field modulation and may therefore be imaged by magnetic neutron scattering (de Gennes and Matricon, 1964), and the first experimental confirmation for the existence of vortices was indeed made by observing neutron diffraction from the VL in niobium (Cribier et al., 1964).

SANS studies of the VL can be considered as crystallography of a two-dimensional system of lines (vortices), providing information about the lattice structure and correlations, as well as the internal structure of the individual scatterers. While there are many similarities between the VL and the skyrmion lattice discussed in Sec.VIII, there are also important differences. Due to flux quantization, the vortex density depends linearly on the magnetic induction, $B$. For a square VL, the magnitude of the fundamental (first order) scattering vector is given by

$$q_0 = 2\pi \sqrt{\frac{B}{\Phi_0}}. \quad (64)$$

For a rhombic VL, i.e., oblique with equal side lengths but an arbitrary opening angle $60^\circ < \beta < 90^\circ$ the scattering vector is $q = (\sin \beta)^{-1/2} q_0$, whereas a distorted triangular (hexagonal) lattice ($\beta < 60^\circ$) will have $q = (2\tan 60^\circ/2)^{1/2} q_0$. From precise measurements of the scattering vector magnitude, it is therefore possible to determine the VL symmetry, or to provide a relation between $B$ and the applied magnetic field $\mu_0 H$.

The scattered intensity is directly related to the amplitude of the magnetic field modulation due to the vortices, quantified by the VL form factor. The latter is given by the Fourier transform of the two-dimensional field modulation, $F(q) = \int B(r) e^{iq\cdot r} \, dr$. The form factor depends on the superconducting penetration depth ($\lambda$) and coherence length ($\xi$), and measuring $|F(q)|$ allows a determination of these characteristic length scales$^4$. This requires a measurement of the integrated scattered intensity, obtained by rotating the relevant VL diffraction peak through the Bragg condition in a rocking curve, as discussed in Sec. VIII.A.4 for the skyrmion lattice. Normalizing the integrated scattered intensity to the incident neutron flux one obtains the VL reflectivity

$$R = \frac{2\pi \lambda^2 \Delta N^2}{16\Phi_0 q^2} |F(q)|^2, \quad (65)$$

where $t$ is the sample thickness, and $\lambda_n$ is the neutron wavelength (Christen et al., 1977; Kemoklidze, 1965). With modern SANS instruments is is possible to measure $|F(q)|$ as small as $0.1 - 0.2$ mT, constituting tiny “ripples” on top of the average magnetic induction.

Early neutron scattering studies of the VL were performed using two-axis diffractometers (Cribier et al., 1964; Schelten et al., 1974, 1971). Such measurements were challenging due to the long VL periodicity $\sim 2\pi/q_0$, exceeding tens of nanometers even for the largest available magnetic fields and leading to scattering angles $< 1^\circ$.

$^4$ To avoid confusion with the London penetration depth $\lambda$, the neutron wavelength is denoted as $\lambda_n$ throughout this section.
This situation was improved using a double crystal geometry to separate the scattering due to the VL from the direct (undiffracted) beam (Christen et al., 1985, 1980). Still, only a few such studies were performed, mostly on niobium where the small $\lambda$ leads to scattering intensities in excess of 10% of the incident beam. The development of dedicated SANS instruments at cold neutron sources changed this situation dramatically. By providing a higher incident flux and allowing the entire scattering plane to be imaged using a two-dimensional position sensitive detector (Fig. 36), these have made SANS an ideal tool for VL studies.

The emphasis in the following reflects the most studied problems/materials and current directions for SANS VL studies, but is not a comprehensive treatise. Shorter and more focused reviews are available elsewhere (Eskildsen, 2011; Eskildsen et al., 2011).

B. Vortices as probes of superconducting materials

Information about the nature of the superconducting state in the host material can be obtained from the VL structure as well as the field and temperature dependence of the scattered intensity.

1. VL symmetry and orientation

The equilibrium VL configuration is determined by the vortex-vortex interaction, and in the ideal isotropic case will have a triangular symmetry (Kleiner et al., 1964; Matricon, 1964). However, the free energy difference between the triangular and square symmetries is small (2%), rendering the VL sensitive to an anisotropy of the screening current plane perpendicular to the applied field. Actual superconducting materials possess a hierarchy of anisotropies that will influence the VL symmetry and/or orientation relative to the crystalline axes. This often leads to degenerate VL orientations relative to the crystalline host, and SANS diffraction patterns that include scattering from two or more domain orientations as seen in Figs. 36(b), 41(a) and 44(c,d).

The simplest example of a nontriangular VL is found in tetragonal superconductors with a fourfold basal plane anisotropy and the applied field parallel to the $c$ axis. With increasing vortex density the contribution of the anisotropy to the free energy becomes dominant, leading to a sequence of VL transitions. A field-driven VL transition from a triangular to hexagonal symmetry was first observed in the borocarbide magnetic superconductor ErNi$_2$B$_2$C (Eskildsen et al., 1997; Paul et al., 1998; Yaron et al., 1996; Yethiraj et al., 1997). Since then, similar transitions have been observed in a range of materials, including V$_3$Si (Yethiraj et al., 1999), the high-temperature cuprate YBa$_2$Cu$_3$O$_7$ (Brown et al., 2004), and the heavy-fermion CeCoIn$_5$ (DeBeer-Schmitt et al., 2006). The triangular-to-square transition is so ubiquitous that it is more noteworthy when it is absent in a superconductor with a fourfold basal plane anisotropy (Gilardi et al., 2004; Kawano-Furukawa et al., 2011; Morisaki-Ishii et al., 2014; Riseman et al., 1998). More recently it was also observed for the skyrmion lattice (Karube et al., 2016), underscoring the similarities between the VL and SkL.

Theoretically, nonlocal corrections to the London model describe a VL symmetry evolution driven by a Fermi surface anisotropy (Kogan et al., 1997). The transition is characterized by two critical fields as shown in Fig. 37(a). Here a rhombic (distorted triangular) VL [Fig. 36(b)] undergoes a first-order reorientation transition at $H_1$, followed by a continuous increase of the opening angle towards a second-order transition at $H_2$ above which a square VL [Fig. 36(a)] is observed. Figure 37(b) shows results of a detailed study of the VL in YNi$_2$B$_2$C, where coexisting low- and high-field rhombic phases are observed near the first order reorientation transition (Dewhurst et al., 2005). At higher temperatures the gap between $H_1$ and $H_2$ grows, consistent with increased thermal fluctuations close to the upper critical field (Gurevich and Kogan, 2001). Separate studies of doped samples showed that $H_2$ depends sensitively on the nonlocality range (Gammel et al., 1999).

An additional contribution to a fourfold anisotropy may also come from the superconducting gap, e.g., in materials with non $s$-wave pairing symmetry (Agterberg, 1998; Franz et al., 1997; Ichioka et al., 1999). In YBa$_2$Cu$_3$O$_7$ early SANS studies showed diffraction patterns with a fourfold structure due to pinning to twin boundaries (Forgan et al., 1990; Keimer et al., 1994; Yethiraj et al., 1993b). More recent measurements on de-twinned samples revealed a complex evolution of the VL symmetry and orientation, Fig. 37(c) (White et al., 2008, 2011, 2009). The presence of two, first-order reorientation transitions may naively be attributed to Fermi surface and gap anisotropies separately. However, it has proven difficult to deconvolute these two contributions to the anisotropy (Leos et al., 2015), and a definitive understanding of the VL symmetry in such cases is still
the vortex-vortex interaction potential, giving rise to also applies to the SkL. That said, the “hairy ball” theorem winding number, the topology here governs the behavior of the collective VL. That is, the number of vortices increase. More rigorous models likely existence of VL phases that spontaneously break an intermediate mixed state (IMS) at low fields where flux free Meissner regions and VL regions coexist (Aston et al., 1971; Christen et al., 1977). In the IMS the vortex separation, and hence \( q_0 \), is independent of the applied field, as illustrated in Fig. 38(c). With increasing \( H \), the VL regions grow at the expense of the Meissner ones until they fill the entire sample, after which \( q \) follows the \( \sqrt{H} \)-behavior given by Eq. (64). Upon cooling through the superconducting transition, vortex clusters with a constant periodicity were found to coexist with a regular Abrikosov (or Shubnikov) VL phase, before the latter vanishes at low temperature to produce the IMS state (Pautrat and Brulet, 2014). Due to its inherent inhomogeneity, the IMS may be used as a model system for domain nucleation and morphology. Recently, this was studied using spatially-resolved SANS combined with neutron grating interferometry, to obtain detailed spatial information about the structure of the IMS domains and how they expand, Fig. 38(c) (Reimann et al., 2015).

2. Field and temperature dependence of the VL form factor

The simplest model for the VL form factor is provided by the local London theory. This relates the supercurrent density to the magnetic vector potential at the same point via the penetration depth, extended by a Gaussian cut-off to take into account the finite coherence length (Eskildsen et al., 2011):

\[
F(q) = \frac{B}{1 + q^2 \lambda^2} e^{-cq^2 \xi^2}.
\]

Here, \( c \) is a constant of order unity (Yaouanc et al., 1997). In most cases \( q \lambda \ll 1 \), and Eq. (66) predicts a form factor that decreases exponentially with increasing \( q^2 \sim B \). This is due to an increased core overlap resulting in a rapid reduction of the field modulation, even as the number of vortices increase.
FIG. 38 VL phases in niobium. (a) Phase diagram for applied fields along the fourfold [001] axis (Laver et al., 2009). (b) VL structures observed as a function of field direction (Laver and Forgan, 2010). Isosceles half-unit cells are indicated by filled triangles, and equilateral triangles are colored in red. Thick lines show sudden changes in the VL orientation. (c) Intermediate mixed phase indicated by a constant VL scattering vector below 120 mT at two different positions within the same crystal (Reimann et al., 2015). Gray lines correspond to a vortex density $\propto H$, Eq. (64).

based on the Ginzburg-Landau theory (Clem, 1975; Hao et al., 1991), as well as numerical calculations carried out within the Eilenberger formalism (Ichioka et al., 1999), confirm the monotonic decrease of $F(q)$ while yielding deviations from a pure exponential behavior. For strongly type-II superconductors ($\lambda/\xi \gg 1$) the extended London and Ginzburg-Landau form factors agree at low fields. Measuring $F(q)$ at fields that are simultaneously small, but still large enough to satisfy $q\lambda \gg 1$, will therefore allow an extrapolation to $q = 0$ and an estimate of the penetration depth which is unaffected by vortex core effects.

Figure 39(a) shows the measured form factor for the primary VL reflection in LuNi$_2$B$_2$C (Densmore et al., 2009). For this material $|F(q)|$ is best described by the London model (full line) rather than the more sophisticated models (dashed lines). In addition to the determination of $\lambda$ from the $q = 0$ extrapolation as discussed above, an estimate of the coherence length may be obtained from the slope of $\ln |F(q)|$ versus $\mu_0 H$. Here a $c = \frac{1}{2}$ is commonly used in Eq. (66), which is found to yield reasonable values for $\xi$. This is further justified by a quantitative comparison to numerical results (Ichioka et al., 1999), which suggests an appropriate value of 0.44 at low temperatures (Bowell, 2008; Eskildsen et al., 2011). In cases where more than the primary VL reflection can be measured, a more detailed analysis of the form factor is possible, including comparison to complex models as for Sr$_2$RuO$_4$ (Kealey et al., 2000). In materials such as LuNi$_2$B$_2$C where it is possible to measure a large number of higher-order VL reflections as shown in Fig. 36(a), the real space field distribution may be obtained by $B(\mathbf{r}) = \sum F(q_{hk}) e^{ijq_{hk} \cdot \mathbf{r}}$ and shown in the inset to Fig. 39(a).

A deviation from the conventional field dependence occurs in materials with a strong coupling between the applied field and the quasiparticle spins. This was first reported for TmNi$_2$B$_2$C (DeBeer-Schmitt et al., 2007) and later observed in a more extreme form in CeCoIn$_5$ (Bianchi et al., 2008; White et al., 2010). Results for the latter are shown in Fig. 39(b). At all temper-
atures, the form factor initially increases before reaching a maximum and then decreasing on approaching the upper critical field, $H_{c2}$. Below 500 mK $F(q)$ remains finite up to $H_{c2}$ above which it vanishes abruptly, consistent with the first-order nature of the superconducting transition in CeCoIn$_5$ at low temperatures. The unusual field dependence of $F(q)$ is due to a strong exchange coupling of the conduction electrons and the TmNi$_2$B$_2$C or CeCoIn sublattice moments (Ichioka et al., 2007). The enhanced Pauli paramagnetic effects lead to a polarization of the unpaired quasiparticle spins in the vortex cores and a spatially varying paramagnetic moment, providing an additional contribution to the field variation in the mixed state. Approaching $H_{c2}$, the paramagnetic depairing causes the vortex cores to expand, making them more isotropic. For $H \parallel c$ this suppresses the fourfold anisotropy in the screening current plane, leading to a more isotropic. For $H \perp c$ the re-entrance of the the square VL phase (Bianchi et al., 2008; Das et al., 2012a; Eskildsen et al., 1998)

The temperature dependence of the VL form factor reflects the structure of the superconducting gap, $\Delta$. Specifically, $F(q)$ is proportional to the superfluid density $\rho_s \propto \lambda^{-2}$ given by

$$\rho_s(t) = 1 - \frac{1}{4\pi t} \int_0^{2\pi} \int_0^\infty \cosh^{-2} \left( \frac{\sqrt{\epsilon^2 + \Delta^2(t, \phi)}}{2t} \right) \, d\phi \, d\epsilon,$$

where $t = T/T_C$ is the normalized temperature (Eskildsen et al., 2011; Prozorov and Giannetta, 2006). Here, $T_C$ is the superconducting critical temperature and the gap is in units of $k_B T_C$. The latter can be separated into temperature- and momentum-dependent parts $\Delta(t, \phi) = \Delta_0(t) \Delta_k(\phi)$. The $T$-dependence is given by the weak coupling expression

$$\Delta_0(t) = \Delta_0(0) \tanh \left( \frac{\pi}{\Delta_0(0)} \sqrt{1 - \frac{1}{t}} \right),$$

where $\Delta_0(0)$ is the zero-temperature amplitude (Gross et al., 1986). With increasing temperature $F(q)$ decreases due to thermal excitation of quasiparticles across the gap, and eventually vanishes at $T_C$. For $t \lesssim \frac{1}{2}$ the temperature dependence is dominated by the lowest values of the gap. If $\Delta_k(\phi)$ is large over the whole Fermi surface, few quasiparticles are excited and the scattered intensity $I \propto |F(q)|^2$ is nearly constant, Fig. 40(b). In contrast, for a nodal gap the intensity will vary with temperature even as $t \to 0$, in a manner that depends on the location and dispersion of the nodes. The coherence length changes little for $t \ll 1$ and is often ignored in the temperature dependence, although it would be straightforward to include since within the BCS theory $\xi \propto \Delta_0(t)^{-1}$.

Figure 40(a) shows the VL scattered intensity as a function of temperature in the iron-based superconductor KFe$_2$As$_2$ (Kawano-Furukawa et al., 2011). Here, the intensity has a strong $T$-dependence down to the lowest measured temperatures, indicating a range of gaps that extend down to zero or near-zero values. The best fit to the data is from a single nodal gap coupled with nonlocal effects shown in Fig. 40(c) or to a three-gap model with a small lowest $\Delta$, Fig. 40(d) (Kawano-Furukawa et al., 2011). The nonlocal corrections become important for nodal gaps where the London assumption of a vanishing ratio of $\xi/\lambda$ breaks down. From the temperature dependence alone it is not possible to discriminate between the two scenarios, but, as shall be discussed later, there is strong evidence for multiband superconductivity in KFe$_2$As$_2$. Similar studies were recently used to provide information about the nodal structure of the unconventional superconductor UPt$_3$ (Gannon et al., 2015). Here measurements with different directions of the applied field were compared to models of both the location and dispersion of the gap nodes.

3. Complex order parameters, multigap superconductivity and Pauli limiting

The ability to probe within the mixed (vortex) state allows VL SANS measurements to extract subtle features about the superconducting state.

With three distinct superconducting phases (denoted A, B and C) the heavy-fermion UPt$_3$ is a paradigm for unconventional superconductivity (Joynt and Taillefer, 2002). This material has a hexagonal crystal structure but, within the $E_{2u}$ model, a nodal gap with a fourfold symmetry in the superconducting A and C phases. The
lower rotational symmetry of the gap gives rise to a rotation of the triangular VL, Fig. 41(a) (Huxley et al., 2000). These measurements focused on the A phase which exists at low fields in a narrow temperature range just below $T_C$. SANS studies of UPT$_3$ are challenging due to the large penetration depth, and measurements within the A phase itself are not feasible due to the further loss of intensity when approaching $T_C$. Instead, a quenching technique was applied, where an equilibrium VL is obtained by a field oscillation at $T_Q$ and then cooled to the lower measurement temperature. For UPT$_3$ thermal effects are sufficiently weak that the VL remains in the configuration obtained at the quench temperature, $T_Q$. The diffraction patterns in Fig. 41(a) show how triangular VL domains, oriented with Bragg peaks along the crystalline a axis at low temperature, rotate by $\pm15^\circ$ as the A phase is approached. This allowed a determination of the A phase nodal structure, with antinodes along the a and a$^*$ crystalline axes (Champel and Mineev, 2001; Huxley et al., 2000).

Rotating the applied field away from the principal axis in a uniaxial superconductor introduces a twofold anisotropy, causing a distortion of the VL and Bragg peaks that lie on an ellipse rather than a circle, as shown in the inset to Fig. 41(b) (Campbell et al., 1988; Das et al., 2012c; Gammel et al., 1994; Keimer et al., 1993; Yethiraj et al., 1993a). The VL anisotropy $\Gamma_{VL}$ is given by the major-to-minor axis ratio, and due to flux quantization it is sufficient to measure the location of the VL Bragg peaks along the minor ellipse axis (Kuhn et al., 2017; Rastovski et al., 2013a). For fields parallel to the basal plane $\Gamma_{VL} = \Gamma_{ac}$. Here, $\Gamma_{ac}$ mainly reflects the anisotropy of the Fermi velocity in the ac plane, and in general one expects $\Gamma_{ac} = \lambda_c/\lambda_{ab}$ (Kawano-Furukawa et al., 2013). Even in cases where measurements with an exact in-plane field are not possible, $\Gamma_{ac}$ can be obtained by extrapolation. The ability to measure $\Gamma_{ac}$ is especially useful in materials where this is not equal to the upper critical field anisotropy because $H_{c2}$ is Pauli limited along one or more crystalline directions.

In materials where the superconductivity resides on multiple Fermi surface sheets with different anisotropies one often finds a field and/or temperature dependence of $\Gamma_{ac}$. This was first reported for MgB$_2$ and attributed to a more rapid suppression of the superconductivity on the Fermi surface sheets with the smaller gap (Cubitt et al., 2003). Figure 41(b) shows measurements of $\Gamma_{VL}$ in KFe$_2$As$_2$ for two different angles close to the basal plane (Kuhn et al., 2016). This shows a clear field dependence, indicative of multigap superconductivity in this material. Moreover, $\Gamma_{VL}$ exceeds the upper critical field anisotropy of $\sim 3.3$ and thus shows Pauli limiting for in-plane fields.

In highly anisotropic superconductors spin-flip scattering, due to the transverse VL field modulation, can greatly exceed the non-spin-flip scattering usually used to image the VL (Amano et al., 2014; Kealey et al., 2001; Kuhn et al., 2016; Thiemann et al., 1989). Since the spin flip scattering vanishes for fields within the basal plane, this also allows a precise in situ alignment of the magnetic field. The spin flip scattering was utilized in SANS studies of the anisotropy in Sr$_2$RuO$_4$ which would otherwise not have been possible, Fig. 41(c) (Nakai and Machida, 2015; Rastovski et al., 2013a). This found a $\Gamma_{ac} \approx 60$, greatly exceeding that of the upper critical field (20) indicating Pauli limiting similar to the situation in KFe$_2$As$_2$. This raises questions concerning the order parameter in Sr$_2$RuO$_4$ which is considered to be a p-wave superconductor with equal-spin pairing where Pauli limiting is not expected to occur (Maeno et al., 2012).

C. Vortex matter studies

While the repulsive vortex-vortex interaction favors the formation of a well-ordered VL, thermal effects and/or pinning to imperfections can lead to disordering. The balance between these competing factors determines both the structural and dynamic properties of vortex matter which is interesting in its own right, and as a model system for the behavior of soft matter in general.

1. Structural properties and correlations

Obtaining an ordered VL with a well-defined diffraction pattern often requires “annealing” to remove disorder that may have been frozen in during a field cooling procedure. In materials with weak pinning this can be achieved by applying a transport current (Pautrat et al., 2005; Yaron et al., 1994, 1995) or by a damped, small-amplitude AC field to “shake” the vortices into their equilibrium positions (Levett et al., 2002). In contrast, it is often not possible to achieve an ordered VL in superconductors with strong pinning, and a shaking can even lead to a further disordering of the VL. This is seen in members of the iron-based superconductors where a ring of scattering from a VL powder is observed (Eskildsen et al., 2009; Inosov et al., 2010a,b).

For a well-ordered VL the correlation lengths are determined from SANS, being inversely proportional to the width of the Bragg reflections in reciprocal space (Yaron et al., 1995). A measure of the longitudinal correlation length (vortex “straightness”) is obtained from rocking curves (RC) of the scattered intensity as the VL peaks are rotated through the Bragg condition, shown in Fig. 43(b,c). While the poor resolution in the detector plane makes it difficult to determine the positional order directly, this can be estimated from the RC width, as the longitudinal, translational and orientational correlation lengths are related by the VL elastic constants (Brandt,
A fundamental question is the exact structural characteristics of a dislocation free VL. For any amount of disorder the positional order will decay over some characteristic length scale, where the vortex displacement becomes of the order of the VL spacing. In the so-called Bragg glass model, where the VL is treated as an elastic manifold, the positional order has a weak, power-law decay, giving rise to rocking curves with a decreasing amplitude but no broadening (Cubitt et al., 1993; Klein et al., 2001). This is supported by the measurements in Fig. 42(a) showing a decay positional correlation, i.e. of the integrated intensity multiplied by $q/F(q)^2$, which, from Eq. (65), is expected to be constant. More recent SANS measurements used a transverse magnetic field, where the RC width probes the positional order, coupled with a reverse Monte Carlo analysis of the data (Laver et al., 2008). This found some evidence for a Bragg glass at large length scales, but also a fracturing shown in Fig. 42(b) which leads to RC broadening. Subsequent studies using a time-of-flight approach to SANS yielded consistent results (Pautrat et al., 2012).

2. Relating structural and dynamic properties

Vortex matter exhibits a rich and complex dynamic behavior (Blatter et al., 1994; Brandt, 1995). This includes an increased de-pinning critical current in a region below the upper critical field, known as the peak effect (PE), and VL melting, where SANS can provide complementary structural information. The first report of VL melting observed by SANS was in the high-temperature superconductor Bi$_2$SrCaCuO$_8$+, where an abruptly vanishing intensity was observed at temperatures much less than $T_C$ (Cubitt et al., 1993). Reports of melting, based on the observation of a VL disordering, were also made for niobium where effects of thermal fluctuations are expected to be weak. However, subsequent measurements on ultrapure samples contradicted these findings, revealing an ordered VL in very close proximity to $H_{c2}$ (Bowell et al., 2010; Forgan et al., 2002). The difference may be attributed to differences in sample quality, demonstrating the challenge in eliminating extrinsic effects.

In the case of the PE, the increase in the critical current is commonly associated with a softening of the VL and an ensuing order-disorder transition, which better allows it to conform with the pinning landscape (Gammel et al., 1998; Joumard et al., 1999). As already discussed, energy barriers may trap the VL in disordered metastable configurations when cooling from the normal state in a constant field. In addition to a transport current or an AC field, an ordering of the VL can in some cases be achieved by a simple thermal cycling (Daniilidis et al., 2007; Marziali Bermudez et al., 2017). Conversely, the application of a field oscillation will introduce an edge “contamination” of the VL at surfaces where vortices enter and leave the sample (Hanson et al., 2011).
associating the PE with a bulk order-disorder transition is widely accepted, this is not universal. This is due to some observations of an ordered VL above the PE (Pautrat et al., 2009, 2007) and studies which indicate that this is caused by surface rather than bulk pinning (Pautrat et al., 2012). Here, one should note that few SANS measurements are performed under the same exact conditions (transport currents, AC fields) as those used to determine the PE. This emphasizes the need for simultaneous transport measurements when drawing connections between the structural and dynamic properties of vortex matter. Recent measurements combined SANS with in situ linear ac susceptibility measurements as shown in Fig. 43 (Marziali Bermudez et al., 2015). These studies explored the behavior in the complex transitional region adjacent to the PE, where pinning can both decrease or increase by the application of transport currents or AC magnetic fields.

3. Ordered, nonequilibrium VL phases

A final example of complex vortex matter behavior is the recent discovery of well-ordered, metastable VL phases (Das et al., 2012b). In MgB$_2$ the triangular VL undergoes a $30^\circ$ continuous rotation transition, similar to the one observed in UPt$_3$ and shown in Fig. 41(a), but here attributed to a Fermi surface anisotropy coupled with the two-gap nature of the superconducting state (Cubitt et al., 2003; Hirano et al., 2013). The diffraction patterns in Fig. 44 show examples of two of the phases in this material: A VL aligned with Bragg peaks along the crystalline a axis (F phase), and rotated away from a resulting in two degenerate domain orientations (L phase). Cooling or warming across the equilibrium F-L phase transition leaves the VL in robust metastable states, and an external perturbation such as a field oscillation is required to drive it to the ground state, as shown in Fig. 44. The metastability is notably not due to pinning (Rastovski et al., 2013b), but represents a novel collective vortex phenomenon most likely stabilized by VL domain boundaries. More recently, a similar phenomena was observed for skyrmion lattices where different phases could be stabilized depending on the temperature- and field-history (Makino et al., 2017). Looking ahead, the VL in MgB$_2$ could possibly be used as a model system for nonequilibrium phase transitions.

X. CONCLUDING REMARKS

We have presented recent applications of magnetic SANS to study a wide range of magnetic materials, such as soft magnetic Fe-based nanocomposites, hard magnetic Nd–Fe–B-based permanent magnets, magnetic steels, nanoparticles and ferrofluids, magnetic oxides and complex alloys, skyrmion lattices (SkL), spiral magnetic structures, and vortex lattices (VL) in type-II superconductors. The final paragraphs are devoted to a brief discussion of future developments, related to scientific grand challenges, theoretical and simulation work, as well as to SANS instrumentation.

Whilst for bulk ferromagnets the theoretical framework of magnetic SANS has recently been developed (Metlov and Michels, 2015, 2016; Mettus and Michels, 2015; Michels, 2014; Michels et al., 2016; Michels and Weissmüller, 2008), the theoretical description of magnetic SANS for magnetic nanoparticles in a nonmagnetic matrix is still in its infancy. This comprises the description of spin arrangements in particles beyond the
single-domain (supermoment) approximation and interparticle correlations in dense interacting particle systems. Future research will therefore concentrate on the analytical and/or numerical micromagnetic computation of the magnetic SANS cross section of a dispersion of magnetic nanoparticles in a nonmagnetic matrix—the classical prototypical sample microstructure in many magnetic SANS experiments. There is ample theoretical (e.g., Berger et al., 2008; Gatel et al., 2015; and Vivas et al., 2017) as well as experimental (e.g., Disc et al., 2012; Günther et al., 2014; and Krycka et al., 2014) evidence that nanosized magnetic particles are not homogeneously magnetized, and the question thus arises whether the standard expression for the cross section, Eq. (23), is still adequate to describe magnetic SANS. Intraparticle spin disorder may be due to the interplay between different magnetic interactions such as surface anisotropy and dipolar interaction, deviations from ellipsoidal particle geometry, or to the presence of crystal defects and antiphase boundaries (Nedelkoski et al., 2017). Heterogeneous particles offer an additional degree of freedom to tailor magnetic properties; for instance, in multiphase core/shell particles, the spin structure may show varying magnetic properties between phases (e.g., AFM core/FM shell). For nanoparticles, boundary conditions for the magnetization at internal and particle-matrix interfaces have to be taken into account, a task which, from the micromagnetic point of view, severely complicates the problem. Nucleation theory (Aharoni, 1996) may provide a guideline for attacking this problem.

For hard magnetic materials, magnetic SANS has allowed a novel perspective where a qualitative examination has been complemented by a semi-quantitative analysis. Although fundamental topics are expected to be further explored, applied research will keep gaining momentum. SANS data suggested that the grain-boundary phase in sintered Nd–Fe–B-based magnets is the main source of spin misalignment within the hard magnetic phase, and whose dimensions correlate with the defect size as observed by SANS. In fact, SANS is still expected to be the only technique to monitor the bulk of the (magnetic) microstructure, relating coercivity and defect dimensions in sintered magnets. Similarly, Nd–Fe–B nanocomposites will increasingly rely on SANS with the aim to address simultaneously composition, processing conditions, and magnetic properties. Future activities in the topic will embrace other engineering magnetic materials, likely with emphasis on Sm–Co compounds (possibly using Sm isotope(s) due to its large neutron absorption) in view of their importance for high-temperature applications combined with a microstructure that resembles the one of Nd–Fe–B counterparts; besides, the magnetic microstructure of ferrites, Alnico, or manganese compounds have not yet been studied using SANS.

SANS investigations on steels take advantage of a perfect correspondence between the size scale most relevant for the mechanical properties of advanced steels and the sensitivity range of SANS. The reported applications of magnetic SANS in the field of ferromagnetic steels indicate the need of an improved understanding of the magnetic properties of nanostructures such as oxide nanoparticles or irradiation-induced solute atom clusters. This will be achieved by refining models of magnetism in complex systems at the nm-scale and by using SANS to calibrate the models. The consideration of both magnetic and nuclear SANS is critical for the separation of magnetic and morphological aspects of the nanostructures and may require magnetic-field-dependent SANS measurements.

Numerical micromagnetic computations can take into account the full nonlinearity of Brown’s static equations of micromagnetics and, indeed, have provided fundamental insights into magnetic SANS. A particular advantage of micromagnetic simulations resides in their flexibility regarding microstructure variations (particle-size distribution, texture, magnetic materials parameters, etc.); it is also rather straightforward to “switch on” and “off” certain magnetic interactions in the simulations and to test in this way their impact on the neutron scattering. In view of the continuously increasing power of modern computers, a further understanding of magnetic neutron scattering may also be expected from the development of efficient micromagnetic algorithms, so that true macroscopic samples, with dimensions of the order of several microns, can be simulated. In general, we believe that the combination of experimental scattering data with large-scale numerical computations will become more and more important. Future simulation work addresses the inclusion of nuclear scattering into the micromagnetic approach, which would allow for the computation of polarization-dependent nuclear-magnetic interference terms, yielding a more complete description of the SANS cross section.

Concerning the study of complex systems, it is clear that SANS has played, and continues to play, an important role in the elucidation of nano and microscale magnetic inhomogeneity in complex materials systems. SANS is in fact ideally suited to obtaining bulk-averaged information on magnetic inhomogeneity in such materials, and has played a key role in the realization that magnetic and electronic inhomogeneity is widespread across numerous materials classes. Future application of the technique to new materials where such issues are emerging will hopefully take place, including topological magnetic materials, doped magnetic Mott insulators, heavy fermion systems, or strong spin-orbit oxides such as iridates.

The identification of further compounds with skyrmionic structures and the diverse mechanisms of their stabilization, often connected to topological decay and emergence is expected to be of high relevance for SANS. Not only the manipulation of SkL phases, but also of noncollinear magnetic structures, by means of electrical currents and fields, microwave radiation, and
resulting nonequilibrium physics will be a focus of research, in line with their application in future logical or storage devices; here, multiferroic properties are of particular interest. Apart from SANS, skyrmionic structures in thin films and nanostructured patterns will continue to be of high importance. Despite intense theoretical effort (e.g., Barker and Tretiakov, 2016; Göbel et al., 2017; and Zhang et al., 2016), the experimental proof for AF skyrmion states is still lacking. Here, key complications are (i) the unambiguous detection of topological properties by means of Hall measurements, which are impossible in the mostly insulating compounds. In particular, recent theoretical work indicates that AF skyrmions only show a topological spin Hall effect (Barker and Tretiakov, 2016). (ii) The high momentum transfer of AF-based SkL structures imposes severe limitations to diffraction measurements, in particular for SANS. Also the search for skyrmions in magnetically frustrated centrosymmetric materials with high lattice symmetry has recently been in the focus of intense theoretical effort (Hayami et al., 2017; Leonov and Motovoy, 2015; Okubo et al., 2012; Wang et al., 2015). Here, experimental proof is still scarce.

While SANS has been highly successful for VL studies, important unresolved scientific problems remain, which will require further developments of the technique before they can be addressed. Prominent examples include the prospects for direct imaging of exotic vortex phases, such a multiply-quantized vortices in bulk superconductors or the inhomogeneous Fulde-Ferrell-Larkin-Ovchinnikov state. Such phases often exist in narrow regions of the phase diagram which may be difficult to reach experimentally, suffer from weak scattering from the VL, or be complicated by the need for an unconventional scattering geometry. Attempts to resolve these exotic states may require or be aided by a combination of SANS with techniques that allow an in situ tuning of the superconducting state itself. Importantly, this must extend beyond electric currents and DC or AC magnetic fields, which typically only affect the VL. Of particular interest are the prospects of performing VL studies under uniaxial strain or hydrostatic pressure. Studies of vortex matter have recently been advanced by the use of stroboscopic SANS, or skyrmion lattices (Mühlbauer et al., 2016), or systems out of equilibrium. Regarding theoretical work, the analytical and numerical extension of the present static micromagnetic approach to include magnetization dynamics (Landau-Lifshitz-Gilbert equation) represents a major challenge.

In contrast to kinetic studies of driven dynamics with stroboscopic techniques such as TISANE, the spin-echo variant MIEZE (Gähler et al., 1992; Hank et al., 1997) allows the combination of the superior q-resolution of SANS and the extreme energy resolution of spin echo, in particular allowing for (ferro)magnetic samples and samples in magnetic fields. MIEZE will open the pathways to study entirely new physics, e.g., FM fluctuations at phase transitions, quantum phase transitions and frustrated spin liquids, or the melting of SkL and vortex lattices (e.g., Haslbeck et al., 2018; Kindervater et al., 2017; and Martin, 2018).

In view of the upcoming SANS instrumentation at the European Spallation Source (ESS) (Jaksch et al., 2014), it will become possible to study ever smaller samples at extreme environments (temperature, field, pressure). In terms of neutron flux at ESS, the SANS instrument SKADI is estimated to provide a flux of \( \approx 5 \times 10^9 \text{cm}^{-2}\text{s}^{-1} \) at 8 m collimation, which is about a factor of five larger than the one of the currently best-performing instrument in the world, the D22 at the ILL (Jaksch, 2018; Jaksch et al., 2014). Moreover, the increased dynamical range will facilitate in situ and \textit{in operando} measurements of irreversible or kinetic processes.

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