Colloquium: Zoo of quantum-topological phases of matter

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Zoo of quantum-topological phases of matter

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What are topological phases of matter? First, they are phases of matter at zero temperature. Second, they have a non-zero energy gap for the excitations above the ground state. Third, they are disordered liquids that seem have no feature. But those disordered liquids actually can have rich patterns of many-body entanglement representing new kinds of order. This paper will give a simple introduction and a brief survey of topological phases of matter. We will first discuss topological phases that have topological order (*i.e.* with long range entanglement). Then we will cover topological phases that have no topological order (*i.e.* with only short-range entanglement).

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I. ORDERS AND SYMMETRIES

Condensed matter physics is a branch of science that study various properties of all kinds of materials. Usually for each kind of materials, we need a different theory (or model) to explain its properties. After seeing many different type of theories/models for condensed matter systems, a common theme among those theories start to emerge. The common theme is the *principle of emergence*, which states that the properties of a material are mainly determined by how particles are organized in the material. This is quite different from the point of view that the properties of a material should be determined by the components that form the material. In fact, all the materials are made of same three components: electrons, protons and neutrons. So we cannot use the richness of the components to understand the richness of the materials. The various properties of different materials originate from various ways in which the particles are organized. The organizations of the particles are called orders. Different orders lead to different phases of matter, which in turn leads to different properties of materials.

Therefore, according to the principle of emergence, the key to understand a material is to understand how electrons, protons and neutrons are organized in the material. Based on a deep insight into phase and phase transition, Landau (1937) developed a general theory of orders as well as transitions between different phases of matter. He pointed out that the reason that different phases (or orders) are different is because they have different symmetries. A phase transition is simply a transition that changes the symmetry. Introducing order parameters that transform non-trivially under the symmetry transformations, Ginzburg and Landau (1950) developed the standard theory for phases and phase transitions, where different phases of matter are classified by a pair of groups ($G_\Psi \subset G_H$). Here G_H is the symmetry group of the system and G_Ψ the unbroken symmetry group of the equilibrium state.

Landau's theory is very successful. Using symmetry and the related group theory, we can classify all of the 230 different kinds of crystals that can exist in three dimensions. By determining how symmetry changes across a continuous phase transition, we can obtain the critical properties of the phase transition. The symmetry breaking also provides the origin of many gapless excitations, such as phonons, spin waves, etc., which determine the low-energy properties of many systems (Goldstone, 1961; Nambu, 1960). Many of the properties of those excitations, including their gaplessness, are directly determined by the symmetry.

As Landau's symmetry-breaking theory has such a broad and fundamental impact on our understanding of matter, it became a corner-stone of condensed matter theory. The picture painted by Landau's theory is so satisfactory that one starts to have a feeling that we understand, at least in principle, all kinds of orders that matter can have. One feels that we start to see the beginning of the end of condensed matter theory.

II. NEW WORLD OF CONDENSED MATTER PHYSICS

However, through the researches in last 30 years, a different picture starts to emerge. It appears that what we have seen is just the end of beginning. There is a whole new world ahead of us waiting to be explored. A peek into the new world is offered by the discovery of fractional quantum Hall (FQH) effect (Tsui *et al.*, 1982). Another peek is offered by the discovery of high T_c superconductors (Bednorz and Mueller, 1986). Both phenomena are completely beyond the paradigm of Landau's symmetry breaking theory. Rapid and exciting developments in FQH effect and in high T_c superconductivity resulted in many new ideas and new concepts. Looking back at those new developments, it becomes more and more clear that, in last 30 years, we were actually witnessing an emergence of a new theme in condensed matter physics. The new theme is associated with new kinds of orders, new states of matter and new class of materials beyond Landau's symmetry breaking theory. This is an exciting time for condensed matter physics. The new paradigm may even have an impact in our understanding of fundamental questions of nature – the emergence of elementary particles and the four fundamental interactions, which leads to an unification of matter and quantum information.[1]

The emergent new field of quantum-topological matter has developed very fast. Many new terms are introduced, but some of them can be very confusing:

Pt.1: Some *Haldane phases* are *topological*, while some other Haldane phases are not topological. Although, the Haldane phase for spin-1 chain is topological, it is actually a *product state* with no *topological order*.

Pt.2: *Topological insulators* and *topological superconductors* (*i.e.* with $T^2 = (-)^{N_F}$ time-reversal symmetry and weak interactions) has no *topological order*. It is wrong to characterize topological insulators as insulators with conducting surface.

Pt.3: What is the difference between *quantum spin Hall state* and *spin quantum Hall state*? Are they *topological insulator*?

Pt.4: "*SPT state*" is the abbreviation for both *symmetry protected trivial state* and *symmetry protected topological state*. The two mean the same.

Pt.5: 3+1D *textbook s-wave superconductors* have no topological order, while 3+1D *real-life s-wave superconductors* have a *Z_2-topological order*.

Pt.6: 2+1D *$p+$ip fermion paired state* and *integer quantum Hall states* (IQH) do not have any fractionalized topological excitations. Some people regard them as *long-range entangled* (*i.e. topologically ordered*) while others regard them as *short-range entangled*.

Pt.7: What are the difference between *Chern insulator*, *quantum anomalous Hall state*, and *integer quantum Hall state*? What are the difference between *fractionalized topological insulator* and *topological order*?

Pt.8: There is a very active search for *Majorana fermions* with *non-abelian statistics*. But should Majorana *fermion* be a fermion that carries Fermi statistics? Is Majorana fermion the *Bogoliubov quasiparticle* in a superconductor?

In this paper, we will clarify those notions.

III. TOPOLOGICALLY ORDERED PHASES

A. Chiral spin liquids and topological order

After the discovery of high T_c superconductors in 1986 by Bednorz and Mueller (1986), some theorists believed that quantum spin liquids play a key role in understanding high T_c superconductors (Anderson, 1987). This is because spin liquid can leads to a so called spin-charge separation: an electron disintegrates into two quasiparticles – a spinon (spin-1/2 charge-0) and a holon (spin-0 charge-e). Since holon is not fermion, its condensation can leads to superconductivity - a novel mechanism of high T_c superconductors. Thus many people started to construct and study various spin liquids.[2]

However, despite the success of Landau symmetry-breaking theory in describing all kind of states, the theory cannot explain and does not even allow the existence of spin liquids (with odd number of electrons per unit cell).

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[1] See Foerster *et al.* (1980) and Baskaran and Anderson (1988) for emergence of gauge interactions, Levin and Wen (2006b); Wen (2002a, 2003) for unification of gauge interactions and Fermi statistics, and Wen (2013b); You *et al.* (2014); and You and Xu (2015) for emergence of chiral fermions.

[2] See Affleck and Marston (1988); Affleck *et al.* (1988b); Baskaran *et al.* (1987); Dagotto *et al.* (1988); and Rokhsar and Kivelson (1988)

This leads many theorists to doubt the very existence of spin liquids. In early proposals of spin liquid, the spinons are gapless and are confined at long distance by the emergent gauge field (Baskaran and Anderson, 1988), adding support to the opinion that the spin liquid is just a fiction and does not actually exist.[3]

In 1987, Kalmeyer and Laughlin (1987) introduced a special kind of spin liquids – chiral spin liquid – in an attempt to explain high temperature superconductivity. In contrast to many other proposed spin liquids at that time, the chiral spin liquid was shown to have deconfined spinons (as well as deconfined holons) and correspond to a stable zero-temperature phase.[4] At first, not believing Landau symmetry-breaking theory fails to describe spin liquids, people still wanted to use symmetry breaking to characterize the chiral spin liquid. They identified the chiral spin liquid as a state that breaks the time reversal and parity symmetries, but not the spin rotation and translation symmetries (Wen *et al.*, 1989). The chiral spin liquid is also characterized by its perfect heat conducting edge and quantized spin-Hall conductance.

However, Wen (1989) quickly realized that there are many different chiral spin liquids (with different spinon statistics and spin-Hall conductances) that have exactly the same symmetry. So symmetry alone is not enough to characterize different chiral spin liquids. This means that the chiral spin liquids contain a new kind of order that is beyond symmetry description . This new kind of order was named **topological order**.

Just like any concepts in physics, the concept of topological order is also required to be defined via measurable quantities, which are called **topological invariants**. The first discovered topological invariants (Wen, 1990b) that define topological order were (1) the *robust ground state* degeneracy on torus and other closed space manifolds (*i.e.* with no boundary), (2) the non-abelian geometric phases (the *modular matrices*) of the degenerate ground states, (3) the *chiral central charge* c of the edge states.[5] It was conjectured that those macroscopic topological invariants, or more generally, "the total gauge structures (the Abelian one plus the non-Abelian one) on the moduli spaces of the models defined on generic Riemann surfaces Σ_g completely characterize (or classify) the topological orders in 1+2 dimensions" (Wen, 1990b).

Microscopically, topological order is a property of a local quantum system whose total Hilbert space have a tensor product decomposition $\mathcal{H}^{\text{tot}} = \bigotimes_i \mathcal{H}_i$, where $\mathcal{H}_i$ is the Hilbert space on each *site*. Such a tensor product decomposition is a part of the definition of a local system, which also satisfies the condition of short-range interaction between *sites*. Relative to such a tensor product decomposition, a *product state* is defined to be a state of the form $|\Psi\rangle = \bigotimes_i |\Psi_i\rangle$, where $|\Psi_i\rangle \in \mathcal{H}_i$. In this paper, only the tensor products of on-site states, $|\Psi_i\rangle$, are called product states. With such a definition of local quantum systems, *topological order is defined to describe gapped quantum-liquids*[6] *that cannot be deformed into a product state without gap-closing phase transitions*. Such quantum liquids are said to have **long-range entanglement** (Chen *et al.*, 2010; Kitaev and Preskill, 2006; Levin and Wen, 2006a). Long-range entanglement is the microscopic origin of topological order. A gapped state that can be deformed into a product state smoothly is **short-range entangled** and has no topological order. In particular, a product state has no topological order.

One may wonder: why do we need such a complicated way to characterize topological order. Is the quantized Hall conductance a more direct and simpler way to characterize topological order, at least for quantum Hall states (see Sec. III.B)? In fact, quantized Hall conductance is due to a combined effect of $U(1)$ symmetry (*i.e.* particle-number conservation) and topological order (*i.e.* long-range entanglement). If we break the $U(1)$ symmetry, quantum Hall states still have topological order, even though the Hall conductance is no longer well defined. How to characterize topological order in such a situation? The above characterization based on ground state degeneracy and non-abelian geometric phases does not require symmetries and provides *a complete characterize of topological orders in 2-dimensions*.

We like to mention that the term "topological" in topological order and in topological insulators/superconductors has totally different meanings. In topological order, the term is motivated by the low energy effective theory of the chiral spin liquids, which is a $U(1)$ Chern-Simons theory – a topological quantum field theory (Witten, 1989). Here, "topological" really means long-range entangled, which is a property of many-body wave functions. We may call it **quantum topology**. While in topological insulators/superconductors,

[3] Now we realized that even those gapless spin liquid can exist as algebraic spin liquid without quasiparticles (Chung *et al.*, 2001; Fradkin *et al.*, 2003; Hermele *et al.*, 2004; Rantner and Wen, 2001, 2002; Senthil *et al.*, 2004).

[4] Recently, chiral spin liquid was shown to exist in Heisenberg model on Kagome lattice with J_1-J_2-J_3 coupling (Gong *et al.*, 2015; He and Chen, 2015).

[5] The central charge c of the edge states is related to a gravitational response of the system described by a gravitational Chern-Simons 3-form ω_3: $\mathcal{L} = \frac{2\pi c}{24}\omega_3$, where $d\omega_3 = p_1$ is the first Pontryagin class (Abanov and Gromov, 2014; Bradlyn and Read, 2015; Gromov *et al.*, 2015). c can be measured via the thermal Hall conductivity $K_H = c\frac{\pi k_B^2}{6\hbar}T$ (Kane and Fisher, 1997).

[6] Zeng and Wen (2015) and Swingle and McGreevy (2016) introduced the notion of *gapped quantum-liquids* to describe a simple kind of gapped states: the states that can enlarge themselves by dissolving product states. Only gapped quantum-liquids have quantum field theory descriptions at long distances. 3D gapped states obtained by stacking 2D quantum Hall states and Haah (2011) cubic code are examples of gapped non-quantum-liquids.

the term corresponds to **classical topology** which is a property of continuous manifold, related to the difference between sphere and torus. The vortex in superfluid, the Chern number, and the Z_2 index in topological insulators/superconductors belong to classical topology, which represent a very different phenomenon. In fact, "topological" in topological insulators/superconductors really means "symmetry protected" (see Sec. IV).

B. Quantum Hall states

However, soon after the proposal of chiral spin liquid, experiments indicated that high-temperature superconductors do not break the time reversal and parity symmetries and chiral spin liquids do not describe high-temperature superconductors (Lawrence *et al.*, 1992). Thus the concept of topological order became a concept with no experimental realization.

But long before the discovery of high T_c superconductors, Tsui *et al.* (1982) discovered FQH effect, such as the filling fraction $\nu = 1/m$ Laughlin (1983) state

$$\Psi_{\nu=1/m}(\{z_i\}) = \prod (z_i - z_j)^m \, \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2} \tag{1}$$

where $z_i = x_i + \mathrm{i} y_i$. People realized that the FQH states are new states of matter. At first, influenced by the previous success of Landau's symmetry breaking theory, people used order parameters and long range correlations to describe the FQH states (Girvin and MacDonald, 1987; Read, 1989; Zhang *et al.*, 1989), which result in the Ginzburg-Landau-Chern-Simons effective theory of quantum Hall states. But in quantum Hall states, there is no off-diagonal long range order in any local operators, and thinking about it can mislead some people to wrong directions, such as looking for Josephson effect in quantum Hall states.

If we concentrate on physical measurable quantities, we will see that all those different FQH states have exactly the same symmetry and conclude that we cannot use Landau symmetry-breaking theory and local order parameters to describe different orders in FQH states. In fact, just like chiral spin liquids, FQH states also contain a new kind of orders beyond Landau's symmetry breaking theory. Different FQH states are also described by different topological orders (Wen and Niu, 1990). The better way to see the essence of FQH states is via topological invariants such as robust ground state degeneracy and modular matrices, as well as the non-trivial edge states (Halperin, 1982; Wen, 1990a). Thus the concept of topological order does have experimental realizations in FQH systems.

One of the most striking properties of FQH states is their fractionalized excitations, that can carry fractional statistics (Arovas *et al.*, 1984; Halperin, 1984)[7] and, if particle number conserves, fractional charges (Laughlin, 1983; Tsui *et al.*, 1982)[8].

We know that a point-like excitation above the ground state is something that can be trapped by a local change of the Hamiltonian near a spatial point $\boldsymbol{x}$. But some times, the local change of the ground state near $\boldsymbol{x}$ cannot be created by local operators. In this case, we refer the corresponding local change of the ground state as a **topological excitations**. It is those topological excitations that can carry fractional statistics/charge.

We note that the presence of any topological excitations imply a presence of topological order in the ground state. But the reverse is not true, the absence of any topological excitations may not imply the absence of topological order in the ground state. In fact, the E_8 bosonic state and the IQH states are states with topological order but no topological excitations.

Regarding to **Pt.6** in Sec. II, some people define those states with no topological excitations as short-range entangled (Kitaev, 2011). However, since those states have non-zero *chiral central charges c* for the edge states, they cannot smoothly change to product state without phase transition. Thus, they are topologically ordered states distinct from the trivial product states. Those topological orders with no topological excitations are called **invertible topological orders** [9], and some people refer them as long-rang entangled (Chen *et al.*, 2010). Regarding to **Pt.7**, *IQH state* (von Klitzing *et al.*, 1980), *Chern insulator* (Hofstadter, 1976; Thouless *et al.*, 1982), *quantum anomalous Hall state* (Haldane, 1988), are just different names for the same fermionic invertible topological order with integer *chiral central charge c*. Also, *fractionalized topological insulator* is same as *topological order*, but may have an additional time reversal symmetry.

C. Non-abelian Quantum Hall states

In addition to the Laughlin states, more exotic non-abelian FQH states were proposed in 1991 by two independent works. Wen (1991b) pointed out that the FQH

[7] The possibility of fractional statistics in 2+1D was pointed out by Leinaas and Myrheim (1977) and Wilczek (1982). The relation to braid group was discussed by Wu (1984).

[8] Fractional charge has been directly observed via quantum shot noise in tunneling current (de Picciotto *et al.*, 1997)

[9] For every invertible topological order $\mathcal{C}$, there exist another topological order $\mathcal{D}$ – the inverse, such that stacking $\mathcal{C}$ and $\mathcal{D}$ on top of each other give us a gapped state that have no topological order, *i.e.* belong to the phase of product states.

states described by wave functions

$$\Psi_{\nu=\frac{n}{m}}(\{z_i\}) = [\chi_n(\{z_i\})]^m,$$
$$\text{or } \Psi_{\nu=\frac{n}{m+n}}(\{z_i\}) = \chi_1(\{z_i\})[\chi_n(\{z_i\})]^m \quad (2)$$

have topological excitations with **non-abelian statistics**[10] of type $SU(n)_m$ (which is denoted as $A(n-1)_m$ in https://www.math.ksu.edu/∼gerald/voas/) (Lan and Wen, 2017). This result was obtained via the low energy $SU(m)_n$ effective Chern-Simons theory of the above states, plus the level-rank duality. Here χ_n is the fermion wave function of n-filled Landau levels. We note that the $\nu = 1/3$ Laughlin state is given by

$$\Psi_{\nu=1/3}(\{z_i\}) = [\chi_1(\{z_i\})]^3. \quad (3)$$

So $[\chi_n(\{z_i\})]^m$ and $\chi_1(\{z_i\})[\chi_n(\{z_i\})]^m$ are generalizations of the Laughlin state (Jain, 1991). They both have non-trivial edge states described by $U(1)\times SU(n)_m$ Kac-Moody current algebra (Blok and Wen, 1992).

In the same year, Moore and Read (1991) proposed that the FQH state described by Pfaffian wave function

$$\Psi_{\nu=1/2} = \mathrm{Pf}\Big[\frac{1}{z_i - z_j}\Big]\mathrm{e}^{-\frac{1}{4}\sum|z_i|^2}\prod(z_i - z_j)^2. \quad (4)$$

has excitations with non-abelian statistics of Ising-type (or $SU(2)_2$-type). Its edge states were studied numerically (Wen, 1993) and were found to be described by a $c = 1$ chiral-boson conformal field theory (CFT) plus a $c = 1/2$ Majorana fermion CFT. Such a result about the edge states supports the proposal that the Pfaffian state is non-abelian, since the edge for abelian FQH states always have integer chiral central charge c. Later, the non-abelian statistics in Pfaffian wave function was also confirmed by its low energy effective $SO(5)$ level 1 Chern-Simons theory (Wen, 1999) (denoted as $B2_1$ in https://www.math.ksu.edu/∼gerald/voas/), as well as a plasma analogue calculation (Bonderson *et al.*, 2011).

It is possible that the $SU(2)_2$-type of non-abelian state is realized by $\nu = 5/2$ fractional quantum Hall samples (Dolev *et al.*, 2008; Radu *et al.*, 2008; Willett *et al.*, 1987).

D. Superconducting states (with dynamical electromagnetism)

It is interesting to point out that long before the discovery of FQH states, Onnes discovered superconductor in 1911 (Onnes, 1911). The Ginzburg-Landau theory for

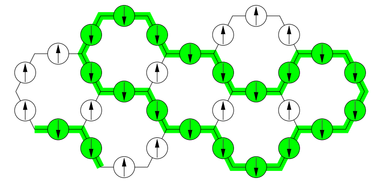


FIG. 1 The strings in a spin-1/2 model. In the background of up-spins, the down-spins form closed strings.

symmetry breaking phases is largely developed to explain superconductivity. However, the superconducting order, that motivates the Ginzburg-Landau theory for symmetry breaking, itself is not a symmetry breaking order. Superconducting order (in real life with dynamical $U(1)$ gauge field) is an order that is beyond Landau symmetry breaking theory. Superconducting order (in real life) is an topological order (or more precisely a Z_2 **topological order** or Z_2 gauge theory) (Hansson *et al.*, 2004; Wen, 1991c). The real-life superconductor has string-like topological excitation that can be trapped by modifying Hamiltonian along a loop. Such a string-like topological excitation is the $\frac{hc}{2e}$-flux loop, since the electromagnetic $U(1)$ gauge field is dynamical. The presence of string-like topological excitation indicate the superconductor has a topological order. The textbook superconductors usually do not contain the dynamical $U(1)$ gauge field, and do not contain string-like topological excitation that can be trapped by modifying Hamiltonian along a loop. This explains **Pt.5** in Sec. II.

It is quite amazing that the experimental discovery of superconducting order did not lead to a theory of topological order. But instead, it led to a theory of symmetry breaking order, that fails to describe superconducting order itself.

E. Z_2-spin liquid in 2+1D

Since chiral spin liquid breaks the time reversal symmetry, while high T_c superconductors do not break the time reversal symmetry. So chiral spin liquid does not appear in high T_c superconductors. This motivated people to look for other spin liquids with deconfined spinons and holons that do not break time reversal symmetry. This leads to the theoretical discovery of 2+1D **Z_2-spin liquid** (Read and Sachdev, 1991; Wen, 1991a) described by effective Z_2 gauge theory (Kogut, 1979) (*i.e.* has a Z_2-topological order). The construction can be easily generalized to obtain 3+1D Z_2-spin liquid, which will have a Z_2 topological order identical to an s-wave superconductor discussed above. Later, an exact soluble toric code model was constructed to realize the Z_2 topological order (Kitaev, 2003). Since then, the Z_2-topological order is also referred as "toric code".

The Z_2-spin liquid of spin-1/2 on Kagome lattice may

[10] Wu (1984) has setup a general theory and braid group for quantum statistics in two dimensions, and Goldin *et al.* (1985) pointed out that such a setup contains non-abelian representations of braid group, which correspond to non-abelian statistics. More complete description of non-abelian statistics are given by Witten (1989) and Kitaev (2006).

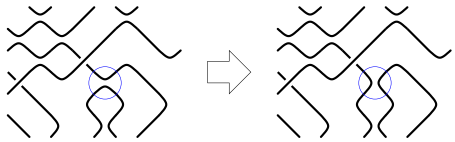


FIG. 2 In string liquid, strings can move freely, including reconnecting the strings.

be realized by Herbertsmithite(Helton *et al.*, 2007), as suggested by recent experiments by Fu *et al.* (2015) and Han *et al.* (2016). The early numerical calculation of Yan *et al.* (2011) suggested the spin-1/2 Heisenberg model on Kagome lattice is gapped, but details of the results are inconsistent with Z_2-topological order, which led people to suspect that the model is gapless. A more recent numerical calculation suggests the model to have a Z_2-spin liquid ground state with long correlation length (10 unit cell length) (Mei *et al.*, 2017), while several other calculations suggest gapless $U(1)$ spin liquid ground states (He *et al.*, 2016; Jiang *et al.*, 2016; Liao *et al.*, 2016). More experimental and theoretical studies are needed to settle the issue.

F. Quantum liquids of non-oriented strings

If we do not require spin rotation symmetry, one can use string liquid to construct a state with Z_2-topological order (Kitaev, 2003). String liquids are long-range entangled (hence topologically ordered). We will see how long-range entanglement in topological order leads to fractional statistics and topological degeneracy.

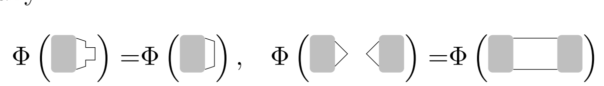
1. Local "dancing" rules in string liquids

Given a spin-1/2 system, if we pick a particular spin-up spin-down configuration, we will get a product state. To construct a highly entangled state, one may consider a equal-weight superposition of all spin-up spin-down configurations. But this does not work. We get a product state with all spins in x-direction. So one idea to get a highly entangled state is to a partial sum. For example, we can view up-spins as background and lines of down-spins as the strings (see Fig. 1). The simplest topologically ordered state in such a spin-1/2 system is given by the equal-weight superposition of all closed strings:(Kitaev, 2003) $|\Phi_{Z_2}\rangle = \sum_{\text{all closed strings}} |\ \rangle$.

To obtain other topological orders, we may consider a different superposition of strings. But those superpositions should all be determined by local rules, so that there is a local Hamiltonian that can produce a given superposition. What are those local rules that give rise to the string liquid $|\Phi_{Z_2}\rangle = \sum_{\text{all closed strings}} |\ \rangle$? The first rule is that, in the ground state, the down-spins are always connected with no open ends. To describe the second rule, we need to introduce the amplitudes of close strings in the ground state: $\Phi(\ )$. The ground state is given by

$$\sum_{\text{all closed strings}} \Phi(\ )\, |\ \rangle . \tag{5}$$

Then the second rule relates the amplitudes of close strings in the ground state as we change the strings locally:



(6)

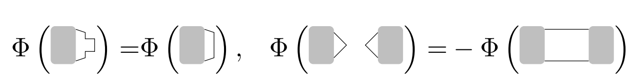
In other words, if we locally deform/reconnect the strings as in Fig. 2, the amplitude (or the ground state wave function) does not change.

The first rule tells us that the amplitude of a string configuration only depend on the topology of the string configuration. Starting from a single loop, using the local deformation and the local reconnection in Fig. 2, we can generate all closed string configurations with any number of loops. So all those closed string configurations have the same amplitude. Therefore, the local dancing rule fixes the wave function to be the equal-weight superposition of all closed strings:

$$|\Phi_{Z_2}\rangle = \sum_{\text{all closed strings}} |\ \rangle . \tag{7}$$

In other words, the local dancing rule fixes the global dancing pattern.

If we choose another local dancing rule, then we will get a different global dancing pattern that corresponds to a different topological order. One of the new choices is obtained by just modifying the sign in eqn. (6):



(8)

We note that each local reconnection operation changes the number of loops by 1. Thus the new local dancing rules gives rise to a wave function which has a form

$$|\Phi_{\text{Semi}}\rangle = \sum_{\text{all closed strings}} (-)^{N_{\text{loops}}} |\ \rangle , \tag{9}$$

where N_{loops} is the number of loops. The wave function $|\Phi_{\text{Semi}}\rangle$ corresponds to a different global dance and a different topological order.

2. Emergence of Fermi and fractional statistics

Why the two wave functions of non-oriented strings, $|\Phi_{Z_2}\rangle$ and $|\Phi_{\text{Semi}}\rangle$ (see eqn. (7) and eqn. (9)), have topological orders? This is because the two wave functions give rise to non-trivial topological properties. The two wave functions correspond to different topological orders since they give rise to different topological properties. In this section, we will discuss two topological properties: emergence of fractional statistics and, in next section, topological degeneracy on torus.

The two topological states in two dimensions contain only closed strings, which represent the ground states. If the wave functions contain open strings (*i.e.* have non-zero amplitudes for open string states), then the ends of the open strings will correspond to point-like topological excitations above the ground states. Although an open string is an extended object, its middle part merge with the strings already in the ground states and is unobservable. Only its two ends carry energies and correspond to two point-like particles.

We note that such a point-like particle from an end of string cannot be created alone. Thus an end of string correspond to a topological point defect, which may carry fractional quantum numbers. This is because an open string as a whole always carry non-fractionalized quantum numbers. But an open string corresponds to *two* topological point defects from the two ends. So we cannot say that each end of string also carries non-fractionalized quantum numbers. Some times, they do carry fractionalized quantum numbers.

Let us first consider the defects in the $|\Phi_{Z_2}\rangle$ state. To understand the fractionalization, let us first consider the *spin* of such a defect, to see if the *spin* is fractionalized or not (Fidkowski *et al.*, 2009). Note that, here the *spin* is not the spin of the spin-1/2 that form our model. The *spin* is the orbital angular momentum of an end. We use different fonts to distinguish them. An end of string can be represented by

$$|\uparrow\rangle_{\text{def}} = |\uparrow\rangle + |\curvearrowright\rangle + |\wr 0\rangle + \dots \qquad (10)$$

which is an equal-weight superposition of all string states obtained from the deformations and the reconnections of $\uparrow$.

Under a 360° rotation, the end of string is changed to $|\circlearrowright\rangle_{\text{def}}$, which is an equal weight superposition of all string states obtained from the deformations and the reconnections of $\circlearrowright$. Since $|\uparrow\rangle_{\text{def}}$ and $|\circlearrowright\rangle_{\text{def}}$ are always different, $|\uparrow\rangle_{\text{def}}$ is not an eigenstate of 360° rotation and does not carry a definite *spin*.

To construct the eigenstates of 360° rotation, let us make a 360° rotation to $|\circlearrowright\rangle_{\text{def}}$. To do that, we first use the string reconnection move in Fig. 2, to show that $|\circlearrowright\rangle_{\text{def}} = |\circlearrowright\rangle_{\text{def}}$. A 360° rotation on $|\circlearrowright\rangle_{\text{def}}$ gives us $|\uparrow\rangle_{\text{def}}$. We see that the 360° rotation exchanges $|\uparrow\rangle_{\text{def}}$ and $|\circlearrowright\rangle_{\text{def}}$. Thus the eigenstates of 360° rotation are given

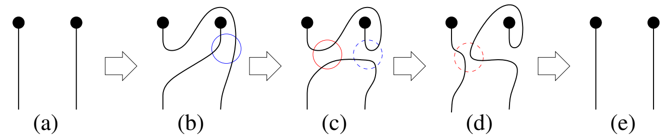


FIG. 3 Deformation of strings and two reconnection moves, plus an exchange of two ends of strings and a 360° rotation of one of the end of string, change the configuration (a) back to itself. Note that from (a) to (b) we exchange the two ends of strings, and from (d) to (e) we rotate of one of the end of string by 360°. The combination of those moves do not generate any phase.

by $|\uparrow\rangle_{\text{def}} + |\circlearrowright\rangle_{\text{def}}$ with eigenvalue 1, and by $|\uparrow\rangle_{\text{def}} - |\circlearrowright\rangle_{\text{def}}$ with eigenvalue -1. So the particle $|\uparrow\rangle_{\text{def}} + |\circlearrowright\rangle_{\text{def}}$ has a *spin* 0 (mod 1), and the particle $|\uparrow\rangle_{\text{def}} - |\circlearrowright\rangle_{\text{def}}$ has a *spin* 1/2 (mod 1).

If one believes in the *spin*-statistics theorem, one may guess that the particle $|\uparrow\rangle_{\text{def}} + |\circlearrowright\rangle_{\text{def}}$ is a boson and the particle $|\uparrow\rangle_{\text{def}} - |\circlearrowright\rangle_{\text{def}}$ is a fermion. This guess is indeed correct. Form Fig. 3, we see that we can use deformation of strings and two reconnection moves to generate an exchange of two ends of strings and a 360° rotation of one of the end of string. Such operations allow us to show that Fig. 3a and Fig. 3e have the same amplitude, which means that an exchange of two ends of strings followed by a 360° rotation of one of the end of string do not generate any phase. This is nothing but the *spin*-statistics theorem.

The emergence of Fermi statistics in the $|\Phi_{Z_2}\rangle$ state of a purely bosonic spin-1/2 model indicates that the state is a topologically ordered state. We also see that the $|\Phi_{Z_2}\rangle$ state has a bosonic quasi-particle $|\uparrow\rangle_{\text{def}} + |\circlearrowright\rangle_{\text{def}}$, and a fermionic quasi-particle $|\uparrow\rangle_{\text{def}} - |\circlearrowright\rangle_{\text{def}}$. The bound state of the above two particles is a boson (not a fermion) due to their mutual semion statistics. Such quasi-particle content agrees exactly with the Z_2 gauge theory which also has three type of topological excitations, two bosons and one fermion. In fact, the low energy effective theory of the topologically ordered state $|\Phi_{Z_2}\rangle$ is the Z_2 gauge theory and we will call $|\Phi_{Z_2}\rangle$ a Z_2-topologically ordered state (Read and Sachdev, 1991; Wen, 1991a).

Next, let us consider the defects in the $|\Phi_{\text{Semi}}\rangle$ state. Now

$$|\uparrow\rangle_{\text{def}} = |\uparrow\rangle + |\curvearrowright\rangle - |\wr 0\rangle + \dots \qquad (11)$$

and a similar expression for $|\circlearrowright\rangle_{\text{def}}$, due to a change of the local rule for reconnecting the strings (see eqn. (8)).

TABLE I **Topologically ordered states with long range entanglement**. Here 1B refers to 1-dimensional bosonic system, 2F 2-dimensional fermionic system, *etc* . The second column indicates the presence of fractionalized point-like excitations. The third column indicates the presence of non-abelian statistics. The fourth column indicates whether the boundary must be gapless, or can be gapped, or for some must be gapless and for others can be gapped.

Topological order	**Frac. exc.**	**Non-ab. sta.**	**Boundary**	**Classification/comment**
1F Majorana chain	No	Not any	Maj. zero mode	$\mathbb{Z}_2$ (Z_2^f symm. breaking)
2B bosonic E_8 state	No	No	Gapless	Invertible topological order
2B chiral spin liquid	Semion	No	Gapless	Spin quantum Hall state
2B Z_2-spin liquid	Fermion	No	Gapped	Z_2-gauge/toric-code
2B double-semion state	Fermion	No	Gapped	Z_2-Dijkgraaf-Witten
2B string-net liquids	Yes	Yes	Gapped	Unitary fusion category
2F $p+\mathrm{i}p$ fermion paired state	No	No	Gapless	Invertible topological order
2F integer quantum Hall states	No	No	Gapless	$\mathbb{Z}$ (invertible topological order)
2F Laughlin states 2F Halperin states	Yes	No	Gapped/gapless	K-matrix (symmetric, integral)
2F $\chi_1\chi_2^2$ state	Yes	$SU(2)_2$	Gapless	Cannot do universal TQC
2F χ_2^3 state	Yes	$SU(3)_2$	Gapless	Can do universal TQC
2F Pfaffian state	Yes	$SU(2)_2$	Gapless	Cannot do universal TQC
2F Z_3 parafermion state	Yes	$SU(2)_3$	Gapless	Can do universal TQC
2F string-net liquids	Yes	Yes	Gapped	Unitary super fusion category
3+1D superconductor	Fermion	Not any	Gapped	With dynamical $U(1)$ gauge field
3B string-net liquids	Fermion	Not any	Gapped	Symmetric fusion category
3B Walker-Wang model	Fermion	Not any	Gapped	Pre-modular tensor category
3B all-boson topo. order	Boson	Not any	Gapped	Pointed fusion 2-category

Using the string reconnection move in Fig. 2, we find that $|\circlearrowleft\rangle_{\text{def}} = -|\circlearrowleft\rangle_{\text{def}}$. So a 360° rotation, changes $(|\uparrow\rangle_{\text{def}}, |\circlearrowleft\rangle_{\text{def}})$ to $(|\circlearrowleft\rangle_{\text{def}}, -|\uparrow\rangle_{\text{def}})$. We find that $|\uparrow\rangle_{\text{def}} + \mathrm{i}|\circlearrowleft\rangle_{\text{def}}$ is the eigenstate of the 360° rotation with eigenvalue $-\mathrm{i}$, and $|\uparrow\rangle_{\text{def}} - \mathrm{i}|\circlearrowleft\rangle_{\text{def}}$ is the other eigenstate of the 360° rotation with eigenvalue i. So the particle $|\uparrow\rangle_{\text{def}} + \mathrm{i}|\circlearrowleft\rangle_{\text{def}}$ has a *spin* $-1/4$, and the particle $|\uparrow\rangle_{\text{def}} - \mathrm{i}|\circlearrowleft\rangle_{\text{def}}$ has a *spin* $1/4$. The *spin*-statistics theorem is still valid for $|\Phi_{\text{Semi}}\rangle_{\text{def}}$ state, as one can see form Fig. 3. So, the particle $|\uparrow\rangle_{\text{def}} + \mathrm{i}|\circlearrowleft\rangle_{\text{def}}$ and particle $|\uparrow\rangle_{\text{def}} - \mathrm{i}|\circlearrowleft\rangle_{\text{def}}$ have fractional statistics with statistical angles of semion: $\pm\pi/2$. Thus the $|\Phi_{\text{Semi}}\rangle$ state contains a topological order. We will call such a topological order a **double-semion topological order** (Freedman *et al.*, 2004; Levin and Wen, 2005).

It is amazing to see that the long-range quantum entanglement in string liquid can gives rise to fractional *spin* and fractional statistics, even from a purely bosonic model. Fractional *spin* and Fermi statistics are two of most mysterious phenomena in natural. Now, we can understand them as merely a phenomenon of long-range quantum entanglement. They are no longer mysterious.

3. Topological degeneracy

The Z_2-topological order has another important topological property: topological degeneracy (Read and Chakraborty, 1989; Wen, 1991a). Topological degeneracy is the ground state degeneracy of a gapped many-body system that is robust against any local perturbations as long as the system size is large (Wen and Niu,

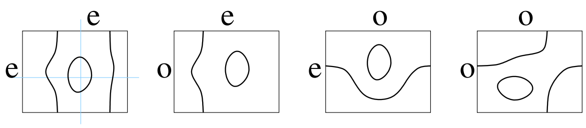


FIG. 4 On a torus, the closed string configurations can be divided into four sectors, depending on even or odd number of strings crossing the x- or y-axes.

1990). It implies the presence of topological order.

Topological degeneracy can be used as protected qubits which allows us to perform topological quantum computation.(Kitaev, 2003) It is believed that the appearance of topological degeneracy implies the topological order (or long-range entanglement) in the ground state. Many-body states with topological degeneracy are described by topological quantum field theory at low energies.

The simplest topological degeneracy appears when we put topologically ordered states on compact spaces with no boundary. We can use the global entanglement pattern to understand the topological degeneracy. We know that the local rules determine the global entanglement pattern. On a sphere, the local rules determine a unique global entanglement pattern. So the ground state is non-degenerate. However on other compact spaces, there can be several global entanglement patterns that all satisfy the same local rules. In this case, the ground state is degenerate.

For the Z_2-topological state on torus, the local rule relate the amplitudes of the string configurations that differ by a string reconnection operation in Fig. 2. On a torus, the closed string configurations can be divided into four sectors (see Fig. 4), depending on even or odd number of strings crossing the x- or y-axes. The string reconnection move only connect the string configurations among each sector. So the superposition of the string configurations in each sector represents a different many-body wave functions. Since those many-body wave functions are locally indistinguishable, they correspond to different degenerate ground states. Therefore, the local rule for the Z_2-topological order gives rise to four fold degenerate ground state on torus.

Similarly, the double-semion topological order also gives rise to four fold degenerate ground state on torus.

G. Table of some topological orders

In table I, we list some topological orders in bosonic and fermionic systems in various dimensions. The simplest one in the table is the 2+1D **IQH states** (von Klitzing *et al.*, 1980). Some entries in table I have not been discussed above. In particular, the **string-net liquids** for bosonic systems (Levin and Wen, 2005) and fermionic systems (Bhardwaj *et al.*, 2016; Gu *et al.*, 2015) allow us to obtain all 2+1D topological orders with gappable boundary (Kitaev and Kong, 2012; Lan and Wen, 2014). It reveals that 2+1D bosonic topological orders are classified by **unitary fusion categories** (Etingof *et al.*, 2005), while 2+1D fermionic topological orders are classified by **unitary super fusion categories**. For more general 2+1D bosonic topological orders, it was conjectured (Wen, 1990b), and became more and more clear (Keski-Vakkuri and Wen, 1993; Kitaev, 2006; Rowell *et al.*, 2009; Wen, 2016), that they are classified by the **modular matrices** S, T (which encode **unitary modular tensor categories** (MTC) (Moore and Seiberg, 1989)) plus the chiral central charge c of the edge states. Physically, the so called *MTC can be viewed as a set of topological excitations, together with the data that describes the fusion and braiding of those excitations.*

Many topological orders have fractionalized excitations (see the second column of table I), some 2+1D topological orders even have non-abelian excitations (see the third column of table I). In 1+1D fermion systems and 2+1D boson/fermion systems, there are even topological orders that have no fractionalized excitations (the second column with an “No” entry). Those topological orders are called invertible topological orders (Freed, 2014; Kapustin, 2014a; Kong and Wen, 2014), and their non-trivialness is reflected in their non-trivial boundary states which has a gravitational anomaly (Kong and Wen, 2014; Wen, 2013a).

Regarding **Pt.8** in Sec. II, we note that the fermions are fractionalized topological excitations in bosonic systems. But they are local non-topological excitations in fermionic systems. For example **Majorana fermions** are local non-topological excitations in fermionic superconductor (with spin-orbital coupling and no dynamical $U(1)$ gauge field), since they are antiparticles of themselves. Therefore, Majorana fermions are indeed fermions with Fermi statistics. They are not particles with non-abelian statistics. In fact, Majorana fermions are the familiar Bogoliubov quasiparticles in superconductors which were discovered long time ago. So what people are looking for, in the intensive experimental search, is not the Majorana fermion first introduced by Majorana, but instead **Majorana zero mode**, that can appear, for example, at the end of an 1D p-wave superconductor (Kitaev, 2001), or at the center of a vortex in a 2D $p + \mathrm{i}p$ fermion paired state (Read and Green, 2000; Senthil *et al.*, 1999). Majorana zero mode is not Majorana fermion. In fact, it is not even a particle. It is a *property* of a particle, just like the mass is a property of a particle. If the mobile particle carries a Majorana zero mode, then the particle will have a non-abelian statistics (Ivanov, 2001). So one should not mix Majorana zero mode with Majorana fermion.

We also like to mention that the $SU(2)_2$-type of non-abelian statistics in the $\chi_1\chi_2^2$ FQH state and the Pfaffian state contain a non-abelian quasiparticle that carries an Majorana zero mode. Such a particle has an internal degrees of freedom of half of a qubit (*i.e.* quantum dimension $d = \sqrt{2}$).[11]

Last, this paper only discusses topological phases at zero temperature. Phases beyond Landau symmetry breaking order also exist for $T \neq 0$, which are not reviewed here since they requires a different theoretical framework.

IV. SPT STATES: NON-TRIVIAL SYMMETRIC PRODUCT STATES

One expects gapped product states that have neither symmetry breaking order nor topological order to be trivial, in the sense that all those states belong to one single phase. In this section, we will see that in fact those states can belong to several different phases if there is a symmetry, and thus can be non-trivial.

A. Gapped integer-spin chain: Haldane phases

The ground state of the $SO(3)$ symmetric antiferromagnetic spin-1/2 Heisenberg chain

$$H = \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} \tag{12}$$

cannot break the $SO(3)$ spin rotation symmetry due to quantum fluctuations.(Mermin and Wagner, 1966) What is the nature of this symmetric ground state? The Beth ansatz approach, bosonization, and Lie-Schultz-Mattis theorem (Lieb *et al.*, 1961) all indicate the ground state of spin-1/2 Heisenberg chain behaves almost like a spontaneous $SO(3)$ symmetry breaking state: spin-spin correlation has an slow algebraic decay (in contrast to exponential decay for a typical disordered system) and the chain is gapless (as if having an Goldston mode (Goldstone, 1961)). This result led people to believe that all spin-S chain are also gapless and have algebraic decaying spin-spin correlation, since for $S > 1/2$, the spins have even weaker quantum fluctuations than the spin-1/2 chain,

In 1983, Haldane considered spin fluctuations in 1+1D space-time that have non-trivial "winding" number in $\pi_2(S^2)$. He realized that the spin configuration with "winding" number ± 1 has a phase factor -1 if the spin is half-integer and a phase factor 1 if the spin is integer. So the half-integer spin chain and integer spin chain may have different dynamics. Haldane conjectured (Haldane, 1983) that the spin chain is gapped if the spin is integer, despite it has weaker quantum fluctuations than spin-1/2 chain. If the spin is half-integer, then the spin chain is gapless. The gapped ground state of an integer spin chain is called a Haldane phase. At that time, people believed the Haldane phase to be a trivial disordered phase, just like the product state formed by spin-0 on each site.

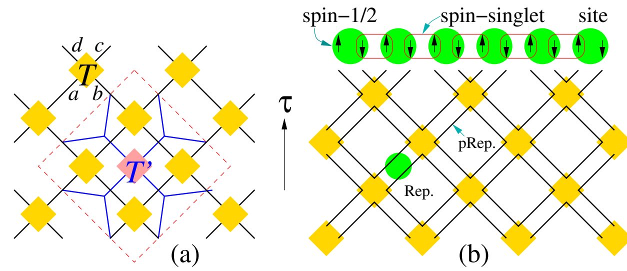


FIG. 5 (a) A tensor network representation of the partition function $Z = \mathrm{Tr}\, e^{-\tau H}$ obtained from path integral for a 1+1D quantum system. Each vertex is a rank-4 tensor T_{abcd} where each leg corresponds to an index. The range of the index is the dimension of the tensor T. The partition function Z is obtained as a product of all tensors, with the common indices on the edges linking two vertices summed over (which corresponds to the path integral). We can combine four tensors T to form a new tensor T' and obtain a new coarse-grained tensor network that produces the same partition function Z. After many coarse-graining iterations, we obtain a fixed-point tensor T^{fix} that characterizes a quantum phase. (b) The fixed-point tensor of spin-1 Heisenberg chain has a corner-double-line structure. It gives rise to the fixed-point wave function of an ideal $SO(3)$-SPT state.

However, such an opinion was put in doubt by an exact soluble integer spin chain. It was shown that, for the exactly soluble model (Affleck *et al.*, 1988a), the boundary of the integer spin-S chain carries degenerate degrees of freedom of spin-$S/2$. Since the gapless edge excitations for 2+1D FQH states implies a bulk topological order, people start to wonder that maybe the similar picture applies to one lower dimensions: the gapped 1+1D ground states of integer spin chains also have topological orders due to the gapless spin-$S/2$ boundary.

But this point of view seems incorrect. The gapless boundary of a 2+1D chiral topological order is actually a bulk property, since gaplessness is robust against any modifications on the boundary. This is why the gapless boundary reflects a bulk topological order. However, gapless spin-$S/2$ boundary of spin-S chain can be easily gapped by applying a Zeeman field at the boundary. This seems to suggest that the gapped ground state of integer spin chain to be trivial.

B. Haldane phases are topological only for odd-integer-spin

What is the nature of the Haldane phase for integer spin-S chain? Topological or not topological? This question bothered me for 15 years, until we used tensor-entanglement-filtering renormalization

[11] An physical explanation of quantum dimension can be found in Kitaev (2006) and Wen (2016).

(TEFR) approach (see Fig. 5a) to study spin-1 XXZ chain (Gu and Wen, 2009):

$$H = \sum_i J\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2 \tag{13}$$

Unlike density matrix renormalization group (DMRG) approach (White, 1992), TEFR approach gives us a simple fixed-point tensor. We found that the fixed-point tensor has a corner-double-line structure (with degenerate weights) when $U \approx 0$ (see Fig. 5b), and the fixed-point tensor becomes a dimension-1 trivial tensor when $U \gg J$ (see Fig. 5a where the indices of T are all equal to 1).

The ground state for $U \gg J$ is a product state of $|S_i^z = 0\rangle$ which is consistent with trivial dimension-1 fixed-point tensor. The corner-double-line fixed-point tensor for $U = 0$ corresponds to a fixed-point wave function that contains 4 states per site (increased from 3 states of spin-1, see Fig. 5b). The 4 states form the $3 \oplus 1$ dimensional representation of $SO(3)$, which can be viewed as two spin-1/2 representations (the projective representations of $SO(3)$)

$$3 \oplus 1 = 2 \otimes 2. \tag{14}$$

In such a fixed-point wave function, the two spin-1/2's on neighboring sites form a spin singlet. The total fixed-point wave function is the product state of those spin singlets (see Fig. 5b). We discovered that, just like the $U \gg J$ limit, *the spin-1 Haldane phase is also a short-range entangled state equivalent to a product state. It is not topological despite the fractionalized spin-1/2 boundary.*

However, non-topological does not mean trivial. We find that, for spin-1 chain, the corner-double-line structure even appear for the follow generic Hamiltonian

$$\begin{aligned} H = &\sum_i [J\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2] \\ &+ \sum_i B_x S_i^x + B_z S_i^z + B_x'[S_i^x(S_{i+1}^z)^2 + S_{i+1}^x(S_i^z)^2] \end{aligned} \tag{15}$$

when $U, B_{x,z}, B_x' \approx 0$. This suggests that the corner-double-line structure is stable against any perturbations with time reversal symmetry T^* (which is the usual time reversal plus a 180° spin-S^y rotation) and spacial reflection symmetry[12]. On the other hand, the corner-double-line structure can be destroyed by perturbations that break those symmetries. This suggest that the spin-1 Haldane phase, characterized by the corner-double-line tensor (or the dimmerized fixed-point wave function) is a stable phase, distinct from the product state of $|S_z = 0\rangle$'s, as long as we do not break those symmetries. We conclude that the Haldane phase of spin-1 chain is non-trivial despite it is a product state that does not spontaneously break any symmetry! This is a new state of matter and we propose the concept of **symmetry protected trivial (SPT)** order to describe this new state of matter. SPT orders is characterized by the corner-double-line fixed-point tensors with degenerate weights (or the dimmerized fixed-point wave function). Later, Pollmann *et al.* (2010) also showed that SPT orders can be characterized via the entanglement spectrum. It is interesting to see that even product states without spontaneous symmetry breaking can be non-trivial. However, the spin-1 Haldane phase at that time has already been widely referred as a topological phase. So we gave the term "SPT order" another representation "**symmetry protected topological** order"[13].

It is very important to regard SPT states as short-range entangled, not topological (in the sense of orange-vs.-donut). This correct way of thinking leads to a complete classification of all 1D gapped interacting phases (Chen *et al.*, 2011a; Schuch *et al.*, 2011), in terms of projective representations of the symmetry group(Pollmann *et al.*, 2010) one year later and the systematic group cohomology theory of SPT phases in higher dimensions two years later (Chen *et al.*, 2013b). In particular, the projective-representation classification of 1+1D SPT phases indicates that *only the odd-integer-spin Haldane phases are the $SO(3)$-SPT phases, while the even-integer-spin Haldane phases are not the $SO(3)$-SPT phase just like the product state of spin-0's* (Pollmann *et al.*, 2012). So Haldane phases can be topological or non-topological depending on the spin to be odd or even integer. This explains the **Pt.1** in Sec. II.

C. An Z_2-SPT state in 2+1D

After realizing SPT states to be product states, it becomes easy to construct SPT states in any dimension. We just need to write a product state in some complicated form, and then try to find all the twisted way to implement the symmetry.

First, we need to introduce the concept of **on-site symmetry**, which is usually referred as global symmetry. Relative to the tensor product decomposition $\mathcal{H}^{\text{tot}} = \bigotimes_i \mathcal{H}_i$ of the total Hilbert space, a symmetry transformation is on-site if it has a tensor product decomposition $U = \prod_i U_i$, where U_i is the symmetry transformation acting on $\mathcal{H}_i$. The notion of on-site symmetry is stressed in Chen *et al.* (2011a,c), which is a key to understand SPT states.

[12] In fact, the corner-double-line structure is stable against any perturbations with time reversal symmetry T^* *or* spacial reflection symmetry (Pollmann *et al.*, 2012).

[13] After long debates, we eventually used the second less-accurate representation in our paper.

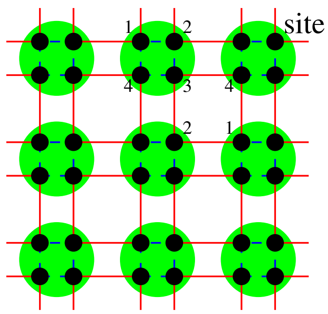


FIG. 6 The filled dots are qubits (or spin-1/2's). A big disk (with four dots inside) represents a site. The dash line connecting dots 1, 2 represents the phase factor CZ_{12} in the Z_2 global symmetry transformation. In the Z_2-SPT state, the four spins in a plaquette (connected by solid lines that form a square) is described by $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$.

The first lattice model that realizes (Chen *et al.*, 2011c) a 2+1D SPT state has four qubits (or spin-1/2 spins) on each site (see Fig. 6). A complicated product state is given by

$$|\Psi_0\rangle = \bigotimes_{\text{plaquette}} \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle) \tag{16}$$

where $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$ is the wave function for the four spins in the plaquette (see Fig. 6). Note that the four spins in $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$ are on four different sites.

One way to introduce a Z_2 symmetry is to define the transformation on each site to be the spin flipping:

$$U_X = \sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4, \quad U_X^2 = 1. \tag{17}$$

Obviously, $|\Psi_0\rangle$ is invariant under such a spin flipping Z_2 transformation. But for such a Z_2 symmetry, $|\Psi_0\rangle$ is not a SPT state.

There is another way to define Z_2 symmetry (on each site, see Fig. 6), but this time with a twist:

$$U_{CZX} = U_X U_{CZ}, \tag{18}$$

where the ± 1 phase twist, U_{CZ}, is a product of CZ_{ij} that acts on the two spins at i and j: $CZ_{ij} = -1$ when acts on $|\downarrow\downarrow\rangle$ and $CZ_{ij} = 1$ otherwise. More specifically

$$\begin{aligned} U_{CZ} &= \prod_{j=1,2,3,4} CZ_{j,j+1} \\ &= \prod_{j=1,2,3,4} \frac{1 + \sigma^z_{j+1} + \sigma^z_j - \sigma^z_{j+1}\sigma^z_j}{2}, \end{aligned} \tag{19}$$

where $j = 5$ is the same as $j = 1$. It is a non-trivial exercise but one can indeed check that $U_{CZX}^2 = 1$. $|\Psi_0\rangle$ is invariant under such a twisted spin flipping Z_2 transformation since all the ± 1 CZ_{ij} factors cancel each other.

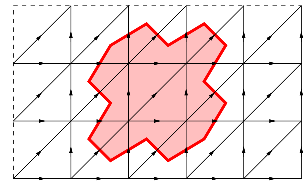


FIG. 7 A 2D lattice on a torus. A $g \in G$ transformation is performed on the sites in the shaded region. The g transformation changes the Hamiltonian term on the triangle (ijk) across the boundary from H_{ijk} to H^g_{ijk}.

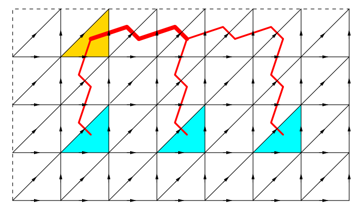


FIG. 8 Three identical monodromy defects (three lower triangles) from $G = Z_3 = \{0, 1, 2\}$ symmetry twist. The think lines are 1-cuts, and the thick line is a 2-cut. The g-cuts can be relocated by local Z_3 transformations as in Fig. 7. The single upper triangle can also be relocated by local Z_3 transformations. Thus it is not a monodromy defect.

For the new Z_2 symmetry, $|\Psi_0\rangle$ is an SPT state (Chen *et al.*, 2011c). In fact, one can construct an exactly soluble lattice Hamiltonian, which is symmetric under the new symmetry and has $|\Psi_0\rangle$ as its unique gapped group state.

The above construction has been generalized to higher dimensions and arbitrary compact symmetry group via group cohomology theory: for each element in $\mathcal{H}^{d+1}(G; \mathbb{R}/\mathbb{Z})$, we can construct an $d + 1$D SPT state protected by G-symmetry. But one thing remain unclear: how to see those constructed state to be a G-SPT state?

D. Probing SPT orders

An SPT state is almost trivial. For example, all the correlations are short ranged and featureless, as well as all the bulk excitations are local excitations without fractionalization. So, it is not easy to see the non-trivialness of a SPT state. One way to reveal the non-trivialness is to probe the boundary (Chen *et al.*, 2011c):
The boundary of a SPT state cannot be gapped and non-degenerate if the symmetry is not broken explicitly.
This because the effective symmetry on the low energy boundary degrees of freedom must be non-on-site, and the non-on-site property for the boundary theory exactly corresponds to and classify the anomaly in global symmetry (Wen, 2013a). This implies the boundary of a SPT

TABLE II **SPT states with short-range entanglement**. Here 1B refers to 1-dimensional bosonic system, 2F 2-dimensional fermionic system, *etc* . Also T represents the time reversal symmetry, which generates the group Z_2^T for bosonic systems, and Z_4^T for electron systems. This is because $T^2 = (-)^{N_F}$ is the fermion-number-parity operator for electron systems. The last column describes the degenerate state at the end of 1D SPT phases, or other SPT-probes for higher dimensions.

SPT order	Symm.	Classification	Chain-end/SPT-probe
1B spin-1 Haldane phase	$SO(3)$	$\mathcal{H}^2(SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	Spin-1/2
1B spin-1 Haldane phase	Z_2^T	$\mathcal{H}^2(Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	Kramer doublet
1B symm. gapped phases	G	$\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$	Proj. rep. of G
1F ins. w/ coplanar spin order	$U^f(1) \rtimes Z_2^T$	$\mathbb{Z}_2$	Kramer doublet
1F topo. superconductor	Z_4^T	$\mathbb{Z}_2$	charge-0 Kramer doublet
1F G^f-SPT phases	G^f	$\mathcal{H}^2(G^f, \mathbb{R}/\mathbb{Z})$	Proj. rep. of G^f
2B Z_n-SPT states	Z_n	$\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$	Z_n-dislocation has frac. statistics/Z_n-charge
2B SPT insulator	$U(1)$	$\mathcal{H}^3(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$	Even-int. Hall conductance
2B T-symm. SPT insulator	$U(1) \rtimes Z_2^T$	$\mathcal{H}^3(U(1) \rtimes Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	π-flux has Kramer doub.
2B spin quantum Hall states	$SO(3)$	$\mathcal{H}^3(SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$	Quantized spin Hall cond.
2B T-symm. SPT spin liquid	$Z_2^T \times SO(3)$	$\mathcal{H}^3(Z_2^T \times SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	
2B G-SPT states	G	$\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$	
2F quantum spin Hall states	$U^f(1) \times U^f(1)$	$\mathbb{Z}$	Spin-charge Hall cond.
2F topological insulator	$[U^f(1) \rtimes Z_4^T]/Z_2$	$\mathbb{Z}_2$	π-flux carries charge-0 Kramer doublet
2F topo. superconductor	Z_4^T	$\mathbb{Z}_2$	π-flux carries charge-even Kramer doub.
2F G^f-SPT states	G^f without T	Chiral central charge $c = 0$ modular extensions of sRep(G^f)	
3B T-symm. SPT states	Z_2^T	$\mathcal{H}^4(Z_2^T, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^2$	
3B T-symm. SPT insulator	$U(1) \rtimes Z_2^T$	$\mathcal{H}^4(U(1) \rtimes Z_2^T, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^3$	A monople is a fermion
3B T-symm. SPT spin liquid	$Z_2^T \times SO(3)$	$\mathcal{H}^4(Z_2^T \times SO(3), \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^4$	
3B G-SPT states	G without T	$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$	
3B G-SPT states	G with T	$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2$	
3F topological insulator	$[U^f(1) \rtimes Z_4^T]/Z_2$	$\mathbb{Z}_2$	A monople carries half-integer charge
3F topo. superconductor	Z_4^T	$\mathbb{Z}_{16}$	

state to be either symmetry breaking, gapless, and/or topologically ordered.

Another way to detect the non-trivialness of a SPT state is to twist the symmetry and measure the ground state response under the twisted symmetry (Levin and Gu, 2012). To understand how to twist the symmetry, let us assume that a 2D lattice Hamiltonian for a SPT state with symmetry G to have a form (see Fig. 7) $H = \sum_{(ijk)} H_{ijk}$, where $\sum_{(ijk)}$ sums over all the triangles (ijk) in Fig. 7 and H_{ijk} acts on the states on site-i, site-j, and site-k. H and H_{ijk} are invariant under the global G transformations.

Let us perform a *local* $g \in G$ transformation which only acts on the sites in the shaded region in Fig. 7. Such a local transformation will change H to $\tilde{H}$. However, only the Hamiltonian terms on the triangles (ijk) across the boundary of the shaded region are changed from H_{ijk} to H^g_{ijk}. Since the G transformation is an unitary transformation, H and $\tilde{H}$ have the same energy spectrum. In other words the boundary (called the g-cut) in Fig. 7 (described by H^g_{ijk}'s) does not cost any energy.

Now let us consider a Hamiltonian on a lattice with some g-cuts (see Fig. 8) $\tilde{H} = \sum_{(ijk)}{}' H_{ijk} + \sum_{(ijk)}^{g\text{-cut}} H^g_{ijk}$, where $\sum'_{(ijk)}$ sums over the triangles not on the cut and $\sum_{(ijk)}^{g\text{-cut}}$ sums over the triangles that are divided into disconnected pieces by the g-cut. The triangles at the ends of the cut have no Hamiltonian terms. We note that the cut carries no energy. Only the ends of cut cost energies. So the Fig. 8 corresponds to three monodromy defects. If the g is a generator of G, then the end of g-cut will be called elementary monodromy defect. We like to point out that dislocation in a crystal is an example of monodromy defect of translation symmetry. It has been used to detect SPT phases protected by translation symmetry (the so called weak topological phases) (Ran *et al.*, 2009; Slager *et al.*, 2014; Teo and Kane, 2010). We also like to point out that the above procedure to obtain $\tilde{H}$ is actually the “gauging” of the G symmetry (Levin and Gu, 2012). $\tilde{H}$ is a gauged Hamiltonian that contain three G vortices at the ends of the cut.

Using the above monodromy defects, we can detect the Z_n-SPT order (Wen, 2017):
*n **identical** elementary monodromy defects in a 2+1D Z_n-SPT state on a torus always carry a total Z_n-charge m, if the Z_n SPT state is described by the m^{th} cocycle in $\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})$.*
The total Z_n-charge of n identical monodromy defects allows us to completely characterize the 2+1D Z_n SPT states. Another way to probe the Z_n-SPT order is to use the statistics of the monodromy defects (Levin and Gu, 2012):
The statistical angle θ_M of an elementary monodromy defect satisfies $\mathrm{mod}(\frac{\theta_M}{2\pi}, \frac{1}{n}) = \frac{m}{n^2}$ for a Z_n-SPT state characterized by $m \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$.

This way of probing an SPT state is like using the modular extensions of Rep(G) to probe the G-SPT order (Lan *et al.*, 2016, 2017b). (The so called *modular extension* can be viewed as including all the monodromy defects and considering their statistics.) It has been shown that the modular extensions of Rep(G) one-to-one correspond to the elements in $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ (Drinfeld *et al.*, 2007; Lan *et al.*, 2016). So the modular extensions can fully characterize $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$. In other words, measuring the abelian and/or non-abelian statistics among the monodromy defects and the local excitations described by Rep(G), allows us to fully detect the G-SPT order in 2+1D for any unitary symmetry G. The similar idea also applies to 3+1D SPT states (Lan *et al.*, 2017a; Wang and Levin, 2014). If the symmetry group contain $U(1)$, one can also use the $U(1)$ monopoles to probe the 3+1D SPT states (Metlitski *et al.*, 2013; Wen, 2014; Ye and Wen, 2014). A systematic discussion to probe all SPT orders in any dimensions can be found in Hung and Wen (2014).

E. Table of some SPT states

In table II, we list bosonic/fermionic SPT states for various symmetries and in various dimensions. For bosonic SPT states with on-site symmetry G, a partial classification was first given by the group cohomology of the symmetry group $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ where d is the space dimension (Chen *et al.*, 2013b). Later, it was pointed out the group cohomology description is incomplete when $d = 3$ and when G contains time reversal symmetry (Vishwanath and Senthil, 2013; Wang and Senthil, 2013). Then, it was realized that bosonic SPT states can all be classified by generalized group cohomology $\mathcal{H}^{d+1}(G \times SO_\infty, \mathbb{R}/\mathbb{Z})/\Gamma$. This implies that in 1+1D and 2+1D, bosonic SPT states are classified by $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$ and $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ respectively. In 3+1D, bosonic SPT states are classified by $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ if the on-site symmetry G does not contain time reversal, and by $\mathcal{H}^4(G, (\mathbb{R}/\mathbb{Z})_T) \oplus \mathbb{Z}_2$ if G contains time reversal. Recent work also generalizes the cohomology classification of bosonic SPT states to translation and point-group symmetries (Hermele and Chen, 2016; Hsieh *et al.*, 2014; Lake, 2016; Song *et al.*, 2017; Thorngren and Else, 2016; You and Xu, 2014).

For non-interacting fermionic SPT states (Bernevig *et al.*, 2006; Fu *et al.*, 2007; Kane and Mele, 2005b; Moore and Balents, 2007; Qi *et al.*, 2008; Roy, 2006), there is a related classification of non-interacting gapped states based on K-theory(Kitaev, 2009) or nonlinear σ-model of disordered fermions (Schnyder *et al.*, 2008) (see Tables III and IV). But such a classification does not apply to interacting fermions. For interacting fermionic SPT states (Wang *et al.*, 2014; Wang and Senthil, 2014), there is a systematic understanding based on group super cohomology theory (Gaiotto and Kapustin, 2015; Gu and Wen, 2014; Kapustin and Thorngren, 2017; Wang and Gu, 2017), if the total symmetry group has a form $G^f = G_b \times Z_2^f$. Here Z_2^f is the fermion-number-parity symmetry which is always present for fermion systems. Recently, a complete classification for all 2+1D fermionic SPT states was found for generic on-site symmetry G^f which does not contain time reversal (Lan *et al.*, 2016): *2+1D fermionic SPT phases are classified by the modular extensions of sRep(G^f).* Here sRep(G^f) is the symmetric fusion category formed representations of G^f where the representations with non-trivial Z_2^f action are fermions. Last, we would like to mention that, in addition to the cohomological and categorical approach, there is also a cobordism approach for bosonic/fermionic SPT states, which can lead a classifying result for all dimensions and for some simple symmetries (Kapustin, 2014a,b; Kapustin *et al.*, 2015).

Regarding to **Pt.3** in Sec. II, quantum spin Hall effect refers to quantized transverse S^z-spin current induced by force acting on electric charges (*i.e.* a quantized mixed-electro-spin Hall conductance) (Bernevig and Zhang,

TABLE III **Classification of the gapped phases of noninteracting fermions** in d-dimensional space, for some symmetries. The space of the gapped states is given by $C_{p+d \text{ mod } 2}$, where p depends on the symmetry group. The distinct phases are given by $\pi_0(C_{p+d \text{ mod } 2})$. "0" means that only trivial phase exist. $\mathbb{Z}$ means that nontrivial phases are labeled by nonzero integers and the trivial phase is labeled by 0. $U^f(1)$ means that the π rotation is $(-)^{N_F}$. Z_4^f is generated by C satisfying $C^2 = (-)^{N_F}$. Adapted from Wen (2011).

Symm. group	$C_p\vert_{\text{for } d=0}$	class	$p\backslash d$	0	1	2	3	4	5	6	7	example
$U^f(1)$ Z_4^f	$\frac{U(l+m)}{U(l)\times U(m)}\times\mathbb{Z}$	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	(Chern) insulator; supercond. with collinear spin order
$U^f(1)\times Z_2^T$ $Z_4^f\times Z_2^T$	$U(n)$	AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	supercond. w/ real pairing and S_z conserving spin-orbital coupling

TABLE IV **Classification of gapped phases of noninteracting fermions** in d spatial dimensions, for some symmetries. The space of the gapped states is $R_{p-d \text{ mod } 8}$, where p depends on the symmetry. The phases are classified by $\pi_0(R_{p-d \text{ mod } 8})$. $\mathbb{Z}_2$ means that there is one nontrivial and one trivial phases labeled by 1 and 0. Note that $\frac{U^f(1)\rtimes Z_4^T\times Z_4^f}{Z_2^2}$ is the symmetry group generated by time reversal T, charge conjugation $c\to \mathrm{i}\sigma^y c^\dagger$ and charge conservation. Adapted from Wen (2011).

Symm. group	$U^f(1)\rtimes Z_2^T$	$Z_2^T\times Z_2^f$	Z_2^f $Z_2\times Z_2^f$	Z_4^T $Z_4^T\times Z_2$	$[U^f(1)\rtimes Z_4^T]/Z_2$ $[Z_4^f\rtimes Z_4^T]/Z_2$	$\frac{U^f(1)\rtimes Z_4^T\times Z_4^f}{Z_2^2}$	$SU^f(2)$	$\frac{SU^f(2)\times Z_4^T}{Z_2}$
$R_p\vert_{\text{for } d=0}$	$\frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times\mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
class	AI	BDI	D	DIII	AII	CII	C	CI
$d=0$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$d=1$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
$d=2$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$d=3$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$d=4$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$d=5$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$d=6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$d=7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
Example	insulator w/ coplanar spin order	supercond. w/ coplanar spin order	supercond.	supercond. w/ time reversal	insulator w/ time reversal	insulator w/ time reversal and intersublattice hopping	spin singlet supercond.	spin singlet supercond. w/ time reversal

2006; Kane and Mele, 2005b), while spin quantum Hall effect refer to quantized transverse S^z-spin current induced by force acting on "S^z-charge" (*i.e.* a quantized spin-Hall conductance). They have a vanishing charge-Hall and thermo-Hall conductances. Under such definitions, the quantum spin Hall states (Bernevig and Zhang, 2006; Kane and Mele, 2005a) and topological insulators in 2+1D (Kane and Mele, 2005b) (both appear in table II) are different fermionic SPT states. They even have different symmetries: quantum spin Hall states have $[U_\uparrow(1)\times U_\downarrow(1)]^f$ symmetry, while topological insulators $[U^f(1)\rtimes Z_4^T]/Z_2$ symmetry[14].

Even though topological insulator arises from the studies of quantum spin Hall effect, it is incorrect to think topological insulator to be due to quantum spin Hall effect. In particular, Kane and Mele (2005b), in "Z_2 Topological Order and the Quantum Spin Hall Effect", concluded that even without quantum spin Hall effect, an insulator can still be non-trivial. This led to the notion of topological insulator. This is a very surprising discovery which started the very active field of topological insulator. Despite the term "Topological Order" in the title, the topological insulator is a short-range entangled SPT state. It has no topological order as introduced by Wen (1990b) and Wen and Niu (1990), which involves long-range entanglement. This explains the **Pt.2** in Sec. II. Kane and Mele (2005b) only deal with non-interacting fermions in 2+1D. Soon, it was shown that the 2+1D topological insulator is stable against weak interactions (Wu *et al.*, 2006; Xu and Moore, 2006).

[14] The superscript f means that the $U(1)$ groups contain Z_2^f as a subgroup. $U_{\uparrow,\downarrow}(1)$ is the symmetry of $\uparrow,\downarrow$-spin conservation, and $U^f(1)$ is the symmetry of charge conservation. Z_4^T is the group generated by time reversal transformation T, that satisfies $T^2=(-)^{N_F}$ and $(-)^{N_F}$ is the fermion-number-parity. After the discovery of the Z_2-topological invariant and the 2+1D topological insulator (Kane and Mele, 2005b), quantum spin Hall state, some times, was also defined as 2+1D topological insulator. Such a quantum spin Hall state has no quantum spin Hall effect nor spin quantum Hall effect, since even the S^z current is not conserved.

With regard to the second part of **Pt.2**, many popular articles characterize topological insulator as an insulator with conducting surface. Such a characterization is incorrect, since both trivial insulator and topological insulator can some times have conducting surfaces, and other times have insulating surfaces (for interacting electrons) (Chen *et al.*, 2013a; Wang *et al.*, 2013). Maybe it is more correct to say "topological insulator is an insulator with conducting surface when electrons interact weakly". But even when electrons interact weakly, both trivial insulator and topological insulator can have conducting surfaces. We need to measure the surface Fermi surface to be sure (Hsieh *et al.*, 2008), but it does not work for 2+1D topological insulator. So a more accurate characterization of 2+1D topological insulator is that the charge-0 time-reversal symmetric π-flux must be a Kramer doublet (Qi and Zhang, 2008; Ran *et al.*, 2008).

V. TOWARDS A CLASSIFICATION OF ALL GAPPED PHASES

Only for a few times in history, we have completely classified some large class of matter states. The first time is the classification of all spontaneous symmetry breaking orders, which can be classified by a pair of groups:

$$(G_\Psi \subset G_H), \tag{20}$$

where G_H is the symmetry group of the system and G_Ψ, a subgroup of G_H, is the symmetry group of the ground state. This includes the classification of all 230 crystal orders in 3-dimensions.

The second time is the classification of all gapped 1-dimensional quantum states: *gapped 1-dimensional quantum states with on-site symmetry G_H can be classified by a triple:(Chen* et al.*, 2011a; Schuch* et al.*, 2011)*

$$[G_\Psi \subset G_H;\ \text{pRep}(G_\Psi)], \tag{21}$$

where $\text{pRep}(G_\Psi)$ *is a projective representation of* G_Ψ.

The third time is the classification of all gapped quantum phases in 2+1D. Since early on, it was conjectured that that *all 2+1D bosonic topological orders (without symmetry) are classified by S, T modular matrices (plus other gauge connections) (Wen, 1990b), or more precisely by a pair* (Kitaev, 2006; Rowell *et al.*, 2009; Wen, 2016):

$$(\text{MTC}, c), \tag{22}$$

where MTC is a unitary modular tensor category and c is the chiral central charge c of the edge states. Recently, the above result is generalized to fermion systems: *2+1D fermionic topological orders are classified by a triple:*(Lan *et al.*, 2016)

$$[\text{sRep}(Z_2^f) \subset \text{BFC};\ c], \tag{23}$$

where $\text{sRep}(Z_2^f)$ is the **symmetric fusion category** (SFC) formed by the representations of the fermion-number-parity symmetry Z_2^f where the non-trivial representation is assigned Fermi statistics, and BFC is a unitary **braided fusion category**.

For quantum systems with symmetry, we have the following result: *all 2+1D gapped bosonic phases with a finite unitary on-site symmetry G_H, are classified by* (Barkeshli *et al.*, 2014; Lan *et al.*, 2016)

$$[G_\Psi \subset G_H;\ \text{Rep}(G_\Psi) \subset \text{BFC} \subset \text{MTC};\ c], \tag{24}$$

where $\text{Rep}(G_\Psi)$ is the SFC formed by the representations of G_Ψ where all representations are assigned Bose statistics, and MTC is a minimal modular extension of the BFC. The above classification include symmetry breaking orders, SPT orders, topological orders, and *symmetry-enriched topological orders* (SET) described by *projective symmetry group* (Wen, 2002b). SET orders of time-reversal/reflection symmetry are classified by Barkeshli *et al.* (2016). Some more discussions on SET orders can be found in (Chang *et al.*, 2015; Hung and Wan, 2013; Hung and Wen, 2013; Lu and Vishwanath, 2013; Mesaros and Ran, 2013; Xu, 2013).

We have a similar result for fermion systems: *all 2+1D gapped fermionic phases with unitary finite on-site symmetry G_H^f are classified by* (Lan *et al.*, 2016)

$$[G_\Psi^f \subset G_H^f;\ \text{sRep}(G_\Psi^f) \subset \text{BFC} \subset \text{MTC};\ c], \tag{25}$$

where $\text{sRep}(G_\Psi^f)$ is the SFC formed by the representations of G_Ψ^f where some representations are assigned Fermi statistics. But we are still struggling to obtain a systematic theory of topological order in 3+1D, 28 years after the introduction of the concept.

Those results imply that the long-range entanglement in 2+1D is described by an unfamiliar mathematics – tensor category theory. This is the mathematics for the quantum topology, and it is the quantum topology (instead of classical topology) that forms the mathematical foundation of topological order (*i.e.* long-range entanglement). This explains the title of this paper "quantum-topological phases of matter", which really means "highly-entangled phases of matter".

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