Lattice tests of beyond standard model dynamics

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Over the last few years lattice techniques have been used to investigate candidate theories of new physics beyond the Standard Model. This review gives a survey of results from these studies. Most of these investigations have been of systems of gauge fields and fermions that have slowly-running coupling constants. A major portion of the review is a critical discussion of work in this particular subfield, first describing the methods used, and then giving a compilation of results for specific models.

CONTENTS

I. INTRODUCTION

A. What does the title mean?

With the recent discovery of the Higgs boson (Aad *et al.*, 2012; Chatrchyan *et al.*, 2012), the Standard Model's particle spectrum seems to be complete. But, is the particle at 126 GeV really the Standard Model Higgs, a fundamental scalar field, or is it something else? And, what about experimental observations that do not have a Standard Model explanation? Examples of such physics include neutrino masses and oscillations, the origin of the matter-antimatter asymmetry of the Universe, the nature of dark matter, and of dark energy. For that matter, why are the fundamental parameters of the Standard Model what they are? Why is the electron so light? Attempts to answer these questions are what is generically meant by "beyond Standard Model" physics.

"Lattice" refers to lattice gauge theory, which is a collection of analytic and numerical techniques for studying quantum field theories. In principle, a lattice calculation starts with a Lagrangian and a cutoff and ends with a fully nonperturbative prediction for some observable. Lattice methods have become a standard technique to study nonperturbative properties of the theory of the strong interactions, Quantum Chromodynamics or QCD. Most of the information we have about the particle spectrum of baryons and mesons, and of many hadronic matrix elements relevant to Standard Model tests, come from lattice calculations.

And why put "Lattice" and "beyond Standard Model" in the same title? For almost forty years, phenomenologists have conjectured that some beyond Standard Model physics might be nonperturbative. For example, the Higgs boson might not be fundamental; it could be a composite object held together by some new kind of strong force. About eight years ago, several physicists with lattice tool kits realized that the techniques they used for QCD might be applied to studies of candidate beyond Standard Model systems. The field became very active. This review is an attempt to describe the systems that were studied, the techniques that were used, and the results that were obtained.

B. A lattice perspective on issues relevant to beyond Standard Model physics

Before going beyond the Standard Model, we should visit the Standard Model itself. (See Logan (2014) for a more pedagogic introduction.) It has a product gauge symmetry $SU(3)\times SU(2)\times U(1)$ encoding respectively color (the strong interactions), weak isospin and weak hypercharge: the last two gauge symmetries break spontaneously, resulting in massive W and Z bosons and leaving the unbroken $U(1)$ symmetry of electromagnetism. Quarks and leptons fall into three generations; the left and right-handed (negative and positive helicity) fermions have different electroweak couplings.

In particular, the left-handed leptons and quarks form a doublet of weak isospin

$$E_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L ; \qquad Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{1}$$

while the right-handed particles e_R, u_R, d_R are singlets. This is for the first generation of fermions; there are identical terms for the second and third generations. The Lagrangian has three parts

$$\mathcal{L}_{SM} = \mathcal{L}_g + \mathcal{L}_\Phi + \mathcal{L}_m. \tag{2}$$

$\mathcal{L}_g$ holds the kinetic terms for the fermions and gauge bosons

$$\begin{aligned}\mathcal{L}_g = &\sum_{j=1}^{3} (\bar{E}_L^j (i\not{D}) E_L^j + \bar{Q}_L^j (i\not{D}) Q_L^j \\ &+ \bar{e}_R^j (i\not{D}) e_R^j + \bar{u}_R^j (i\not{D}) u_R^j + \bar{d}_R^j (i\not{D}) d_R^j) \\ &- \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} W_{\mu\nu}^2 - \frac{1}{4} B_{\mu\nu}^2\end{aligned} \tag{3}$$

in terms of the field strengths of the gluons F, the $SU(2)$ weak fields W and the $U(1)$ field B. D is the covariant derivative. The index j runs over the three generations of fermions. $\mathcal{L}_g$ conceals three parameters, the three Standard Model gauge couplings. These gauge invariant interactions make up the part of the Standard Model that is most well tested: from $\mathcal{L}_g$ follows all of electrodynamics, asymptotic freedom, parity violation in the weak interactions, and much else.

Equation 3 describes a set of massless gauge bosons. Electroweak symmetry is spontaneously broken, that is, the symmetries of the Lagrangian are not respected by the vacuum. The mechanism for doing this is contained in $\mathcal{L}_\Phi$. In the Standard Model this is achieved by the Higgs field Φ, a single scalar field whose components form a complex doublet of weak $SU(2)$. Three of the four components of Φ, which would be Goldstone bosons, are "eaten" by the $SU(2)$ and $U(1)$ gauge fields to give the massive W^+, W^- and Z_0. The fourth component becomes the Higgs particle. The potential is arranged to accomplish this. In the Standard Model, it is just

$$\mathcal{L}_\Phi = |D_\mu \Phi|^2 - V(\Phi) \tag{4}$$

and

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \tag{5}$$

The sign of the quadratic term is taken by hand to be negative, to insure spontaneous symmetry breaking. $V(\Phi)$ is characterized by two parameters μ and λ. Φ develops a vacuum expectation value

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{6}$$

where $v^2 = \mu^2/\lambda$. The Higgs and Goldstone masses are found by considering small fluctuations around the minimum,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ v + H + i\phi_3 \end{pmatrix} \tag{7}$$

and the Higgs mass is $m_H^2 = 2\mu^2 = 2\lambda v^2$. The measured masses of the W, Z, and the Higgs (126 GeV) and the known value for the $SU(2)$ and $U(1)$ couplings g and g' tell us that $v = 246$ GeV and $\lambda = 0.13$, $\mu^2 = (89\text{ GeV})^2$.

$\mathcal{L}_m$ generates the masses of quarks and leptons. A mass term couples left-handed and right-handed fermions to each other. This coupling, if present, would violate gauge invariance, because left-handed and right-handed fermions transform differently under $SU(2)$. However, a trilinear coupling of the Higgs, a left-handed fermion, and a right-handed fermion is consistent with gauge invariance. Thus the Standard Model's mass term is, schematically,

$$\begin{aligned}\mathcal{L}_m = &- \lambda_d^{ij} \bar{Q}_L^i \cdot \Phi d_R^j - \lambda_u^{ij} \epsilon^{ab} \bar{Q}_{La}^i \cdot \Phi_b^\dagger u_R^j \\ &- \lambda_l^{ij} \bar{E}_L^i \cdot \Phi e_R^j + h.c.\end{aligned} \tag{8}$$

When Φ gets its vacuum expectation value v, this trilinear interaction generates (generalized) mass terms for the quarks and leptons, parametrized by the elements of the three complex matrices λ^{ij}. This is usually done in terms of the Cabibbo - Kobayashi - Maskawa matrix, but we will not need this in what follows. $\mathcal{L}_\Phi$ and $\mathcal{L}_m$ are the parts of the Standard Model that are most often replaced or augmented by some new beyond Standard Model physics. Most often, $\mathcal{L}_\Phi$ is replaced by some new mechanism to break electroweak symmetry, perhaps without a Higgs boson. Finding something to replace $\mathcal{L}_m$ which does not involve a Higgs field and also does not introduce unwanted (unobserved) new physics is often a serious issue.

The phrase "beyond Standard Model" has been around in the literature almost as long as the phrase "Standard Model." Why is that? There are several reasons.

First, there is known physics that is not part of the Standard Model: neutrino masses and mixings, dark matter, the origin of the Universe's matter - anti-matter asymmetry, the origin of inflation, dark energy.

Second, the large number of parameters in the Standard Model seems excessive for a fundamental theory. The Standard Model has many couplings: for three generations of particles, there are mass terms for the six quarks and three charged leptons and four Cabibbo-Kobayashi-Maskawa matrix elements responsible for mixing quark mass and weak interaction eigenstates. Additionally, $\mathcal{L}_\Phi$ holds two Higgs couplings (λ and μ), and $\mathcal{L}_g$ contains three gauge couplings. Could there not be something underneath, for which the Standard Model is just a low energy effective field theory, and from which its couplings are derived?

Third, there are issues of principle with the Higgs sector. To begin, what is the origin of electroweak symmetry breaking? In the Standard Model, the Higgs potential is simply postulated to have a negative quadratic term, in order to induce spontaneous symmetry breaking. This seems arbitrary.

The next issue is the "naturalness" or "hierarchy" problem. Imagine that there is some new scale in nature, an ultraviolet (UV) "cutoff scale" Λ for the Standard Model. If this scale is very high, why is the electroweak scale so low compared to it? The problem is the instability of the Higgs mass against radiative corrections. There is a quadratic dependence of the shift in the Higgs mass on Λ. Any new physics scale Λ induces a shift in the squared mass of the Higgs, which is a value that is order $g^2\Lambda^2$, in size. (g^2 is a generic label for one of the Standard Model couplings.)

This effect can be seen in the Standard Model itself. At one loop, the shift in the mass term from its bare value (value at the cutoff) μ_0 is

$$\mu^2 - \mu_0^2 = \frac{\lambda}{8\pi^2}\Lambda^2 - \frac{3y_t^2}{8\pi^2}\Lambda^2 + \frac{3(3g^2+g'^2)}{16\pi^2}\Lambda^2. \quad (9)$$

The three terms on the right side of the equation come from of the Higgs self-interaction, the effect of top quarks, and the interaction of the Higgs with W and Z particles. y_t is the top quark Yukawa coupling and g and g' are the $SU(2)$ and $U(1)$ gauge couplings. I neglected the other quarks, because their masses, and hence their Yukawa couplings, are so much smaller than the top quark's. If Λ is very large, μ_0 must be delicately tuned to set μ to its low-energy value of 89 GeV. This could be the case, but it also seems arbitrary.

Alternatively, imagine setting the Higgs mass to its known value, and setting the couplings to their observed values. Solve Eq. 9 for Λ assuming $\mu_0 = 0$. This gives $\Lambda \sim 5$ TeV. Is this a hint of a new physics scale?

Before the discovery of the Higgs, it was hoped that the Higgs mass itself would indicate a value for the scale of new physics (which I will label Λ). The situation for a large Higgs mass is called the "triviality bound," while too small a Higgs mass led to an "instability bound." Either situation might have hinted at a low value for Λ. Both bounds come from looking at the one-loop beta function for the Higgs self coupling. Including only the Higgs self interaction and the top-quark Yukawa coupling in the equation for the running coupling ($t = \log Q^2$) gives

$$\frac{d\lambda}{dt} = \frac{3}{4\pi^2}[\lambda^2 + \lambda y_t^2 - y_t^4]. \quad (10)$$

The "triviality bound" arises from the fact that the scale dependent Higgs self-interaction becomes stronger as the momentum scale increases. At some point, it might become so strongly interacting that perturbation theory would break down. If we neglect everything but the self-interaction of the Higgs, we can integrate Eq. 10 to find

$$\lambda(Q) = \frac{\lambda(Q_0)}{1 - \frac{3}{4\pi^2}\lambda(Q_0)\log\frac{Q^2}{Q_0^2}}. \quad (11)$$

As the energy scale grows, so does $\lambda(Q)$. To prevent $1/\lambda(Q)$ from vanishing, the Higgs mass must not become too large. Replacing $\lambda(Q_0)$ by $m_H^2/2v^2$ in Eq. 11 gives

$$\frac{m_H^2}{v^2} < \frac{8\pi^2}{3\log\frac{\Lambda^2}{v^2}}. \quad (12)$$

The other bound is the "instability bound." Including only the top-quark Yukawa coupling in Eq. 10 gives

$$\frac{d\lambda}{dt} = -\frac{3}{4\pi^2}y_t^4. \quad (13)$$

Then

$$\lambda(\Lambda) - \lambda(v) = -\frac{3}{4\pi^2}y_t^4\log\frac{\Lambda^2}{v^2}. \quad (14)$$

To keep the vacuum stable, we need $\lambda(\Lambda) > 0$. Preventing this from happening gives a lower bound on the Higgs mass,

$$\frac{m_H^2}{v^2} > \frac{3}{2\pi^2}y_t^4\log\frac{\Lambda^2}{v^2}. \quad (15)$$

These two bounds combine to give the "Higgs chimney;" if the Higgs mass were too small or too large, the scale Λ would become low, and the Standard Model would signal its own upper limit. Unfortunately (or fortunately), 126 GeV is in the middle, and there seems to be no need for a nearby new physics scale for stability. The Standard Model could simply be the low energy limit of delicately arranged dynamics at some high cutoff scale.

As it stands, the Standard Model is a renormalizable quantum field theory, which could be valid all the way up to the Planck scale. Its low energy properties are independent of how it is cut off at arbitrarily short distance. It is a logical possibility that its couplings at the cutoff scale could have been fine tuned.

But there is physics beyond the Standard Model (neutrino masses and so on). How can we combine the Standard Model with this new physics in some unified description?

The first possibility is that new physics is far away in energy. To deal with this situation, there is another way to view the Standard Model: it is an effective, low energy theory of Nature, which arises from some as yet unknown dynamics. Choose units so that the Lagrange density has dimensions (energy)^4 or Λ^4 where Λ is a generic energy scale. Give all fields their engineering dimensions to achieve this. Write down the most general $SU(3) \times SU(2) \times U(1)$ symmetric Lagrangian with the field content of the Standard Model and coupling constants that are either dimensionless or (only for μ^2) of positive energy dimension. That Lagrangian is the Standard Model itself. Then, one can imagine adding new terms, which still involve only Standard Model fields (since that is all there is) but with dimensionful couplings. Including these terms (and symmetries), one would write the electroweak Lagrangian as a set of terms $\mathcal{L}_i$ where i is the dimensionality of the operator: generically,

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \ldots. \tag{16}$$

The terms in this expansion have been cataloged. The enumeration was first given by Buchmuller and Wyler (1986). There is only one dimension-5 term, a Majorana mass term for the neutrinos. Bounds on neutrino masses constrain its Λ to be larger than about 10^{13} GeV. At dimension six, there are eighty operators, each with its own Λ. Fortunately, in any given process, only a few of them contribute, and so in principle, one could consider some electroweak process, fit it to a combination of Standard Model plus higher dimensional processes, and see if there is a signal for new physics.

From the theory side, one could take some beyond Standard Model theory, match it to the Lagrangian of Eq. 16 and either make predictions for its couplings or let data constrain the proposed new physics. For a useful discussion of this procedure, see Han and Skiba (2005). This matching might be done via a lattice calculation, if one imagined that new physics was nonperturbative, lived at some very high scale, and was accessible to simulation. This could range from something like a calculation of a decay constant in QCD (like f_π), to an analog of the kaon B - parameter, to a full-blown calculation of the expectation value of some composite operator, plus its associated anomalous dimension.

Alternatively, new physics could involve particles that could actually be observed in the near future in experiments. (This was the hope before the LHC turned on and it is still the hope today.) These new degrees of freedom have to be realized as explicit terms in the Lagrangian. If the new particle arose from nonperturbative dynamics, the appropriate lattice calculation would be of its mass, like the spectroscopic ones done in a lattice QCD simulation.

In either case, the issue is, that the Standard Model by itself is a very accurate description of many things; the new physics has to be carefully concealed. This is a strong constraint on model building (or on the size of coefficients in the effective field theory description). This problem has been well-analyzed in the particle physics literature.

These days, the branching ratios of the 126 GeV particle strongly constrain new physics. (The Review of Particle Properties, Olive *et al.* (2014) has the latest numbers. To give a one-sentence summary, the couplings of the Higgs are broadly consistent with Standard Model expectations, although experimental uncertainties are still large.) Direct searches for new heavy particles also constrain new physics. (See Halkiadakis *et al.* (2014) for a recent summary.) The situation involving the rest of the Standard Model can be described in a few sentences: The most obvious manifestations of new physics in the low energy sector of the Standard Model occur in the vacuum polarization of the $SU(2) \times U(1)$ gauge bosons (Altarelli and Barbieri, 1991; Peskin and Takeuchi, 1990, 1992). There are four such amplitudes involving the physical photon, Z, their mixing, and the W, which are parametrized by a set of $SU(2)\times U(1)$ quantities, conventionally called Π_{QQ} (involving the photon), Π_{ii} for the ith component of weak isospin, and a mixing term Π_{3Q}. All are functions of the squared momentum q^2 flowing through the gauge boson. At low energy, it is sensible to expand these quantities in a power series in q^2/M^2, where M sets the new physics scale. Contributions to the electromagnetic current vanish at $q^2 = 0$ due to the usual Ward identity. Then

$$\begin{aligned}
\Pi_{QQ} &= q^2\Pi'_{QQ}(0) + \ldots \\
\Pi_{3Q} &= q^2\Pi'_{3Q}(0) + \ldots \\
\Pi_{11} &= \Pi_{11}(0) + q^2\Pi'_{11}(0) + \ldots \\
\Pi_{33} &= \Pi_{33}(0) + q^2\Pi'_{33}(0) + \ldots
\end{aligned} \tag{17}$$

There are six unknown on the right side of Eq. 17. Three of them can be fixed by experimental determinations of the fine structure constant α, the Fermi coupling G_F, and the Z-boson mass, leaving three linear combinations, conventionally called S, T and U, to be probes of new physics. They are

$$\begin{aligned}
S &= 16\pi[\Pi'_{33}(0) - \Pi'_{3Q}(0)] \\
T &= \frac{4\pi}{\sin^2\theta_W \cos^2\theta_W m_Z^2}[\Pi_{11}(0) - \Pi_{33}(0)] \\
U &= 16\pi[\Pi'_{33}(0) - \Pi'_{11}(0)].
\end{aligned} \tag{18}$$

A decade ago, Barbieri *et al.* (2004) and Han and Skiba (2005) combined precision electroweak data with an effective field theory analysis of beyond Standard Model couplings, to constrain the scale of new physics. Typical bounds, even then, were that Λ's were in the few TeV range. The recent analysis of Ciuchini *et al.* (2013) pushes the scale for many kinds of new physics up to the 5-15 TeV range. Most of this new physics involves flavor structures which are different from the Standard Model's $\mathcal{L}_m$ (Eq. 8). This is quite a different situation from the relatively low scales needed to address the hierarchy problem.

The absence of new physics for some distance above the Higgs mass itself constrains the kind of Lagrangians we can write down. We are forced to pause and think of symmetry reasons, for why particles could have masses far below the cutoff scale. Unfortunately, the list is short:

- Gauge bosons remain massless due to gauge symmetry
- Chiral symmetry protects the masses of (tree - level massless) fermions from additive renormalization
- Goldstone bosons remain massless because their potentials obey a "shift symmetry:" Parameterizing the Higgs doublet in terms of the physical Higgs H, its vacuum expectation v, and the Goldstones θ_a,

$$\begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = \exp(i\theta_a\tau_a)\begin{pmatrix} 0 \\ v+H \end{pmatrix}. \tag{19}$$

An $SU(2)_L$ transformation just shifts the θ's by a constant. This implies that the Lagrangian can have no term proportional to θ_a^2.

Most New Physics constructions attempt to exploit these general observations. For example, supersymmetry protects the scalars by putting them in multiplets with chiral fermions. In some higher dimensional completions of the Standard Model, the scalars are extra gauge degrees of freedom. In some composite Higgs models, the Higgs begins life as a Goldstone boson.

Why do some people imagine that some of the new physics is nonperturbative? I do not have a good answer. It is quite a contrast with the history of the strong interactions: people have known that nuclear forces were strong from the moment that they knew that nuclei were composite. After all, something had to overcome the protons' Coulomb repulsion to hold the nucleus together. Here, there is no obvious need for a new strong force.

When I ask people why new physics is nonperturbative, the conversations all seem to come back to the hierarchy problem. In principle, strong interactions could solve it. Suppose that the Higgs is a bound state of some new fermion and anti-fermion, with new gauge interactions that are asymptotically free. (Clearly this is a special case of our first two special cases; we need to have gauge fields and massless fermions in our more-fundamental theory – gauge fields to confine and massless fermions so that their bound states are light.) Asymptotic freedom means that the effective interaction between the fermions grows as one moves to lower energy scales and becomes strong at some scale Q, where it confines the fermions into bound states. This is precisely what happens in QCD. Perhaps it is more general. The quadratic dependence on Λ of Eq. 9 is transformed into something smoother. This result comes from the running of the coupling constant from a value $g^2(\Lambda)$ at the cutoff scale, down to scale Q:

$$\frac{1}{g^2(Q)} = \frac{1}{g^2(\Lambda)} + c\log\frac{Q}{\Lambda}. \tag{20}$$

As we run into the infrared, the coupling grows. Suppose Q is the scale where confinement and chiral symmetry breaking is triggered in some unknown way. Masses take their values around this scale. Setting the left hand side of Eq. 20 to zero, and setting $Q = M_H$, we have

$$M_H \sim \Lambda \exp\left(-\frac{1}{cg^2(\Lambda)}\right). \tag{21}$$

The fine tuning of the Higgs system is replaced (hopefully) by some less-fine tuning; we only need to have a weakly-interacting system at some high cutoff scale, and could take that scale to infinity while simultaneously tuning the bare coupling to zero. All masses (except, of course, those of the Goldstone bosons) would have the same overall scale; all would be roughly around the scale where the dynamics became strong. This is what happens in QCD.

Along the way, these new states are ready for discovery at the Large Hadron Collider and hence exciting for experimentalists.

Having raised the possibility of nonperturbative new physics, we are back to the lattice. It is easy to understand why one might want to apply lattice QCD methods to the study of nonperturbative beyond Standard Model candidate models. For many candidates, the field content is similar to QCD: gauge fields and fermions. Asymptotic freedom is a QCD-like feature. If a model were known to be confining it would have a rich spectrum of hadron-like states. A chirally-broken model would have Goldstone bosons ready to be eaten by the W and Z. The physical Higgs would be the analog of the sigma meson. The pseudoscalar decay constant could be used to set the scale for all this new physics, and might be tied to the scale of electroweak symmetry breaking. From their experience with QCD, lattice practitioners might have all the tools to compute the masses, decay constants, and other low energy constants associated with some specific new physics scenario, starting from the Lagrangian.

Perhaps, after this long general introduction, it is time to turn to specifics. But I have to make one more introduction, to set the stage. Lattice models are typically built of gauge fields and fermions. Depending on the gauge group and fermion content, there is a naive expectation for the vacuum structure of these systems, given by the renormalization group. It is useful to pause and, in Sec. II, remind ourselves of this physics. Then, Sec. III is a "review within a review," a set of thumbnail sketches of the many beyond Standard Model systems that have been the targets of lattice investigation. The range of topics in this section is so broad that it is almost impossible to describe coherently. I then review lattice methodology, with attention to issues which arise in the context of beyond Standard Model candidates. This is done in Sec. IV. Most lattice work has involved systems with slowly running couplings. The rest of the review treats this special case in detail. In Sec. V I describe lattice methods used to study slowly-running systems. The division of subject is by method, rather than by specific model. This allows me to illustrate how the generic features of slowly-running systems reveal themselves to lattice probes. Finally, in Sec. VI I describe the status of particular model systems. A few tentative conclusions are presented in Sec. VII.

I should finish the introduction with a few caveats: First, I cut off the literature search on 1 April 2015, but this is an active area of research, and I expect that many things I say will become obsolete. Next, the subject of beyond Standard Model physics is vast. No one could cover it all in a review. I have tried to highlight places where there are lattice stories. And finally, much of the lattice literature appears as short, unrefereed, and often preliminary contributions to the annual International Symposium on Lattice Field Theory series of meetings. I have tried to avoid referring to these articles when a longer, refereed publication is available.

II. A PAUSE FOR CONTEXT: FORMULAS FROM THE RENORMALIZATION GROUP

Imagine that we have an $SU(N_c)$ gauge theory with N_f flavors of massless Dirac fermions in representation R. The gauge coupling is scale dependent. At two loops, the beta function is (Caswell, 1974; Jones, 1974)

$$\beta(g^2) = \frac{dg^2}{d\log\mu^2} = -\frac{b_1}{16\pi^2}g^4 - \frac{b_2}{(16\pi^2)^2}g^6 + \cdots , \quad (22)$$

where

$$b_1 = \frac{11}{3}\,C_2(G) - \frac{4}{3}\,N_f T(R) \quad (23)$$

$$b_2 = \frac{34}{3}\,[C_2(G)]^2 - N_f T(R)\left[\frac{20}{3}\,C_2(G) + 4C_2(R)\right] \quad (24)$$

Here $C_2(R)$ is the value of the quadratic Casimir operator in representation R (G denotes the adjoint representation, so $C_2(G) = N_c$), while $T(R)$ is the conventional trace normalization. μ is a momentum scale.

When the number of fermionic degrees of freedom, basically parametrized by $N_f T(R)$, is small, both b_1 and b_2 are positive (in my conventions). The beta function has a zero, a fixed point, called the Gaussian fixed point, at $g^2 = 0$. The fixed point is infrared unstable; the coupling increases as μ decreases, "under flow into the infrared." It is thought that in this case the coupling increases without bound under flow to the infrared, and it is further presumed that this implies that the vacuum is confining and chirally broken. Examples of such systems are QCD and its near relatives.

When the number of fermionic degrees of freedom is sufficiently large, b_1 changes sign. The scale dependent coupling falls to zero in the infrared. The Gaussian fixed point becomes infrared stable. At long distances the system is believed to be non-interacting, or "trivial," similar to (for example) ϕ^4 theory in dimension $D \geq 4$.

At an intermediate number of fermionic degrees of freedom, it could happen that $b_1 > 0$ and $b_2 < 0$. The system would have an infrared attractive fixed point (IRFP) where the beta function vanishes, $\beta(g_f^2) = 0$. This is often called a "Banks and Zaks (1982) fixed point." Under a change of scale from the ultraviolet to the infrared, the gauge coupling would flow into the fixed point and remain there. In the particle physics literature this is often referred to as "conformal" or "infrared conformal" behavior. We speak of the "conformal window" as the values of N_f, for a given representation, for which the system is neither confining nor trivial.

Inside the conformal window, all correlation functions show a power law behavior at long distances. There are no intrinsic mass gaps, hence no particles. Chiral symmetry is unbroken. This is the analog of the familiar case of a statistical system at a second order critical point. This is certainly nothing like we see in electroweak Nature.

Thus, candidate beyond Standard Model theories must, generally, not be inside the conformal window.

Of course, unless the zero of the beta function is at a very small value of g^2, it is unlikely that the first two terms in the beta function would show it. Why should the higher order terms be small? The smaller the number of fermion degrees of freedom, the larger the value of g^2 becomes, at a place where the beta function vanishes, and the more uncontrolled would be a perturbative calculation. And the beta function is only scheme independent through two loops. To make sense of the story we are trying to tell requires recasting it in the more general language of the renormalization group, outside the narrow statements of perturbation theory, in terms of relevant and irrelevant operators. The investigation of the system would need a better set of tools, perhaps associated with a lattice calculation.

Let us return to that point later. For the time being, just carry the thought: a system might have a quickly running coupling constant, or a slowly running one.

III. THE LANDSCAPE OF MODELS WITH LATTICE INVESTIGATIONS

Let me briefly summarize the particular scenarios for beyond Standard Model physics that have either seen, or might see, lattice studies. I have rewritten this section multiple times, trying to describe them in some kind of coherent order. I do not think I have succeeded in doing this. But I think the problem is that to ask for coherence is impossible. There are many unrelated possibilities for physics beyond the Standard Model. Instead, what I will do is start with QCD, and then move increasingly farther away from it.

A. QCD

A large fraction of lattice QCD literature has beyond Standard Model physics as its back story. The rate for any hadronic weak interaction process – or for some process driven by new physics – typically involves a hadronic matrix element of some operator. These matrix elements are computed on the lattice. This subject is huge; for example, Aoki *et al.* (2014a) is a 179 page review of it. Most of these tests are associated with the flavor structure of the Standard Model, either checking Eq. 8 or looking for modifications to it. The hope, of course, is that the Standard Model rate will show some disagreement with low energy experiment, so revealing the need for new physics.

Some lattice QCD calculations make direct contact with Higgs physics. Lepage *et al.* (2014) recently emphasized the importance of good quality measurements of the strong coupling constant and of the charm and bottom quark masses on precision measurements of the Higgs width.

B. Slightly beyond QCD

A small amount of lattice work has been devoted to systems that are believed to be like QCD. These are systems that are almost certainly confining and chirally broken. Examples of these systems are $SU(N_c)$ gauge theories with $N_c > 3$ and a small number of fermionic degrees of freedom. Usually the physics issues discussed in the literature of these systems are related to QCD rather than beyond Standard Model dynamics. For example, most of the qualitative knowledge about QCD we have comes from the large-N_c expansion of 't Hooft (1974). This knowledge can be – and is being – tested by lattice simulation.

Lucini and Panero (2013) gave a recent review of work on large-N_c QCD. Studies of physical systems with a particle content similar to QCD include the familiar large-N_c limit of 't Hooft (where the fermions are in the fundamental representation and N_f is held fixed and small), or variants such as $SU(N_c)$ gauge theories coupled to a small number of fermions, not in the fundamental representation. The situation with these systems is quite simple to state: large N_c scaling works quite well. A nice example of a comparison, from Bali *et al.* (2013), is shown in Fig. 1. This is a plot of the vector meson mass versus the quark mass, both scaled in units of the square root of the string tension. They have many more examples. Large N_c scaling predicts that meson masses show little dependence on N_c. Decay constants scale as $\sqrt{N_c}$ (for fundamental representation fermions; the scaling is as N_c for two-index representation fermions). At least for $N_c = 3$, and small $N_f = 2-3$, the N_f dependence of masses and matrix elements is small, according to Aoki *et al.* (2014a).

These studies have a role in beyond Standard Model applications. Often, one sees large-N_c arguments quoted in general discussions of composite Higgs systems. For example, one might have a new physics scenario with $SU(N_c)$ gauge fields. One might be interested in the ratio of scale of the masses of excitations to the size of chiral symmetry breaking. This might be parametrized by, say, the pseudoscalar decay constant f_π, which, in turn, might be related to some electroweak parameter, such as the Higgs vacuum expectation value. Often, phenomenological papers just take the known (QCD, real world) numbers and scale them appropriately. This is, of course, just an assumption. These days, one could do a simulation to get the ratio directly. Then, absent some direct experimental measurement, a comparison to large-N_c counting is an appropriate way to put the lattice number into some bigger context.

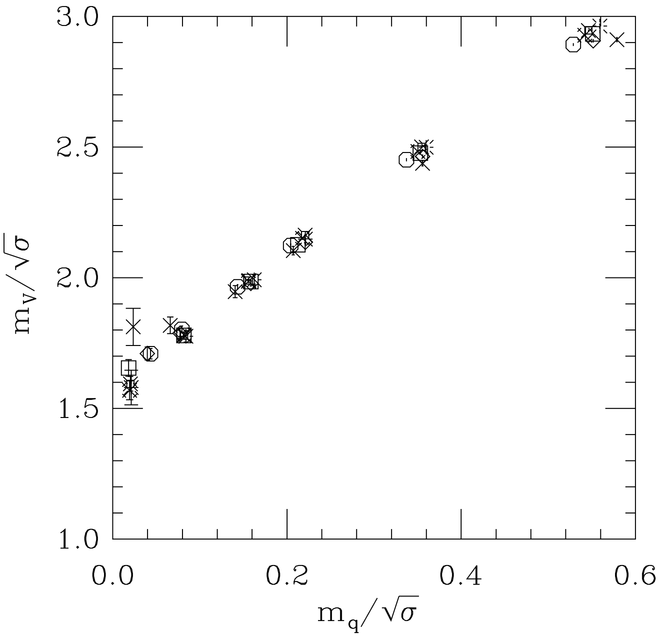

FIG. 1 The vector meson spectrum versus quark mass for different N_c, using data from Bali *et al.* (2013). The data points are crosses for $N_c = 2$, diamonds for $N_c = 3$, octagons for $N_c = 4$, squares for $N_c = 5$, fancy crosses for $N_c = 6$, fancy squares for $N_c = 7$, and bursts for $N_c = 17$,

C. Models with slowly running couplings and the Higgs as a bound state

The oldest scenarios in which the Higgs is not a fundamental scalar date back to Susskind (1979) and Weinberg (1979). The physics idea for generating gauge boson masses is elegant: Imagine having some new gauge fields coupled to massless fermions, whose dynamics is confining and breaks chiral symmetry. (The original model assumed a doublet of quarks, like in QCD, so the pattern of chiral symmetry breaking was $SU(2)\times SU(2) \to SU(2)$.) There will be a set of massless Goldstone bosons. The spontaneously broken symmetry is associated with axial vector currents

$$J^{\mu 5a} = \bar{q}\gamma^\mu\gamma^5\tau^a q \tag{25}$$

and the matrix element of the axial vector current can be parametrized as

$$\langle 0|J^{\mu 5a}|\pi^b\rangle = if_\pi p^\mu \delta^{ab} \tag{26}$$

where in QCD f_π is the pseudoscalar decay constant.

The spontaneous symmetry breaking mechanism that gives mass to the W and Z when they "eat" a Goldstone boson does not require that the eaten Goldstone must be fundamental. Any coupling (L for left-handed, now)

$$\mathcal{L}_{int} = gJ^{\mu La}A^a_\mu \tag{27}$$

will induce a mixing of the gauge boson with the Goldstone through the generation of a term in the vacuum polarization tensor

$$\Pi^{ab}_{\mu\nu} = (\frac{gf_\pi}{2})^2[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}]\delta^{ab}. \tag{28}$$

This is a mass for the vector meson, $m_W = gf_\pi/2$. The vacuum expectation value of the usual Higgs field v in the formula for the gauge boson's mass is replaced by the pseudoscalar decay constant, f_π. To use this dynamics to generate the Standard Model result, we then need to assume that the new dynamics naturally generates a scale $f_\pi = 246$ GeV. More complicated models would scale this equality by an order - unity numerical value. Such a model is referred to, generically, as "Technicolor." The new fermions are called "techniquarks." This subject has an enormous literature. For a review of it, see Hill and Simmons (2003).

The Higgs also generates fermion masses. This is generally awkward to achieve with technicolor. In technicolor models there are just bound states of techniquarks. To generate masses for the quarks and leptons, they must be coupled somehow to the techniquarks. This is commonly done by introducing a new level of dynamics, at some much higher scale, called Extended Technicolor (ETC) interactions (Dimopoulos and Susskind, 1979; Eichten and Lane, 1980). If the ETC gauge bosons are very heavy, they induce four fermion interactions (here, between two capital letter techniquarks and two lower case ordinary Standard Model fermions)

$$\mathcal{L}_4 = (g_E\bar{u}_L\gamma_\mu U_L)\frac{-i}{m^2_{ETC}}(g_E\bar{U}_R\gamma_\mu u_R). \tag{29}$$

Now let the ETC fermions condense. Replacing $U_L\bar{U}_R$ by its vacuum expectation value Σ, we generate a fermion mass

$$m_u = \frac{g_E^2}{m^2_{ETC}}\Sigma. \tag{30}$$

One can make a guess at the scale m_{ETC} using the known quark masses. It must be on the order of 1 - 100 TeV.

This mechanism has a number of phenomenological issues. The first one is that the same interactions that couple two quarks to two techniquarks should also couple four quarks together. This is a problem, because such interactions must be very weak; they give rise to flavor changing neutral currents, which are too large to be consistent with observation if the quarks are to acquire their observed masses. A potential resolution of this problem is called "walking technicolor" (Appelquist and Wijewardhana, 1987a,b; Holdom, 1981, 1985; Yamawaki *et al.*, 1986). To describe it, we have to rewrite Eq. 30 more carefully.

Fermion masses arise from physics at the ETC scale.

Labeling this scale as Λ_{ETC}, a fermion mass is

$$m = \frac{\langle U_L \bar{U}_R \rangle_{ETC}}{\Lambda_{ETC}^2}. \tag{31}$$

The flavor changing neutral current term is also ETC scale physics,

$$\mathcal{L}_{FCNC} \sim \frac{(\bar{s}\gamma_5 d)(\bar{s}\gamma_5 d)}{\Lambda_{ETC}^2}. \tag{32}$$

In Eq. 31 the condensate is scale dependent. $\langle U_L \bar{U}_R \rangle_{TC}$ ought to be a typical electroweak size, say about v^3. Its value at the ETC scale is related to its value at the TC scale by renormalization group running,

$$\langle U_L \bar{U}_R \rangle_{ETC} = \langle U_L \bar{U}_R \rangle_{TC} \exp\left(\int_{\Lambda_{TC}}^{\Lambda_{ETC}} \gamma_m(g_{TC}(\mu)) \frac{d\mu}{\mu} \right) \tag{33}$$

where γ_m is the anomalous dimension of the technifermion mass operator. If the gauge coupling runs very slowly as the energy scale drops from the high ETC scale to the low TC (or electroweak) scale, then γ_m does not change much either, and the soft running expected for a typical QCD-like theory is replaced by a power law. We have

$$\langle U_L \bar{U}_R \rangle_{ETC} = \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}} \right)^{\gamma_m} \langle U_L \bar{U}_R \rangle_{TC}. \tag{34}$$

Slow running is, of course, "walking." Finally, if γ_m is large at the values of g^2's that run slowly, one might be able to have one's cake (generate phenomenologically viable fermion masses) and eat it, too (make Λ_{ETC} large enough to suppress flavor-changing neutral currents).

So many "if's". But the situation for the lattice simulator is pretty well laid out: Does a candidate theory exhibit walking? Is it confining and chirally broken? If so, is its mass anomalous dimension large? If the answer to all these questions is Yes, the perhaps it is a viable technicolor candidate. What is its spectrum and what are its low energy constants?

Technicolor candidates would lie in the confining phase, but very close to the conformal window. To search for them, a first task might be to try to map out the boundary between confining and chirally broken theories, and ones in the conformal window. The relevant parameters are of course the number of colors and the number of flavors of fermions and their representations. Two-loop perturbation theory might be suspect. Higher order terms for the beta and gamma functions have been computed, in $\overline{MS}$ scheme. Pica and Sannino (2011) and Ryttov and Shrock (2011) have used these results to explore the location and properties of the IRFP. My impression of these results is that when the fixed point coupling becomes strong, perturbative predictions for the location of a fixed point and of the value of the critical exponents at the fixed point are not particularly stable.

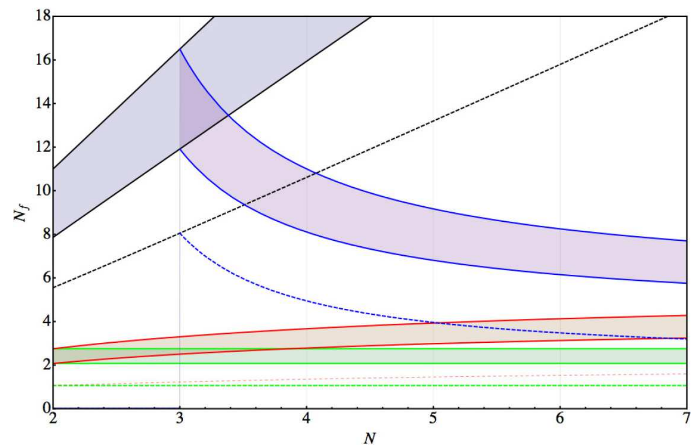

FIG. 2 Conjectured phase diagram (due to Dietrich and Sannino (2007)) for non-supersymmetric theories with fermions in the: i) fundamental representation (blue), ii) two-index antisymmetric representation (purple), iii) two-index symmetric representation (red), iv) adjoint representation (green) as a function of the number of flavors N_f and the number of colors N. The shaded areas depict the corresponding conformal windows from their calculation. The dashed curve represents the change of sign in the second coefficient of the beta function.

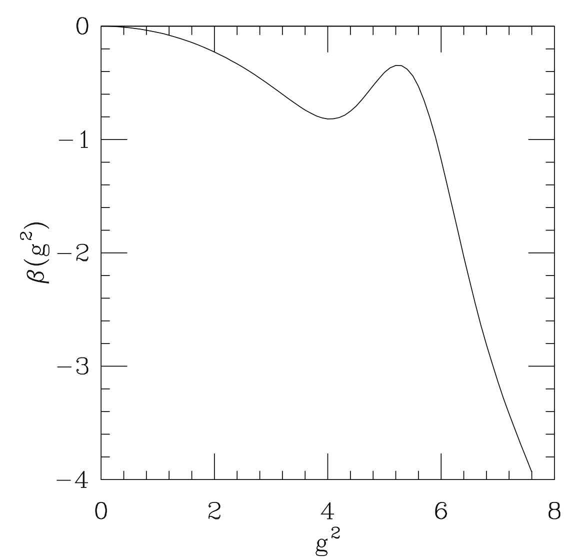

FIG. 3 Artist's conception of a beta function for a walking theory – a negative beta function that approaches the origin, then turns away.

Dietrich and Sannino (2007) combined one-loop running with expectations from solving Schwinger-Dyson relations, to make a map of the $N_c - N_f$ plane for various representations of fermions. Figure 2 shows their prediction for a phase diagram. This figure has served as the target for many lattice calculations.

A cartoon of the expected coupling constant evolution of a walking theory is shown in Fig. 3. The beta function starts out negative, then bends toward zero. Walking occurs at the coupling where the beta function is smallest. At the inflection point, something must make the beta function bend over steeply. What could that be? One possibility (Appelquist *et al.*, 1996; Miransky and Yamawaki, 1997) is that chiral symmetry breaking occurs at a coupling near the cusp. Some of the fermions condense into colorless pions and decouple from the gauge bosons, reducing the effective number of fermionic degrees of freedom and letting the coupling grow. Self-consistent (Schwinger-Dyson) calculations support this scenario, but can they be trusted?

One can imagine theories whose beta function looks like Fig. 3. Several toy models of walking theories have been proposed (Aoki *et al.*, 2014b; Nogradi, 2012). Another way to turn the beta function over might involve introducing extra external scales. (This is different from the technicolor scenario, where the scale of the turning appears dynamically.) The simplest possibility is a system with many flavors of massive fermions. At momentum scales that are much greater than the fermion masses, the fermions behave as if they are massless. But as the energy scale falls below the fermion masses, they decouple from the gauge fields. The effective number of fermions in the beta function changes. One might set the number of flavors large enough, that at sufficiently high energy, the coupling might be arranged to show an IR flow toward a fixed point. As the energy scale drops, the fermions decouple, the coupling runs differently (faster) and the true long distance behavior would cross over to some strongly coupled theory – almost certainly confining and chirally broken. Perhaps parameters could be tuned to produce walking. (See Brower *et al.* (2014) for a recent study of this.)

(This description might be too poetic. Recall (Rodrigo and Santamaria, 1993) that in $\overline{MS}$ schemes, where coupling constants are mass-independent, one has to treat theories with different N_f's as effective field theories and match the running couplings at a scale μ equal to the fermion mass. With an nth order beta function, this has to be done at order $O(n-1)$. The coupling constant steps discontinuously at thresholds, rather than showing a smooth behavior like Fig. 3.)

A more serious issue with Fig. 3 is that it suggests that the physics of strong coupling is described by a single coupling constant. This may not be the case. In QCD, for example, the coupling constant monotonically strengthens as the momentum scale falls. At long distances, the coupling constant loses its utility as a useful quantity, in the sense that one cannot use it to parametrize calculations of interesting observables.

Technicolor has additional issues. Most of them are difficult to quantify because the dynamics of technicolor is strongly interacting.

The range of quark masses is wide, from a few MeV for the up and down quarks, to 173 GeV for the top quark. Complicated constructions seem to be needed to generate all these masses.

The technipions might be eaten by the W and Z, but where are the other particles, the technirho and beyond? One typically imagines that the scale of non-chiral physics is about $4\pi f$. That value is in the range of LHC searches, and, so far, they have not been seen. (Halkiadakis *et al.* (2014) has limits.)

Technicolor has issues with precision electroweak measurements. Usually, the $S-$ parameter is too large. However, is it really possible to compute the technicolor vacuum polarization contribution to the gauge bosons in a reliable way? The literature often falls back on analog QCD calculations, suitably rescaled. [See, for example, the discussion in Contino (2010) and Peskin and Takeuchi (1992).]

In technicolor models, electroweak symmetry breaking does not involve a Higgs boson. If technicolor dynamics were sufficiently QCD-like, one would expect to see a scalar state in the spectrum, in analogy with the situation in QCD. There, the scalar state is the $f_0(500)$, a light ($M \sim 400-550$ MeV) broad ($\Gamma \sim 400-700$ MeV) resonance. The observed Higgs is narrow, so this is an issue for technicolor phenomenology. But, if the candidate theory is not very QCD-like, can QCD analogies be trusted? All of these questions could be addressed by lattice simulations.

I think that technicolor is the only beyond Standard Model scenario that has enough of a lattice literature to justify a detailed review. The subject turned out to be filled with surprises. After completing this survey section, I will return to a detailed discussion of results for these systems.

D. Composite Higgs: the Higgs as a pseudo - Nambu - Goldstone boson

In Sec. III.C electroweak symmetry breaking occurs when the techniquarks form a condensate which transforms non-trivially under $SU(2) \times U(1)$. In such models the condensate scale is the weak scale, around 246 GeV. Another alternative is to arrange that the new physics generates a condensate, but the condensate preserves $SU(2) \times U(1)$. Then, electroweak symmetry breaking could occur at a scale which is much lower than the condensate scale. (From our point of view, the new physics is at a higher scale than the electroweak symmetry breaking scale.) A scalar excitation present at the high scale would develop a vacuum expectation value, and some of its degrees of freedom would eaten by the W and Z.

The earliest discussions of this approach go back to Banks (1984); Dugan *et al.* (1985); Georgi (1986); Georgi and Kaplan (1984); Georgi *et al.* (1984); Kaplan and Georgi (1984); and Kaplan *et al.* (1984). The idea was to make the Higgs a Goldstone boson corresponding to a spontaneously broken global symmetry of a new strongly interacting sector. Gauge and Yukawa interactions of the Higgs explicitly violate the global symmetry and generate a potential, including a mass term, for the Higgs.

Let us approach the issue in a top-down way: We have some new dynamics at a scale Λ. It encodes a global symmetry G which is spontaneously broken to a subgroup H, and the Goldstones are described by a G/H nonlinear sigma model. Some of them are destined to become the real Higgs doublet. At this point, all of the components are massless. They have non-renormalizable interactions parametrized by a scale f, where $4\pi f$ is presumed to roughly equal Λ (the usual connection between chiral and non-chiral dynamics).

Electroweak interactions are introduced by gauging an $SU(2) \times U(1)$ subgroup of G. Electroweak gauge interactions, and the interactions of Standard Model fermions with the Goldstones, explicitly break the shift symmetry. They generate a potential for the Goldstones which has a nontrivial minimum. The task of the model builder is to do this without re-introducing quadratic divergences along the lines of Eq. 9. The literature refers to these models as "composite Higgs models."

Avoiding a mass shift like Eq. 9 is a nontrivial task. One mechanism which could succeed in principle was the idea of collective symmetry breaking (Arkani-Hamed *et al.*, 2002a,b). The resulting systems are called "little Higgs" systems. In these models, electroweak interactions are introduced by gauging a subgroup which is a direct product of several factors, $G_1 \times G_2 \times \ldots$ in such a way that each G_i commutes with a subgroup of G that acts non-linearly on the Higgs. This means that if any one G_i is gauged, the unbroken global symmetry insures that the Higgs remains massless. Only when the full product of G_i's is gauged does the Higgs cease to be a Goldstone boson. The consequence of this dynamics is that the induced mass of the Higgs is proportional to a product of all the gauge coupling constants corresponding to the different G_i factors. The terms in Eq. 9 are not of this form and so they are absent.

Most of the literature of composite Higgs models confines itself to the low energy effective theory of the would-be Goldstones. Explicit examples of such actions may be found, for example, in Azatov *et al.* (2012); Buchalla *et al.* (2015); Contino *et al.* (2013, 2010); and Giudice *et al.* (2007). General surveys, such as Bellazzini *et al.* (2014), organize their discussion in terms of the ratio ξ of two dimensionful parameters, f, as described above, the scale of the nonlinear sigma model, and the Higgs vacuum expectation value $v = 246$ GeV, $\xi = v^2/f^2$.

In partial compositeness scenarios, the top quark and the Higgs share a common dynamics. One such scenario is due to Kaplan (1991): the top quark couples linearly to strong-sector baryons, which, in effect, allows it to couple to the composite Higgs as well. This scenario is used to generate fermion masses. Here the interaction involves the coupling of a Standard Model fermion ψ to a composite operator O,

$$\mathcal{L} = \lambda \bar{\psi} O + h.c. \tag{35}$$

In the original Kaplan (1991) version of this idea, the composite operator O is a three-quark technibaryon so $\mathcal{L}$ is a four-fermi interaction, whose origin is (perhaps) some ETC theory at a yet higher scale. Diagonalizing the resulting mass matrix gives states which are linear superpositions of the fermion and the technibaryon – hence the phrase "partial compositeness." The mixing of a Standard Model fermion with a composite is also used to generate part of the effective potential for the Goldstones.

This is a huge field. However, unlike the systems described in Sec. III.C, it has a tiny lattice literature. Why that is so I will come back to, below. What it means is that this section has a different orientation from the rest of the review. There, the story is "Here is some physics; here is what lattice simulations showed," and the intended audience is (mostly) physicists who did not do the simulations. For this section, my goal is to try to convince lattice physicists that there are interesting issues which can be addressed on the lattice.

A complete technical analysis of the issues facing a lattice calculation remains to be written. Here is my attempt at an overview.

First, the Standard Model gauge group must be a subgroup of the unbroken group H, otherwise the Standard Model gauge fields would develop masses on the scale of $g\Lambda$. Next, phenomenology needs to know the couplings in an effective Lagrangian. Given a specific choice of an ultraviolet completion of a composite Higgs model, the situation might be exactly like QCD: Introduce Σ, the nonlinearly-realized field, $\Sigma \sim \exp(-i\tau^a\pi^a/f)$ (for generic generators τ^a, Goldstone fields π^a and decay constant f). The goal of a lattice calculation would be to start with the ultraviolet completion and compute the effective potential of the nonlinear sigma model, $V_{eff}(\Sigma)$. It is necessary to specify the electroweak quantum numbers of the fields in Σ. For a viable model, four of them have to self-assemble into a complex $SU(2)$ doublet, the Higgs. Typically there will be members of other $SU(2)$ multiplets. The members of other nonsinglet irreducible representations should not condense. (For lattice QCD practitioners, it is standard to assume that the vacuum can be rotated into the identity in flavor space.)

Generically, $V_{eff}(\Sigma)$ receives contributions from the Standard Model gauge bosons and fermions. The gauge boson part comes from the part of the lowest order chiral Lagrangian which is quadratic in the gauge fields, from

$$\mathcal{L} = \frac{f^2}{4} \mathrm{Tr}\, |D_\mu \Sigma|^2 \tag{36}$$

where D_μ is a covariant derivative. The Lagrangian includes a $g^2 W^2 \pi^2$ vertex, which in turn, generates a quadratically divergent contribution to the potential. Typically, though, the energy scale is f, not Λ:

$$V_{eff} = c\frac{g^2 f^2}{16\pi^2}\pi^2 + \ldots . \tag{37}$$

The constant c is calculable on the lattice. The procedure is similar to that for the $\pi^+ - \pi^0$ mass difference in QCD, via the Das, Guralnik, Mathur, Low, Young sum rule (Das *et al.*, 1967). It uses the the difference of the vector and axial current correlators

$$\begin{aligned}\Pi_{\mu\nu}(q) &= \int d^4q \exp(iqx)\langle J^L_\mu(x) J^R_\nu(0)\rangle \\ &\equiv (q^2\delta_{\mu\nu} - q_\mu q_\nu)\Pi^{LR}_T(q^2) + q_\mu q_\nu \Pi^{LR}_L(q^2).\end{aligned} \tag{38}$$

J^L_μ and J^R_μ are the left and right currents $\bar{\psi}\gamma_\mu(1 \pm \gamma_5)\psi$, so the object in the integral is the appropriate difference. The coefficient c is proportional to the integral

$$c \sim \int_0^\infty dq^2 q^2 \Pi^{LR}_T(q^2). \tag{39}$$

Several lattice groups (Appelquist *et al.*, 2011b; Boyle *et al.*, 2010; Shintani *et al.*, 2008) have published calculations of the $\pi^+ - \pi^0$ mass difference using this observable. Contino (2010) gives a pretty explicit description of what to do for a composite Higgs model.

The sign of c can be inferred in advance, without the need for a lattice calculation. This is the phenomenon of vacuum alignment, first described by Peskin (1980) and Preskill (1981), and related to Witten's inequality (Witten, 1983). The contribution of gauge bosons is positive, so the gauge symmetry remains unbroken. Something else must break it. But sometimes, one can use the calculation as an estimate for the masses given by electroweak symmetry breaking to the uneaten, now-pseudo Goldstones.

Typically, the negative term in V_{eff} comes from the fermions. Models vary in details, but many involve partial compositeness: the Standard Model fermions mix with new physics baryons, which in turn can couple to the Higgs.

In published models, the Yukawa couplings are numbers and so the derived value of v depends on them. I am not sure if their actual values are accessible to a lattice calculation, or not. However, they are running couplings, and their anomalous dimensions are related to those of the technibaryon operators. For example, Contino (2010) rewrites Eq. 35 as

$$\mathcal{L} = \sum_n \lambda \langle 0|O|\chi_n\rangle \bar{q}\chi_n + h.c. \tag{40}$$

introducing a tower of composite fermions χ_n. To the lattice practitioner $\langle 0|O|\chi_n\rangle$ is just a baryon creation amplitude. Lattice techniques could be adapted to find its anomalous dimension. There is a recent discussion by Golterman and Shamir (2015) of lattice issues involved in computing partial compositeness observables. The subject needs more theoretical analysis.

Interesting ultraviolet completions require QCD - like theories with different numbers of colors, or quarks in non-fundamental representations, or both. The Littlest Higgs model (Arkani-Hamed *et al.*, 2002a) relies on the non-linear sigma model $SU(5)/SO(5)$. A possible ultraviolet completion is any confining gauge theory with five Majorana fermions in some real representation. The most economical way to realize this scenario is an $SU(4)$ gauge theory, where the two-index antisymmetric representation (AS2) is real. The $SU(5)/SO(5)$ sigma model is also central to the more recent composite-Higgs models of Ferretti (2014); Ferretti and Karateev (2014); and Vecchi (2013). In particular, Ferretti and Karateev (2014) makes the case why the $SU(4)$ theory with AS2 fermions is the most attractive candidate within this approach, whereas Ferretti (2014) elaborates on the phenomenology of this composite-Higgs model. The models of Ferretti (2014) and Ferretti and Karateev (2014) require fermions in the fundamental representation in addition to the AS2 ones, in order to give the top quark a mass via partial compositeness.

Another ultraviolet completion is Barnard *et al.* (2014), with an $Sp(2N)$ gauge group and two representations of fermions. The global symmetry breaking pattern is $SO(6)/SO(5)$.

The pattern of chiral symmetry breaking can be different from QCD. When the fermions in the ultraviolet completion are Dirac fermions in a complex representation, parity and charge conjugation are good symmetries, and the Goldstone bosons associated with chiral symmetry breaking are all pseudoscalars. The Higgs is a scalar, so there is apparently no way it can be a Goldstone boson. However, the fermions associated with the new dynamics could belong to a real or to a pseudoreal representation. Then there is no a-priori distinction between a scalar bilinear or a pseudoscalar one. The quantum numbers will be determined after the fact when the Standard Model quantum numbers of the appropriate fields are assigned. For consistency, the condensate must be a scalar.

The situation was first described by Kosower (1984); Peskin (1980); and Preskill (1981). When the fermions make up a complex representation of the gauge group, the expected pattern of chiral symmetry breaking is $SU(N_f) \times SU(N_f) \to SU(N_f)$. With N_f Dirac fermions (or $2N_f$ Majoranas) in a real representation of the gauge group, the symmetry breaking pattern is $SU(2N_f) \to SO(2N_f)$. With a pseudoreal fermion representation, it is $SU(2N_f) \to Sp(2N_f)$.

There is already a small lattice literature on these systems: see Damgaard *et al.* (2002) and its citations. These early papers observed the pattern of chiral symmetry breaking through regularities in the spectrum of Dirac eigenvalues. Hietanen *et al.* (2014a) and Lewis *et al.* (2012) recently studied the spectroscopy of $SU(2)$ gauge fields and $N_f = 2$ fundamentals, a pseudoreal representation. DeGrand *et al.* (2015) did similar work for $SU(4)$ with $N_f = 2$ two-index antisymmetric (real representation) fermions. There is more to do. Direct calculations of the $V_{eff}(\Sigma)$ are an obvious target for future work.

Finally, why is the lattice literature for this subject so small? I can think of several reasons.

First, lattice simulations are performed discretizing ultraviolet complete Lagrangians. Most of the literature of composite Higgs is concerned with its low energy effective theory. Until recently, there were few examples of ultraviolet completions. For example, the two (very complete) review articles of Bellazzini *et al.* (2014) and Perelstein (2007) total about eighty pages of print, but their combined discussion of ultraviolet completions is only about three pages long.

Second, many of the published ultraviolet completions are difficult venues for lattice simulations: they involve theories in more than four dimensions, or supersymmetry, or both.

Third, the ultraviolet completions typically involve gauge groups with $N_c \neq 3$, or fermions in higher dimensional representations, or Weyl or Majorana fermions rather than Dirac fermions. New code must be written. This should not be a barrier, but historically, it has been.

Fourth, some of the key calculations require lattice fermions with good chiral properties, at least for the valence quarks. An example is $\Pi_T^{LR}(q^2)$. The matching factors converting lattice to continuum regularization for the vector and axial vector currents are different unless the lattice action can support a Ward identity pinning them together.

And last, particularly for some versions of the partial compositeness scenario, one needs to simulate several representations of fermions at once. (Of course, there are interesting physics questions for these systems on their own.)

So far, there is not enough lattice work in this area to justify a review. Perhaps in a few years there will be.

E. Composite dark matter

Not much is known about dark matter other than it exists, that it is long - lived, that its density is about a quarter of the mass density needed to close the Universe, and that it is dark, lacking electromagnetic interactions. In some cases dark matter candidates naturally arise in other models of beyond Standard Model physics: for example, in supersymmetric extensions of the Standard Model that have R-parity as a symmetry, the lightest supersymmetric partner is a dark matter candidate. But there are also many models for dark matter with no direct extension to other physics issues. There is a small speculative literature arguing that dark matter could be strongly interacting, a sort of hidden version of QCD, coupling somehow weakly to Standard Model particles. Early references include Barr *et al.* (1990) and Nussinov (1985) and the recent lattice study by Detmold *et al.* (2014a) of one candidate system lists about thirty phenomenological papers. Not surprisingly, there are lattice studies of composite dark matter models. The literature I know of includes studies of $SU(2)$ gauge theories coupled to $N_f = 2$ flavors of fundamental fermions (Detmold *et al.*, 2014a,b; Hietanen *et al.*, 2014a,b; Lewis *et al.*, 2012) and $SU(4)$ gauge theory with quenched fundamental representation fermions (Appelquist *et al.*, 2014b, 2015a). Most of the work is about the spectroscopy of these systems, mostly their baryon spectroscopy because one is interested in knowing what is likely to be the most stable particle. There is also some discussion about matrix elements appropriate for dark matter detection. [For examples of such a calculation, see Appelquist *et al.* (2013, 2015b).]

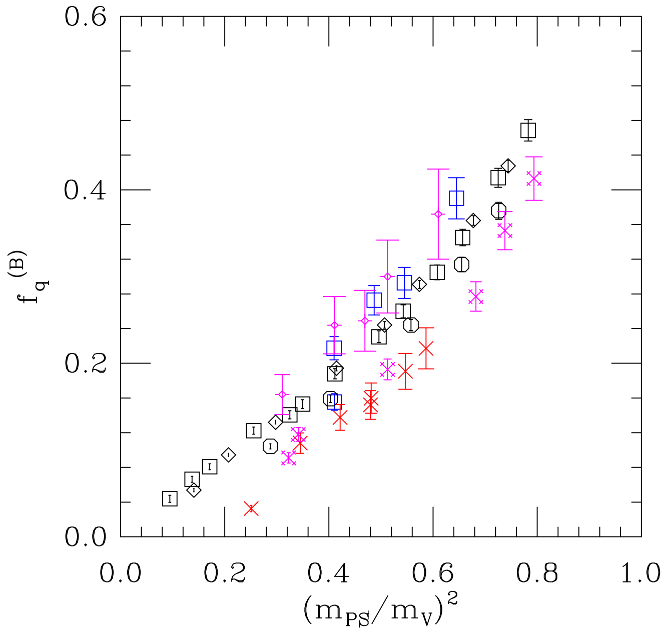

FIG. 4 The quantity $f_q^{(B)}$ defined in Eq. (42), plotted vs the squared pseudoscalar to vector meson mass ratio $(m_{PS}/m_V)^2$. Data shown include quenched fundamental SU(3), SU(5) and SU(7) (black squares, diamonds, octagons), dynamical SU(3) (blue squares), and dynamical SU(4) AS2 (red crosses). This data is from DeGrand *et al.* (2015). Also plotted in purple are results from Appelquist *et al.* (2014b) for quenched fundamental SU(4), for bare gauge coupling $\beta = 11.5$ (fancy diamonds) and $\beta = 12.0$ (fancy crosses).

In the models which have been studied to date, the dominant nuclear interaction with a dark matter particle is through Higgs exchange. The interesting quantity is the matrix element between a nucleon a through its constituent quarks Q and the dark matter baryon B through its constituent fermions q. Schematically, this quantity is proportional to

$$M_a = \frac{y_Q y_q}{m^2_{Higgs}} \sum_q \langle B|\bar{q}q|B\rangle \sum_Q \langle a|\bar{Q}Q|a\rangle\,. \qquad (41)$$

The factors y_Q and y_q are Yukawa couplings. The expectation values in the expression are the QCD sigma term and its dark matter analog. There are several lattice calculations of this quantity, typically given in terms of

$$f_q^{(B)} \equiv \frac{m_q}{M_B}\frac{\partial M_B}{\partial m_q} = \frac{m_q}{M_B}\langle B|\bar{q}q|B\rangle. \qquad (42)$$

A recent compilation (DeGrand *et al.*, 2015) is shown in Fig. 4. (Not shown is data by Detmold *et al.* (2014b) which is for an $SU(2)$ system and is presented over a tiny range of fermion masses, with similar results.) It appears that $f_q^{(B)}$ is reasonably independent of the underlying dynamics.

I believe that dark matter phenomenology does not demand technicolor - like dynamics (a slowly running coupling constant) and so to the lattice practitioner, these systems are QCD - like and are reasonably easy to study. The issue, of course, is motivation for any particular model in the absence of an experimental signal.

F. Dilatonic Higgs

Another possibility to generate a light Higgs is to somehow tune the ultraviolet theory so that its couplings are close to some critical value, where its correlation length diverges. A diverging correlation length is the same thing as a very light particle, which would be a candidate to replace the Higgs. Of course, it also brings along new physics at some higher scale.

The "homework example" for these systems is the mean field behavior of an $O(N)$ spin model with a potential $V(\phi) = a_2\phi^T\cdot\phi + a_4(\phi^T\cdot\phi)^2$, With $a_2 > 0$, the $O(N)$ symmetry is unbroken and all fields have a squared mass $\sim a_2$. The symmetry is spontaneously broken for $a_2 < 0$, the Goldstone bosons are massless and the Higgs has a squared mass $m_H^2 \sim -a_2$. At criticality, where $a_2 = 0$, the Higgs mass also vanishes. That is the state we are interested in.

(Of course, in a better treatment, all masses vanish at criticality where the system experiences scaling behavior. But close to criticality, there should be a light state.)

I used the $O(N)$ model rather than a $Z(2)$ model, where a massless state also appears at criticality, in order to make the point in the symmetry-broken phase there are Goldstone bosons; the scalar channel will have a two-particle branch cut in addition to a Higgs pole. A numerical simulation will have to disentangle the branch cut from the desired signal.

Similar behavior is expected in $N_f = 2$ QCD with massless quarks. Precisely at the critical temperature, the system should exhibit scaling, with power law decay for all correlation functions. Slightly away from criticality, this branch cut behavior should dissolve into a set of resonances, one of which, an isoscalar scalar meson, will be very light.

I have not found any definitive study of such a state in the finite temperature QCD literature. These calculations are technically quite demanding. There are two (related) problems. The first one is that the state has the quantum numbers of the vacuum. A mass M is determined by fitting a correlation function of a source and sink operator separated by a distance t to the functional form

$$\langle O(t)O(0)\rangle \sim A + B\exp(-Mt) + \ldots \qquad (43)$$

The constant term A is only present when the states created by O have vacuum quantum numbers, otherwise it vanishes. When it is nonzero it dominates the mass-dependent term when t becomes large. The second issue is that the calculation involves disconnected diagrams. Think of the state as a $\bar{q}q$ pair. The correlator has a contribution where the source pair annihilates into gluons, which then reconvert at the sink. These correlators are intrinsically noisy. There is, however, one related observation. Cheng *et al.* (2011) have published measurements of the isotriplet scalar screening mass in finite temperature QCD. It shows a dip near the transition temperature, while always remaining greater than the pseudoscalar mass.

The particle physics literature refers to these states as "dilatons." A dilaton is a pseudo-Nambu Goldstone boson associated with scale symmetry breaking. The divergence of the dilatation current is the trace of the energy-momentum tensor. This trace is anomalous in a massless gauge theory and its size is proportional to the beta function. The Goldstone boson which comes from spontaneously broken dilatation symmetry (i.e. the scale put in by renormalization) has a mass proportional to the anomaly, and thus to the beta function. A walking theory has a small beta function, hence a light dilaton. When the fermions acquire masses they couple to the dilaton in proportion to their masses, so the scalar couples to fermions like a Higgs.

Such states have a long citation trail [two of many early papers are Gildener and Weinberg (1976) and Yamawaki *et al.* (1986)] and the idea continues to appear as a beyond Standard Model possibility. They have a somewhat fraught phenomenology. The issue is that our world is not conformal. If the world of very high energy is conformal, there must be a crossover to its behavior, and it is quite difficult to keep such a Higgs from moving up in mass to the scale where the crossover begins. [Compare the discussion in Bellazzini *et al.* (2013) and Marques Tavares *et al.* (2014).]

In the lattice literature, the words "dilatonic Higgs" seem to me to be a shorthand for having a confining and chirally broken system in which there is a scalar particle which is parametrically lighter than the rest of the spectrum (apart from the Goldstones, of course). The first question is to determine whether the system is confining and chirally broken, or not. Such light states could appear in the symmetry-restored phase (as in the $O(N)$ example). They could also appear in a system which is conformal in the zero fermion mass limit. There, all masses fall to zero as $m_H = A_H m_q^\gamma$, but the A_H's can be different in different channels. Then, where does the a 0^{++} state fit into the spectrum, either at nonzero fermion mass, or as the fermion mass is taken to zero? The value at nonzero fermion mass is what is actually measured in a simulation. The zero mass case is an extrapolation. Finite simulation volume is another issue when light particles are involved. It had better be the case that the Higgs candidate is much lighter than everything except the would-be Goldstone modes. Even then, there are other issues: an important one is the ratio of f_π to other mass values. If f_π is set by electroweak physics, the other states must be out of reach of where the LHC has already scanned, or the model is not viable. And if the light state is going to replace the Higgs, its branching ratios had better be close to Standard Model values. I will return to the discussion of lattice results for these states in Sec. VI.

G. Fundamental scalars on the lattice

Lattice studies of strongly coupled scalar fields have a long history, going back into the 1980's. There was a literature about self-interacting scalar fields, of scalar fields interacting with gauge fields, and of scalars interacting with fermions. A major area of research in that era, which extended up to the discovery of the Higgs, was constructing upper and lower bounds on the Higgs mass. My discussion of the issues, around Eqs. 12-15, was quite naive. When the Higgs gets close to its upper bound, or to its lower bound, its interactions (either with itself, or with the top quark) become strong. A perturbative story is suspect. Of course, people were hopeful – perhaps the Higgs would not be found, or it might have been pushed to a mass value where new physics could be nearby. They wanted to make nonperturbative bounds, to get a better indication of where new physics might be. Two papers which studied this, from just before the Higgs discovery, are Fodor *et al.* (2007) and Gerhold and Jansen (2010).

Now that we have the Higgs, the story might be different: suppose there are heavier generations of fermions. Does the observed Higgs mass constrain their masses? Bulava *et al.* (2013) say Yes, and that the maximum allowed mass of a fourth generation quark is about 300 GeV.

In the late 90's, several lattice groups studied the scenario of electroweak baryogenesis (Kuzmin *et al.*, 1985). If the electroweak sector had a strongly first order transition, the metastability associated with the transition would lead to thermal non-equilibrium. This is one of the necessary Sakharov conditions for baryogenesis. The other conditions (baryon number violation, C and CP nonconservation) also exist in the Standard Model, so in principle, the generation of the baryon asymmetry in the early Universe could arise from electroweak interactions. A series of beautiful lattice calculations showed that the transition was a crossover for Higgs masses above 72 GeV. Even at the time, this was already inconsistent with experiment, ruling out the scenario. A recent conference proceedings (Laine *et al.*, 2013) has references to the original literature. Knowing the Higgs mass allows one to refine the calculations, and perhaps constrain other models of baryogenesis. See D'Onofrio *et al.* (2014).

Finally, there is a small recent literature of lattice simulations of the gauge-Higgs sector of the Standard Model itself. The formal issue is that the Standard Model is a gauge theory. Observables must be gauge invariant, gauge invariant observables are represented by composite operators, and composite operators can have very different spectral properties than simple ones. Take QCD as an example. Maas (2013); Maas and Mufti (2014, 2015) have studied these issues. My interpretation of their results is that the weakly coupled Standard Model is still what we think it is, even on the lattice, but that it could have been different.

H. Lattice-regulated supersymmetry

Phenomenological supersymmetric extensions of the Standard Model are, of course, completely perturbative. No lattice calculations are needed to make predictions. But there is also a small literature devoted to lattice-regulated supersymmetry. These are simulations of $\mathcal{N} = 1$ and $\mathcal{N} = 4$ supersymmetric Yang - Mills theory in space - time dimension $D = 4$ and various models in $D = 2$. These papers are not about phenomenology, per se. Rather, the questions are along the line of "does the lattice system exhibit features of supersymmetry?"

People want to put supersymmetric theories on the lattice because many of the nonperturbative features which appear in ordinary (non-supersymmetric) theories, such as spontaneous chiral symmetry breaking, confinement, magnetic monopole condensation, strong coupling - to - weak coupling duality (to name a few) were first studied in a supersymmetric context. It would be useful to have a nonperturbative formulation of these specific systems, which checks these calculations.

Of course, one has somehow to evade the problem that supersymmetry is an extension of the usual Poincaré algebra and is broken completely by naive discretization. However, this is a problem that has been mostly solved.

A good place to begin a literature search is with the review article by Catterall *et al.* (2009a), and with citations to it.

$\mathcal{N} = 1$ probably has the greater literature. This is a system of adjoint Majorana fermions coupled to gauge fields. It is simulated with chiral lattice fermions, such as domain wall fermions, using the rational Hybrid Monte Carlo algorithm (see Eq. 55, below). The supersymmetric limit is the limit of vanishing fermion mass. Some representative papers include Bergner *et al.* (2013); Endres (2009); Fleming *et al.* (2001); Giedt *et al.* (2009); and Kim *et al.* (2011).

$\mathcal{N} = 4$ is much trickier. The issue is, not surprisingly, the scalars. An intricate construction allows one to simulate a theory with a single scalar supercharge. The other fifteen supercharges of $\mathcal{N} = 4$ are broken by the lattice discretization. It is believed that the situation is like the loss of rotational invariance in a usual lattice system: the breaking of the symmetry is due to irrelevant operators. This means that these supersymmetries are recovered in the continuum limit. Exactly how to do that in an efficient way is at present a research problem. A recent paper, Catterall *et al.* (2014), discusses this issue. It has citations to earlier literature. Alternative formulations of lattice supersymmetry include Hanada *et al.* (2014); Honda *et al.* (2013, 2011); Ishii *et al.* (2008); Ishiki *et al.* (2009a,b).

I. Gauge bosons and matter in space-time dimensions $D > 4$

Higher dimensional extensions of the Standard Model have an enormous and rich continuum literature. Lattice studies, however, are very sparse. The fundamental issue is that gauge couplings in $D > 4$ are dimensionful, and hence the systems are non-renormalizable. In fact, the extra-dimensional gauge theory has to be understood only as a low energy excitation of some more fundamental theory. At its cutoff scale (Λ in energy, or for us on the lattice, the lattice spacing a) the effective description breaks down and details of the underlying theory become important. Typically, the systems of interest have compact extra dimensions. Calling their scale L, the effective description only makes sense if the compactification length L is large compared to the cutoff, or $L\Lambda \gg 1$.

Most of the work I know about is in $D = 5$, with $SU(N)$ gauge fields and small N (mostly $N = 2$). The fifth dimension is compact, sometimes orbifolded (Irges and Knechtli, 2007, 2014; Irges *et al.*, 2013; Knechtli *et al.*, 2014), sometimes not (de Forcrand *et al.*, 2010).

Many of the simulations introduce one lattice spacing for the four large dimensions and a different lattice spacing a_5 for the fifth dimension. One cannot take both cutoffs to zero; power divergences appear that cannot be absorbed into a finite number of counterterms. But one can tune one of the dimensions to zero, holding the others fixed. Then one can explore the phase diagram of the system, looking for critical points or lines. At these places, the correlation length ξ diverges, in units of the lattice spacing a, $a/\xi \to 0$. In that sense, the lattice spacing is removed, and a four-dimensional theory is left. Slightly off the critical line, there is a four dimensional theory, but with extra irrelevant operators.

Like lattice supersymmetry, the question here seems to me to be more "Can I make it work?" rather than "What can I do with it?"

IV. LATTICE METHODOLOGY

A. A lightning introduction to lattice calculations

Before we go on, we have to recall how lattice calculations are performed. Good textbooks, for example DeGrand and DeTar (2006) and Gattringer and Lang (2010), provide a detailed introduction to the subject. What follows is a synopsis, the bare minimum the reader who does not do lattice simulations of beyond Standard Model systems needs to know to have a context for the results.

Imagine that we are interested in studying some quantum field theory with lattice techniques. We discretize the system, that is, we replace space and time by a grid of points. We then define field variables that live on the links or sites of the lattice, and construct an action that couples them together. We do this in some way that preserves as many symmetries as possible. Preserving gauge symmetry is vital to maintain current conservation, so nearly all lattice calculations use gauge invariant actions and integration measures. Space-time symmetries and chiral symmetries may be more problematic to enforce, so let us defer a discussion of them for a while. The lattice theory is then an effective field theory defined with an UV cutoff, the lattice spacing a. One can think of this cutoff as being roughly equivalent to an ultraviolet momentum cutoff $\Lambda \sim 1/a$.

The lattice path integral is used as a probability measure to generate configurations of the field variables. For example, the functional integral (or partition function) for a lattice bosonic field ϕ_n has the form

$$Z = \int [d\phi] \, \exp[-S(\phi)] \tag{44}$$

where $S(\phi)$ is some lattice action and $[d\phi] = \prod_n d\phi_n$ is an integration over the values of the field on each lattice site n. Any physical observable $\mathcal{O}$ can be expressed as a function of the field ϕ. Its formal expectation value is

$$\langle \mathcal{O} \rangle = \int [d\phi] \, \mathcal{O}(\phi) \exp[-S(\phi)] \Big/ \int [d\phi] \, \exp[-S(\phi)]. \tag{45}$$

This is just the average of the observable with respect to the measure

$$P(\phi) \propto \exp[-S(\phi)]. \tag{46}$$

Thus, the average value of some observable is an ensemble average over the configurations of field variables. In a lattice calculation, the generation of configurations is done numerically, by some stochastic algorithm. Monte Carlo methods generate a sequence of N random field configurations $\phi^{(k)}$ with a probability distribution given by Eq. 46. The expectation value of the observable is then just the simple average of the observable over the ensemble of configurations:

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{k=1}^{N} \mathcal{O}(\phi^{(k)}). \tag{47}$$

The uncertainty in the observation typically scales like $1/\sqrt{N}$. Lattice correlation functions are then compared to some theoretical model to extract the values of desired observables.

Generic correlation functions measured in a lattice simulation in a finite simulation volume usually show an exponential falloff with distance, characterized by a correlation length ξ. A particle mass m is of course just the inverse of the correlation length.

All lattice calculations are performed with the cutoff present. It is clear that the cutoff is unphysical; we want to remove it from the calculation and present cutoff-independent results. Lattice people talk about taking the momentum cutoff Λ to infinity, or the lattice spacing a to zero, while fixing some fiducial mass scale. This is a shorthand for the requirement that the correlation length measured in units of the cutoff, ξ/a, must diverge. The correlation length will, of course, be a function of the bare parameters that characterize the simulation. Making the correlation length diverge is done by tuning the bare parameters of the theory.

A fiducial scale is needed to set against the correlation length. In lattice simulations, this scale is almost universally taken, as a first step, to be the lattice spacing a itself. When this is done, pure numbers come out of the simulation; all predictions of dimensionful quantities (like masses) appear with an appropriate power of the cutoff (that is, a calculation produces the product $a \times m$). Almost all real lattice Monte Carlo predictions are of dimensionless ratios of dimensionful quantities, like mass ratios. Lattice people like to say that one prediction of a mass determines the lattice spacing, when the value of that mass is fixed by experiment. This is just the statement that $a = ma/m_{expt}$. One then uses this a to make predictions in energy units for other masses or dimensionful quantities.

Recall the usual definition that a running coupling is (infrared) relevant, marginal, or irrelevant with respect to changes of scale, depending on whether it grows, remains almost unchanged, or shrinks, as it is evaluated at longer and longer distance scales. That a coupling is relevant or not can be empirically determined: can it be varied, so that the correlation length grows? If so, it is probably relevant. The increase in the correlation length occurs as the relevant bare coupling is tuned toward its critical value. Most lattice simulations are of theories with one or two relevant couplings. They also have many irrelevant ones, typically arising when the continuum theory is transferred to the lattice.

While the correlation length is finite, the fact that the lattice action is an effective field theory becomes important: one's answers ought to – and generally do – depend on the value of cutoff. One would observe this in measured mass ratios, as a function of the bare parameters in the simulation.

Most lattice simulations are done for asymptotically free theories. Their one or two relevant couplings are the gauge coupling g and fermion masses m. The system has a critical surface in the space of all couplings that encloses a Gaussian fixed point at $g = 0$ and $m = 0$. Tuning the two relevant couplings to zero causes the correlation length, measured in units of a, to diverge.

Much of the lattice language for understanding cutoff effects implicitly makes use of the fact that one tunes g and m to zero to remove them. Focus on the gauge coupling for a moment. The advantage of having an asymptotically free theory is that when the bare coupling is taken smaller and smaller, the short distance behavior of the theory becomes increasingly perturbative and hence increasingly controlled. In particular, field dimensions approach their engineering dimensions. This allows us to parametrize the dependence of an observable on the cutoff scale. It is nearly given by naive dimensional analysis. In an asymptotically free theory, if the lattice spacing were small enough, a typical mass ratio would behave as

$$[am_1(a)]/[am_2(a)] = m_1(0)/m_2(0) + \mathcal{O}(m_1 a) + \mathcal{O}[(m_1 a)^2] + \ldots \tag{48}$$

(modulo powers of $\log(m_1 a)$). The leading term does not depend on the value of the UV cutoff. That is our cutoff-independent prediction. Everything else is an artifact of the calculation.

This is important because it gives control over the calculation. Away from weak coupling, scaling dimensions of operators may be different from their engineering dimensions. Corrections to scaling may not scale with their expected power laws. It may not be possible to identify relevant versus irrelevant operators. Worse, the system may happen to lie in the basin of attraction of other fixed points, or may be susceptible to non-universal lattice-artifact phase transitions which depend on the particular choice of discretization.

Running of the gauge coupling to zero in the UV is generally only observed qualitatively in "spectral" calculations (of masses or matrix elements), through the observation that the correlation length increases ("a goes to zero") as the bare gauge coupling is decreased. This was not the case in the earliest days of lattice simulations, where people attempted to relate a mass to a bare lattice gauge coupling along the lines of Eq. 21. We now know that lattice perturbation theory is much dirtier than its continuum counterpart, and corrections to this naive behavior are large due to lattice artifacts. Instead of this, almost all lattice data is extrapolated to the continuum with an analog of Eq. 48. Nowadays completely separate calculations of non-spectral observables are used to make quantitative statements about running couplings.

It is much easier to see that the mass is a relevant coupling; masses of all multiquark bound states vary strongly as the bare lattice mass is tuned, and only become small as it is tuned to zero.

All lattice gauge theories replace the gauge fields $A_\mu(x)$ by "link variables" connecting adjacent sites. The link variables are group elements

$$U_\mu(x) = \exp igaA_\mu(x). \tag{49}$$

The gauge field functional integration measure is a product of integrals for each link variable over the Haar measure of the gauge group. All lattice actions are traces over products of the U's around some closed path. In the so-called Wilson or plaquette action, this path is the minimal four-link one around a unit square. There are many other possibilities, of course. All these actions, and all fermionic actions, differ from the expected continuum action of fermions coupled to gauge fields by the addition of extra irrelevant operators, so simulations with any of these actions done sufficiently close to the Gaussian fixed point are expected to produce cutoff - independent predictions of the continuum theory. In particular, spacetime symmetries are broken by the lattice discretization, but the operators which break them are irrelevant ones, and these symmetries (such as rotational invariance) are expected to be restored in the naive continuum limit.

B. What systems can be studied on the lattice?

Technical issues associated with putting fermions on the lattice strongly affect how easy it is to simulate any particular theory.

Briefly, there are three generic kinds of lattice fermions. To summarize a (long) textbook discussion, the constraint is the Nielsen-Ninomiya theorem, which says (loosely; this actually not precisely correct) that one cannot write down a well-behaved lattice fermion action that is simultaneously chiral and undoubled. "Doubling" is a shorthand way to say that the lattice system has extra, usually unwanted, fermionic degrees of freedom. These states are the doublers. The three kinds of fermions are

- Wilson fermions and their variants (clover or twisted mass fermions): a four-component spinor sits on each site of the lattice. Their actions contain terms which, while formally irrelevant, explicitly break chiral symmetry. The benefit of this breaking is that the lattice theory has the same number of fermionic degrees of freedom as its continuum analog.

- Staggered fermions maintain some chiral symmetry, but at the cost of introducing doublers. "A single staggered fermion corresponds to four degenerate flavors in the naive continuum limit," we say.

- Domain wall and overlap fermions, which live in five dimensions (domain wall fermions), or are the four dimensional effective field theories of five-dimensional fermions (overlap fermions), remain undoubled and replace the continuum definition of chirality by a more complicated one, called the Ginsparg-Wilson relation. They are theoretically beautiful, exactly encoding Ward identities associated with chiral symmetry. From a practical point of view these fermions are quite expensive to simulate.

A specific fermion action will lie in one of these classses, but beyond that, it will have a variety of different lattice terms, typically different ways of discretizing the derivative operator.

All lattice simulations I know of are of vector theories. Direct simulation of chiral gauge theories, like the Standard Model itself, is quite difficult. Luscher (2000) gives a fairly complete overview of the subject. To even begin, by imagining an ultraviolet regulator for a chiral gauge theory, the theory must be anomaly free. But the consequence is that any consistent regulator that preserves gauge invariance must refer to the fermion representation. This is hard to do; a simple lattice cutoff will not suffice. People who want to study chiral gauge theories on the lattice typically feel that they are forced to use regulators that break gauge symmetry, and then attempt to tune their bare parameters to a critical point which will produce a chiral gauge theory when the correlation length diverges. Golterman (2001) and Golterman and Shamir (2004) describe approaches along these lines.

The lattice introduces additional issues. The doublers which appear in an action with chiral symmetry turn out to have the opposite chirality to their partners; at the end, there will be equal numbers of left- and right-handed fermions. Domain wall or overlap fermions allow one to go farther, and Luscher (2000) describes all-orders perturbative constructions of chiral gauge theories. I do not know of any numerical studies of these systems, though.

The next issue is that $P(\phi)$ (see Eq. 46) has to have a probability interpretation, in order to perform importance sampling. One cannot directly simulate dynamical fermions, because Grassmann variables are not numbers. One must formally integrate them out, leaving behind a functional integral for bosons. Being classical, this can be studied numerically. To see how this is done, consider a system of gauge fields and a single species of fermion. Its partition function is

$$Z = \int [dU][d\bar{\psi}][d\psi] \; \exp[-S_G(U) - \bar{\psi}M(U)\psi] \qquad (50)$$

where $M = D + m$. After integrating out the fermionic Grassmann variables, the partition function becomes

$$Z = \int [dU] \; \exp[-S_G(U)] \mathrm{det} M(U). \qquad (51)$$

The determinant is nonlocal, so computing its change under a change in the gauge field is very expensive. The standard way to deal with this is to simulate the determinant by introducing a set of scalar "pseudofermion" fields Φ. This is done via the formal identity

$$\mathrm{det} M(U) = \int [d\Phi^* d\Phi] \exp[-\Phi^* M^{-1} \Phi]. \qquad (52)$$

Expanding Φ in terms of eigenmodes ψ_j of M and the corresponding eigenvalues λ_j

$$\Phi^* M^{-1} \Phi = \sum_j \langle \Phi | \psi_j \rangle \frac{1}{\lambda_j} \langle \psi_j | \Phi \rangle \qquad (53)$$

exposes a cascade of problems, all arising from the fact that the eigenvalues of lattice Dirac operators are complex and their real parts may not be positive-definite. Individual terms in the exponential can be complex or carry a net negative sign. Then the exponential in Eq. (52) cannot be interpreted as a conventional probability measure.

There are often ways to avoid this. With Wilson fermions, one can show, using lattice symmetries of the action, that simulations of pairs of degenerate mass fermions (i.e., even N_f) give a positive-definite determinant. (Basically, $D^\dagger = \gamma_5 D \gamma_5$, so $(\mathrm{det} D)^2 = \mathrm{det} D^\dagger D$.) Staggered fermions naturally come in multiples of four flavors, and the four flavor combination has a positive determinant.

Often, one wants to have a different fermion content than what is possible in these favorable situations. Odd numbers of flavors require caution. For example, in QCD, one might want to simulate a degenerate up and down quark pair, and a heavier strange quark. One replaces the strange quark's determinant by

$$\mathrm{det} M(U) \to (\mathrm{det}|M(U)|^2)^{1/2}. \qquad (54)$$

This can be simulated with the RHMC ("rational Hybrid Monte Carlo") algorithm, with a pseudofermion action

$$\mathrm{det} H(U)^p \to \int [d\Phi^* d\Phi] \exp[-\Phi^* \sum_j \frac{c_j}{H(U) + d_j} \Phi]. \qquad (55)$$

The determinant could try to change sign during the simulation. That would invalidate Eq. 54. This might not be noticed, nor treated properly, by its approximation, Eq. 55.

There are related issues with staggered fermions, going from the doubled number of degrees of freedom that staggered fermions naturally encode, to the desired counting for a single continuum flavor. One must make the replacement

$$\mathrm{det} M(U) = \mathrm{det} M_{stagg}^{1/4}(U) \qquad (56)$$

to simulate a single continuum flavor. There is a long controversy in the QCD literature about how to correctly deal with this replacement. I believe that the situation is well understood for chirally broken theories simulated in the vicinity of the Gaussian fixed point. (The conference proceedings by Sharpe (2006) is an excellent overview.) Briefly, at nonzero lattice spacing, the action associated with Eq. 56 is nonlocal. Rooted staggered fermions cannot be described by a local theory corresponding to a single Dirac fermion. Associated with this nonlocality, there are all kinds of artifacts, such as negative norm states. However, when chiral symmetry is broken, a low energy theory can be construct which correctly describes the Goldstone sector of the rooted theory. This theory has a set of low energy constants which include those of the continuum theory, plus additional ones. Continuum predictions can be made – and are made – using this more complicated chiral perturbation theory.

Simulations of QCD at nonzero chemical potential are difficult because the fermionic determinant is complex.

Finally, some vocabulary. To label the bare gauge coupling g of an $SU(N_c)$ gauge theory, lattice people work with the quantity $\beta = C/g^2$, where C is a constant. For the plaquette action, $C = 2N_c$. The bare quark mass m_0 in simulations with Wilson or clover fermions is usually replaced by a hopping parameter $\kappa = 1/2(4 + am_0)^{-1}$, and people almost always quote κ rather than am_0.

Patterns of chiral symmetry breaking ("vacuum alignment") for different numbers of colors and fermionic representation were first described by Kosower (1984); Peskin (1980); and Preskill (1981) and were listed in Sec. III.D above. Golterman and Shamir (2014a,b) describe the complications of lattice artifacts for this physics.

C. Lattice issues for beyond Standard Model calculations with slowly running couplings

The situation for a lattice practitioner faced with a proposed nonperturbative extension of the Standard Model is, at first sight, similar to the situation of lattice QCD: Given an ultraviolet complete action that might encode some nonperturbative low energy physics, the way to proceed is as follows:

1. Write down a lattice discretization and simulate it
2. From the simulation, determine the vacuum structure of the system: does it have a mass gap in the infinite volume limit, is it confining, is it chirally broken, is it something else?
3. If the system has a mass gap, compute the spectrum and perhaps appropriately interesting matrix elements
4. Alternatively, use the expectation value of some operator to define a running coupling constant (typically, the scale at which the coupling is measured is set by the size of the simulation volume) and see how it runs
5. From the results of (3) or (4), evaluate the possibility that the action might be a viable scenario for beyond Standard Model physics

Most lattice studies of beyond Standard Model dynamics involved systems with slowly running gauge couplings. As a result, getting beyond item (2) proved to be very difficult. The issue was that all the techniques lattice people had at their disposal were designed for QCD, where the coupling constant runs quickly. Several years later, I think there is a reasonable consensus between different groups about the answer to point (2) for most of the systems that have been studied. However, agreement is not universal and one can find controversy throughout the literature of the subject.

This is quite different from the situation in modern lattice QCD simulation. There, the disagreements are about very specific points, such the particular value of some mass or matrix element. In fact, the flow chart I gave for beyond Standard Model candidate theories already differs from its QCD analog. Lattice QCD simulations never really had to deal with item (2): the vacuum structure of QCD was, broadly speaking, noncontroversial before the first simulations were carried out. Before QCD, experiment showed that strongly interacting matter was composite and chirally broken. After the discovery of asymptotic freedom and before lattice gauge theory was invented, the question was, are asymptotic freedom and confinement related? Confinement was the most important phenomenological feature of the Wilson (1974) formulation of lattice gauge theory. He showed that essentially all lattice gauge theories are confining in their strong coupling limit. The important question then became, does confinement persist in the continuum limit? The earliest numerical simulations of lattice gauge theories by Creutz (1980a,b) showed the coexistence of confinement and asymptotic freedom in a single phase for a non-Abelian gauge theory.

Early analytic lattice work (Blairon *et al.*, 1981; Greensite and Primack, 1981; Kluberg-Stern *et al.*, 1982; Svetitsky *et al.*, 1980; Weinstein *et al.*, 1980) argued strongly that chiral symmetry was broken in the strong coupled limit of lattice QCD, and again, questions of interest were about the value of quantities such as the condensate or the pion decay constant in the continuum limit, not about whether chiral symmetry was actually broken. Lattice QCD very quickly became a subject about numbers, not about qualitative behavior. And so it has remained. Not knowing the answer ahead of time made lattice studies of beyond Standard Model candidates very different from lattice QCD.

The origin of the difficulty in analyzing systems with slowly running couplings is most easily seen from the formula for the one-loop beta function: with a scale change of s, the inverse coupling changes by an amount

$$\frac{1}{g^2(s)} - \frac{1}{g^2(1)} = \frac{b_1}{8\pi^2}\log s. \tag{57}$$

For the $SU(3)$ gauge group with N_f flavors of fundamental representation Dirac fermions, $b_1 = 11 - 2/3N_f$. Consider ordinary QCD, with $N_f = 3$, for which $b_1 = 9$. We know empirically that a scale change of about $s = 10$ causes the system to go from weakly coupled to strongly coupled: this can be seen from the potential between heavy quarks, which is Coulombic at short distance (0.1 fm) but confining at long distance (1 fm). A single lattice simulation with a lattice spacing of say 0.05 fm and a size of 20-40 lattice spacings can capture both ends of this behavior, so that the system can be perturbative at the shortest distance and nonperturbative at the longest distance. Simulations involve the action at the cutoff scale, and if the system is weakly interacting at the cutoff scale, we know what we are doing.

Now consider the case of $N_f = 12$, where $b_1 = 3$. With one loop running, we would need a scale change of $s = 1000$ to make the coupling constant change by the same amount as the $N_f = 3$ system changed with $s = 10$. Such a scale factor cannot be accommodated on any single lattice size which is capable of simulation today or in the foreseeable future.

At this point, a reader objects: You are telling a one-loop story. You are a lattice person working in strong coupling. Why should I believe a one-loop story?

The answer is: Yes, the story could be wrong. But either it is wrong in a favorable way, or an unfavorable way. In a favorable way, physics evolves more rapidly with scale than b_1 suggests (this happens in $N_f = 3$ QCD). This is easy to see in a simulation; you do not need to know about the story. But the physical systems I am thinking about are candidates for walking technicolor. Recall Fig. 3. To the left of the inflection point, the coupling runs more slowly than the one loop formula. b_1 is effectively smaller. The one-loop result for how the coupling changes with scale is too optimistic. Instead of $s = 1000$, suppose the beta function is half the size of b_1. Then you need $s = 10^6$ to see the same change in the coupling.

There are many equivalent ways to state the consequences of having a slowly running coupling in a finite volume lattice simulation:

- In such a theory, if the coupling constant is small at short distances (that is, at the cutoff scale) in any simulation, it remains small at long distance. Then, how can nonperturbative physics appear?

- If the coupling constant is large at long distances, it must be large at the shortest distance (at the cutoff scale) on the lattice. Then, how closely does the lattice theory resemble its continuum analog?

- The coupling constant effectively does not run with scale in any practical simulation volume

- If a simulation does show a potential which has both a Coulomb term and a linear confining piece, it must also be characterized by a quickly running coupling constant, over the range of scales present in the simulation.

Nearly all lattice systems with many fermion degrees of freedom show this generic behavior.

How to deal with lattice artifacts in QCD is under reasonable control. However, that is because continuum QCD is qualitatively well understood. Making sense of the "quarter root trick" (Eq. 56 for staggered fermions) is done in the context of chiral perturbation theory. But, suppose that one is simulating a theory which might not be chirally broken? One might not be able to distinguish a lattice artifact from real physics. In particular, little is known about the universality properties of a rooted theory. Its global symmetries are simply different from those of an unrooted system.

Since most lattice studies involve slow running, we should think a bit more about what to expect. We can continue to do this using perturbation theory. The two loop beta function can be integrated exactly to find a relation between scale and coupling. Defining $\overline{b}_1 = b_1/(16\pi^2)$ and $\overline{b}_2 = b_2/(16\pi^2)^2$, it is

$$\overline{b}_1 \log \frac{\mu^2}{\mu_0^2} = \frac{1}{g^2(\mu)} - \frac{1}{g^2(\mu_0)} - \frac{\overline{b}_2}{\overline{b}_1} \log \left(\frac{\overline{b}_1 + \overline{b}_2 g^2(\mu)}{\overline{b}_1 + \overline{b}_2 g^2(\mu_0)} \right). \tag{58}$$

When the coefficients b_1 and b_2 have opposite signs, there is a fixed point, at $g_f^2 = -\overline{b}_1/\overline{b}_2$. Equation 58 takes the compact form

$$\overline{b}_1 \log \frac{\mu^2}{\mu_0^2} = \frac{1}{g^2(\mu)} - \frac{1}{g^2(\mu_0)} - \frac{1}{g_f^2} \log \left(\frac{g^2(\mu) - g_f^2}{g^2(\mu_0) - g_f^2} \right). \tag{59}$$

Now we can examine some useful limits. If $g^2(\mu)$ and $g^2(\mu_0)$ are both small, the logarithm is small compared to the first terms and we have the familiar one loop running

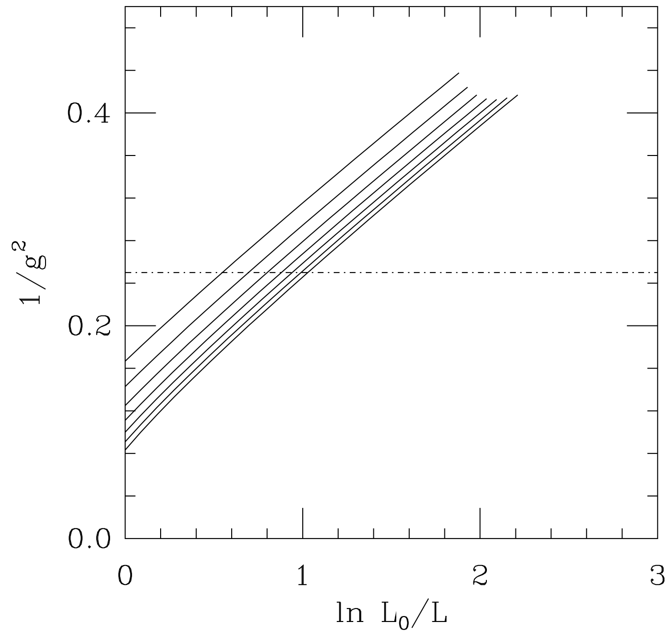

FIG. 5 Running coupling constant from Eq. 58 for a set of initial couplings: $N_c = 3$, $N_f = 3$. The dotted line is a line of constant $1/g^2$.

formula. However, when $g^2(\mu) - g_f^2$ is small, the logarithm is the dominant term, and the coupling evolves in a different (but equally familiar) way:

$$g^2(\mu) - g_f^2 = (g^2(\mu_0) - g_f^2) \left(\frac{\mu}{\mu_0} \right)^{\overline{b}_1 g_f^2} \tag{60}$$

The beta function has a linear zero: $\beta(g^2) \sim -\overline{b}_1(g^2 - g_f^2)$. At ever smaller μ, the coupling runs into the fixed point. This is an infrared attractive fixed point. $\overline{b}_1 g_f^2$ is an example of a critical exponent. In this case we will label it y_g.

Setting $\mu/\mu_0 = L_0/L$, we can define a coupling measured at a distance scale L. This will be useful to anticipate definitions of running couplings used in lattice simulations. We contrast the running of the coupling constant in two cases in Figs. 5-6. The first picture shows the case of $N_c = 3$ and $N_f = 3$ fundamental flavors; the second shows the case for $N_c = 3$ and $N_f = 12$. The initial $g^2(\mu_0)$ is taken to be equally spaced values 1,2,3, ... The fixed point coupling for $N_f = 12$ is at $g^2 = 9.47$. We will come back to the dotted lines in Sec. V.B.

This analysis is incomplete, because it leaves out the fermion mass. Inside the conformal window, the fermion mass is a relevant coupling. In fact, in most simple systems, it is the only relevant coupling, and to make the correlation length diverge, it must be fine-tuned to zero. Its evolution equation has a linear zero, as does the renormalization group equation for g^2 in the vicinity of g_f^2. This is the usual textbook situation for second order criti-

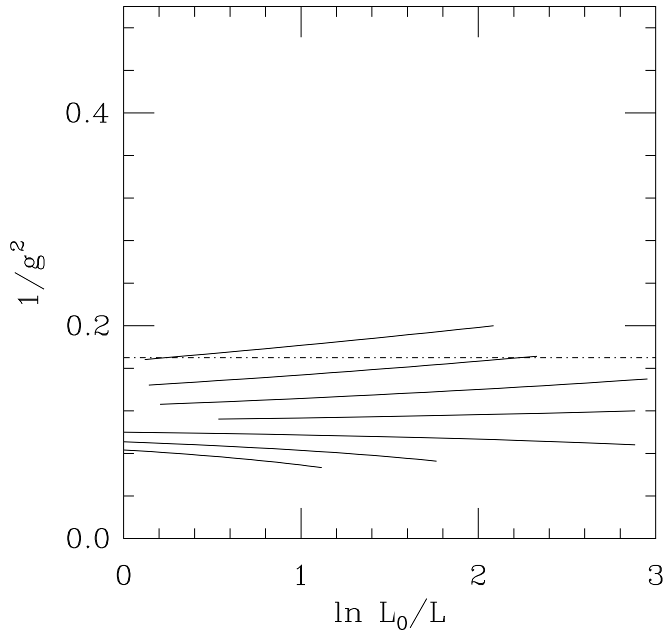

FIG. 6 Running coupling constant from Eq. 58 for a set of initial couplings: $N_c = 3$, $N_f = 12$. The dotted line is a line of constant $1/g^2$.

cality, where the correlation length diverges algebraically,

$$\frac{\xi}{a} \sim (am_q)^{-\frac{1}{y_m}} . \quad (61)$$

(Cardy (1996) is a good reference.) Systems with this behavior are often called "infrared conformal" in the particle physics literature. The quantity y_m is the leading relevant exponent for the system, in statistical physics language. This exponent is related to the anomalous dimension γ_m of the mass operator $\bar{\psi}\psi$, and determines the running of the mass parameter according to

$$\mu \frac{dm(\mu)}{d\mu} = -\gamma_m(g^2)m(\mu). \quad (62)$$

The relation is $y_m = 1 + \gamma_m(g^2)$.

For future reference, in lowest order in perturbation theory,

$$\gamma_m = \frac{6C_2(R)}{16\pi^2} g^2. \quad (63)$$

The actual exponent is $\gamma_m(g_f^2)$. In perturbation theory, it grows as the bottom of the conformal window is approached from above (say, by decreasing the number of flavors).

Inside the conformal window, all couplings other than m_q are irrelevant. Note that the gauge coupling (more properly, the distance of the gauge coupling from its fixed point value) is one of these couplings. Taking the continuum limit has nothing to do with tuning the bare gauge coupling, other than setting it within the basin of attraction of g_f^2. (The literature is occasionally confused about this point.) In the two loop example, that happens naturally, for any value of g^2. In most cases, $0 < g^2 < g_f^2$ is in the basin of attraction of g_f^2. So is a region $g^2 > g_f^2$. This is a curious region, a bit like QED, because the coupling constant becomes larger at shorter distances. But again, tuning m_q to zero is how we take the continuum limit of $a/\xi \to 0$. We expect that values of g^2 which remain in the basin of attraction of g_f^2 cannot become too great, because lattice theories generically confine when the gauge coupling becomes large. Thus, there should be a strongest coupling, a boundary of the basin of attraction. If the boundary were marked by another second order transition, it would be characterized by a UV attractive fixed point, at some g_{UV}. More complicated possibilities can be imagined (Kaplan *et al.*, 2009).

(The absence of a scale in the conformal window should not be confused with the presence of a massless state in the spectrum. In the confining phase, when chiral symmetry is broken, there is an infinite correlation length, the inverse pion mass. But in other channels there is a mass gap, and there are other physical scales, such as f_π.)

The irrelevance of the gauge coupling has the consequence that the location of an infrared attractive fixed point is not physical. Contrast this behavior to that of a relevant coupling, which marks a real qualitative change in long-distance physics: $m_q = 0$ for the fermion mass. A change in the renormalization scheme can shift the location of g_f^2. This means that in the scaling limit ($\xi/a \gg 1$), changes in ξ as the bare g^2 is varied can only be order unity corrections. (For a lattice QCD practitioner used to simulating clover fermions, the situation is similar to what one would find when tuning the clover coefficient.) There is no good reason for ξ/a to increase (or decrease, either) versus increasing g^2. The scaling limit is the limit of vanishing quark mass, or more generally, of the limit that all relevant couplings are taken to their critical values.

One complicating issue in this discussion is that while the gauge coupling is irrelevant, the critical exponent associated with the gauge coupling is often close to zero. (Dimensional analysis "predicts" $y_m \sim 1$, $y_g \sim 0$.) This has an unfortunate practical consequence which I already mentioned: In finite volume simulations, the gauge coupling will evolve so slowly with scale toward its fixed-point value that the system is effectively conformal, regardless of the actual value of the cutoff-scale gauge coupling. One is likely to observe a leading exponent y_m that shows a slow, smooth dependence on bare gauge coupling. This may be very hard to analyze.

Finally, while I have discussed the relevant m_q and irrelevant g^2, we cannot forget all the other couplings. To choose a lattice action is to implicitly fix the initial values of all the irrelevant couplings, too. But these couplings also run. While they run to zero in the long distance limit, it might be that over the range of length scales accessible in a simulation in finite volume, these couplings could exhibit significant running, which could contaminate results. (And remember, far away from the Gaussian fixed point, one may not know what is relevant and what is not. The flow may even find another fixed point.) This is a source of systematic error. It is also an important practical issue which arises when one wants to compare results from two different simulations which are performed with different lattice actions.

And of course, one may not know a priori that one is dealing with a conformal system.

Once the fermion mass becomes large, we expect that the fermions decouple from long - distance dynamics. The most likely scenario in that case is that the system becomes confining, since the fermions no longer screen the gauge fields. One would expect to see a linear potential re - emerge. Probably the lightest excitations would be gluonic in nature, glueballs. It is unknown how much of spontaneous chiral symmetry breaking would remain. Of course, explicit chiral symmetry breaking by the fermion masses would also be present.

More direct applications of the statistical literature to physics in the conformal window can be found in DeGrand and Hasenfratz (2009) and Del Debbio and Zwicky (2010, 2011, 2014).

V. LATTICE RESULTS FOR SYSTEMS WITH SLOWLY RUNNING COUPLINGS – BY METHOD

A. Spectroscopy and related observables

Spectroscopy can, in principle, distinguish between systems which are confining and chirally broken, and systems which are nearly conformal. So, let us imagine doing a simulation. Recall that, at any nonzero m_q, the system is "ordinary," not conformal, with a mass gap, regardless of what happens at $m_q = 0$. It will have a discrete spectrum.

Collect spectroscopic data (probably one might begin with a similar set of bare couplings, perhaps one bare gauge coupling and several fermion masses). Is the spectrum of excitations QCD-like? That is, as the fermion mass is made smaller, does the pseudoscalar state become much lighter than the vector state? Does the pseudoscalar mass extrapolate to zero with the fermion mass, like $m_\pi^2 \propto m_q$? Do other masses extrapolate to nonzero values at $m_q = 0$? Is the vector meson mass different from the axial vector mass? Is the static potential linear at long distance?

If the answer to these questions is Yes, then probably the system is confined and chirally broken and, given the discussion in the last section, it probably also has a quickly running coupling constant.

(The question "Is the vector meson mass different from the axial vector mass?" refers to the fact that in a system in which chiral symmetry is unbroken, opposite parity states are degenerate, being related to each other by chiral rotations.)

Now simulate at weaker bare gauge coupling. Does it seem that the correlation length grows, while the good features seen so far appear to maintain themselves? Is it possible to move to ever weaker coupling without encountering a discontinuity, a phase transition between the strong coupling phase and some new phase? If there is a transition, is it induced by the size of the lattice? If the answer to these questions is Yes, then confinement and chiral symmetry breaking probably coexist with asymptotic freedom.

In a system inside the conformal window, spectroscopy would be qualitatively different. All masses would track towards zero as the fermion mass were made smaller. The system would not exhibit spontaneous chiral symmetry breaking; the vector and pseudoscalar mesons would not behave particularly differently. In a system with only explicit chiral symmetry breaking, the spectrum is parity - doubled, in the $m_q \to 0$ limit, so that one would observe approximate equality of the vector meson and axial vector meson mass, and of the and scalar and pseudoscalar masses. If there was a nonzero string tension at nonzero fermion mass, one would expect that it would go to zero with the fermion mass.

We can illustrate these differences with an example. Figure 7 shows the pseudoscalar mass and the vector meson mass, as a function of the quark mass, from a typical calculation in quenched $SU(3)$, at each of two different gauge couplings. All of the parameters are given in units of the lattice spacing a. The separation of the pseudoscalar and vector meson at small quark mass is apparent. The data sets labeled by octagons and squares are collected at a smaller gauge coupling than the data sets shown by crosses and diamonds. It is clear that if one uses the zero fermion mass limit of the vector mass to define the lattice spacing, then weaker coupling corresponds to smaller lattice spacing.

Figure 8 is a presentation of spectroscopy which is more common in the QCD literature. The data is identical to Fig. 7, but I am plotting the squared pseudoscalar mass. Its approximate linearity is the qualitative signal that chiral symmetry is broken, the Gell-Mann, Oakes, Renner formula in action.

Contrast this case with that of a data set from a simulation of $SU(2)$ gauge theory with $N_f = 2$ adjoint representation fermions. I have plotted data from Bursa *et al.* (2011a). Many other collaborations including Catterall *et al.* (2009b); Catterall and Sannino (2007); and Hietanen *et al.* (2009a) have similar results. The particular lattice system that was simulated had a strongly coupled phase which is chirally broken and a weakly coupled

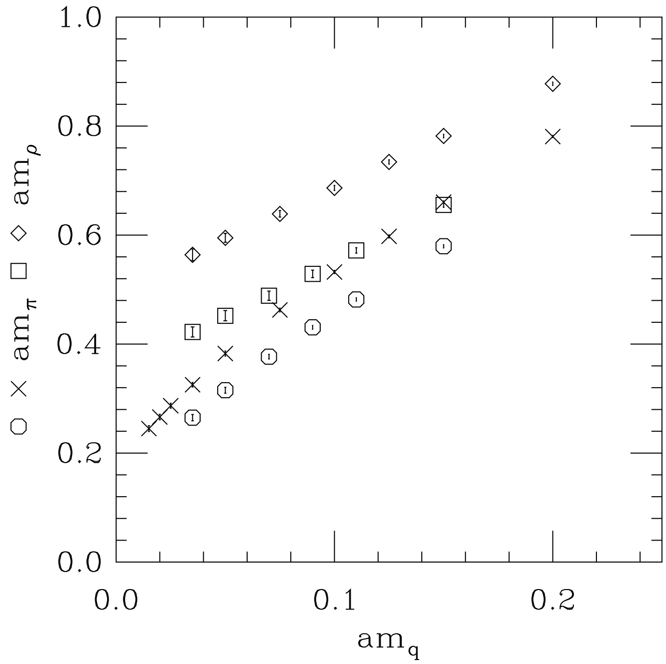

FIG. 7 Pseudoscalar and vector meson masses, in lattice units, from quenched $SU(3)$ gauge theory. Octagons and squares are data from a weaker gauge coupling simulation; crosses and diamonds from stronger coupling. [This is raw data from DeGrand (2004).]

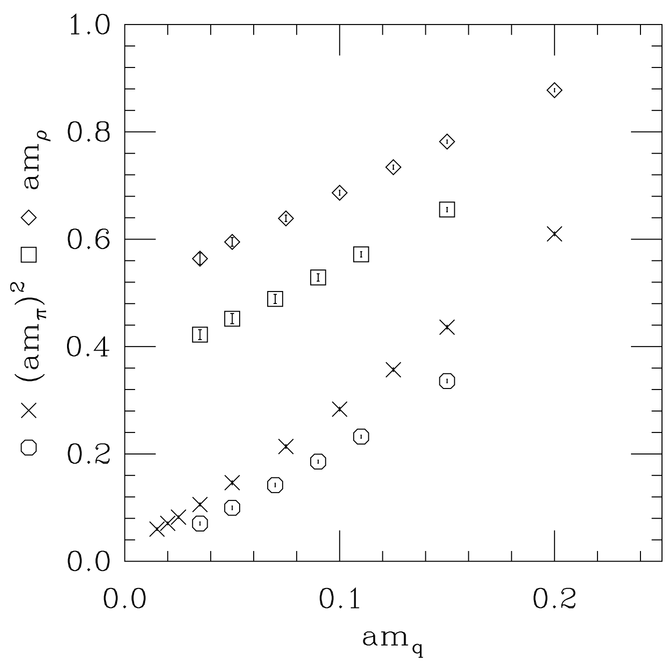

FIG. 8 Squared peudoscalar and vector meson masses, in lattice units, from quenched $SU(3)$ gauge theory. Octagons and squares are data from a weaker gauge coupling simulation; crosses and diamonds from stronger coupling. The data is identical to what is shown in Fig. 7.]

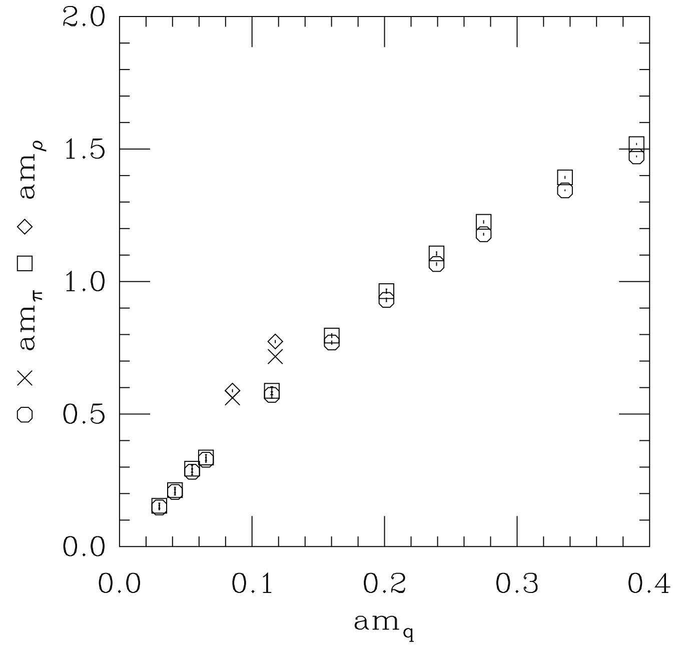

FIG. 9 Pseudoscalar and vector meson masses, in lattice units, from $SU(2)$ gauge theory coupled to $N_f = 2$ adjoint fermions. Octagons and squares are data from a weaker gauge coupling simulation; crosses and diamonds from stronger coupling. This is data from Bursa *et al.* (2011a), a $12^3 \times 24$ volume lattice at weaker coupling and a $24^3 \times 64$ volume at stronger coupling.

phase which is almost certainly conformal in the zero quark mass limit. Data from the strong coupling phase qualitatively resembles that in Fig. 7. But in weak coupling, shown in the figure, the pseudoscalar and vector masses never separate, and while there is some dependence of the bound state mass on bare gauge coupling, there is strong dependence on the fermion mass.

Unfortunately, a data set may not be so clean-cut.

Recall, that first, at nonzero m_q, even a would - be conformal system is “ordinary,” with a mass gap, regardless of what happens at $m_q = 0$. In heavy quark systems the pseudoscalar and vector states are nearly degenerate. If the fermion mass is too large, it may be impossible to distinguish systems which are trending conformal in the zero mass limit from confining ones.

The Gell-Mann, Oakes, Renner dependence of pseudoscalar mass on fermion mass, $m_\pi^2 \propto m_q$, which is a signal of chiral symmetry breaking, is only the leading behavior in a chiral expansion. Higher orders might be important. With large N_f, one-loop chiral logs can be huge. For example, the non-analytic correction to the pseudoscalar decay constant f (for symmetry breaking pattern $SU(N_f) \times SU(N_f) \to SU(N_f)$) is

$$\frac{\delta f_{PS}(m)}{f} = \frac{N_f}{2}\left(\frac{m}{4\pi f}\right)^2 \ln\frac{m^2}{\Lambda^2}. \qquad (64)$$

Such terms might overwhelm any naive (analytic) expansion of lattice data to the zero mass limit.

Finite volume affects spectroscopy in many ways. Imagine first that we do have a confining, chirally broken system in infinite volume. When the volume is large, in the sense that $M_H L$ for any hadron mass M_H and system size L, is large, the dominant effect of finite volume is from pion loops. Instead of returning to the point of emission, the pions scatter "around the world" or "off image charges." The typical situation is that finite volume effects go as $\exp(-m_\pi L)$. When the observable in question has a chiral logarithm in its expansion, as in the case of f_{PS} above, the coefficient of the logarithm is also the coefficient of the finite volume correction. If that coefficient is large, there will be large finite volume corrections.

Next, suppose that $m_\pi L << 1$ but $m_H L >> 1$ for all other states, and $f_\pi L >> 1$ as well. This is called the "epsilon regime." Symmetries cannot break in finite volume, so the condensate $\Sigma(V)$ will fall from its infinite volume value, Σ, to zero with the quark mass. The relevant dimensionless variable is $\zeta = m_q \Sigma V$ with $V = L^4$. If one is certain one is in this situation, one can use measurements of the condensate and of correlators in various channels to extract chiral observables. For example, the finite-volume condensate is

$$\Sigma(V) = \Sigma f(\zeta) \sim m_q \Sigma^2 V + \ldots \tag{65}$$

But suppose one is not certain? One might interpret the vanishing of $\Sigma(V)$ as evidence for infrared conformality.

In between the epsilon regime and the large volume "p-regime" there is another regime, the "delta-regime," where the pseudoscalar correlator shows a rotor spectrum. Naive chiral behavior is once again absent.

And notice the repeated use of the words "extrapolate to $m_q = 0$." Even in QCD, that is a nontrivial task.

Typically, a finite simulation volume can induce phase transitions in a lattice system. For an asymmetric box, with $N_t < N_x$, the short time direction implies a finite temperature, $T = 1/(aN_t)$. Typically, simulations of confining systems in these asymmetric volumes will show a phase transition from a strong-coupling confined phase to a weak-coupling deconfined and chirally restored phase. The weak coupling phase is (usually) analytically connected to the Gaussian fixed point, which is (usually) where we might want to tune, to take the continuum limit. So, is the continuum limit confined?

One way to test this is to vary N_t and see if the transition moves in bare parameter space. If it moves to weaker coupling, in a way that T remains roughly constant in physical units, the transition is, most likely, a finite temperature transition, and the zero temperature phase is likely to remain confined, while still analytically connected to the Gaussian fixed point. If the transition remains fixed in bare coupling as the lattice volume is varied, and the system is deconfined on the weak coupling side, the transition is a "bulk transition." The weak coupling phase is analytically connected to the Gaussian fixed point and the system has a continuum limit that is deconfined. So there is a test: does the deconfinement transition move appropriately with N_t?

The problem with this test is that even a bulk transition moves a bit when the volume is small. And there is another problem: how can one tell that the motion is consistent with a finite temperature transition, anyway? Often, one imagines analyzing a formula like Eq. 21, where M_H is T_c and $g^2(\Lambda)$ is the bare lattice gauge coupling. But typically, one is simulating at strong coupling, and asymptotic freedom does not work well as a descriptor of physics at strong coupling. One really has to compute the value of some other observable M with the dimensions of a mass, typically in a zero temperature simulation at the same bare parameters, and look for variation of T/M as N_t is varied. It is Eq. 48 all over again. (This is how the deconfinement or chiral restoration crossover temperature in QCD is determined.) This is rapidly becoming an expensive proposition.

Inside the conformal window, and in infinite volume, tuning the fermion mass m_q to zero causes the correlation length to diverge algebraically, as in Eq. 61. One might hope to use this functional dependence as a diagnostic. However, no simulation is ever done in infinite volume. The system size L is also a relevant parameter since the correlation length only diverges in the $1/L \to 0$ limit. When the correlation length measured in a system of size L (call it ξ_L) becomes comparable to L, ξ_L will saturate at L even as m_q vanishes. Equivalently, bound state masses will become independent of the fermion mass when it is small. This is what non-Goldstone excitations are expected to do in a confining system. Again, the finite volume can induce confusion between "confining" and "infrared conformal" behavior. Simulations with the same bare parameters, but at several volumes, are needed to sort out this behavior. We will return to a detailed description of the necessary analysis in Sec. V.C.

Regardless of whether the data looks confining or looks conformal, for a definitive answer, one needs simulations at many values of the bare coupling. This is quite similar to the case in precision QCD, where one has to extrapolate to $a = 0$ to produce a cutoff - free number. But it is expensive. In QCD, when one is doing something new, one might attempt to simulate at one value of a (or perhaps one value of the bare gauge coupling and several quark masses) and to present results with the claim that the lattice spacing is small enough, and the lattice volume is large enough, that at the level of accuracy, only small quantitative changes in the numbers are expected. But this is for a system whose gross behavior is reasonably well understood. When the properties of the system are unknown, making inferences based on data from one bare gauge coupling is risky.

B. Running coupling constants from observables

A running coupling constant is determined in a lattice simulation by measuring some observable that has a perturbative expansion (that gives the coupling), at some convenient length scale, such that variation of the observable with the length scale gives the running.

In QCD, the physical problem whose solution is desired is "What is α_s at the Z-pole?" This is a needed ingredient in precision tests of the Standard Model. In the lattice calculations reported in the Review of Particle Properties (Olive *et al.*, 2014), the length scale is taken from some low energy observable computed on the lattice, and the running to the Z uses perturbation theory, rather than treating the running as something to be determined. However, there are lattice techniques that (at least for QCD) are capable of doing the running completely nonperturbatively. These are the ones that have been adapted to beyond Standard Model systems.

Recall that generic beyond Standard Model candidates have two potentially relevant couplings, a fermion mass and the gauge coupling. Coupling constant evolution takes place in (at least) a two dimensional space. This is difficult to deal with in a simulation, and so lattice studies typically begin by setting one of the relevant couplings to zero. This coupling is usually the fermion mass, and the theorist considers the gauge coupling in isolation. This is (often) not an easy place to do simulations. At a minimum, one must use boundary conditions for which the massless Dirac operator is invertible.

With an action with good enough chiral properties (staggered, domain wall, or overlap fermions) the massless limit is simply achieved, by setting the bare fermion mass to zero. With Wilson fermions, one must tune the bare mass so that a derived mass is zero. The fermion mass whose vanishing signals the chiral limit is the so-called Axial Ward Identity (AWI) fermion mass, defined through the ratio of correlators

$$am_q = \frac{1}{2}\frac{\partial_4 \left\langle A_4^b(t)\mathcal{O}^b(0)\right\rangle}{\left\langle P^b(t)\mathcal{O}^b(0)\right\rangle}. \tag{66}$$

Here $A_4^b(t) = \bar\psi\gamma_5\gamma_4\tau^b\psi$ is the time component of the local axial vector current with flavor b, taken at zero spatial momentum on the time slice t; $P^b(t)$ is the local pseudoscalar density. The operator $\mathcal{O}^b(0)$ is a source. Any calculation that must be done at $am_q = 0$ is carried out along a line called the critical kappa line, $\kappa_c(\beta)$ vs β.

However the coupling is determined, it must be analyzed. To contrast the issues encountered while studying slowly running versus quickly running couplings, it is instructive to return to Figures 5-6. The first figure shows the situation for a quickly running coupling; the second figure is the situation with twelve fundamentals, and slow running. The two problems in data analysis are to determine the shape of the $1/g^2(L)$ versus $\ln L_0/L$ curves, and to show that the determination is free of cutoff effects.

The key to doing this is to take L as the independent variable. $g^2(L)$ is the coupling defined at scale L. Interpret the L_0 in $\ln L_0/L$ as the cutoff. In a lattice calculation, L_0 is the lattice spacing a. $L_0/L = 1/N$ the number of lattice points the simulation length is divided into. Do a simulation at some value of the bare parameters. This gives a point in the figure. Now change L_0, holding the bare simulation parameters fixed. In a perfect world, we might imagine doing this infinitesimally. The running coupling will shift along the solid lines in the figure.

Next, change L to some new value L', and change the bare parameters. Tune them, until $g^2(L') = g^2(L)$. These couplings are connected by the dotted line in Fig. 5. Now the question is, how does the slope of the line change as L is changed. Or, what are the slopes of the two curves? Are they different? And if they are different, can they be used to extrapolate to tiny L_0?

In Fig. 5, the slopes are the same, by construction. But the picture also shows that when the beta function is large, it is easy to shift the bare coupling by a large amount and match renormalized couplings at a dense set of L's. In a real simulation, the line is replaced by a set of points at (integer) L's. The slope is typically replaced by measurements at two L values related by a common scale change: $L = 6 - 12$, $L = 8 - 16$, and so on. One would then have a set of measurements of the slope as a function of $g^2(L)$, at many values of L_0. One could then proceed to an extrapolation to the continuum limit.

This shifting and matching is reasonably straightforward to perform when the coupling runs quickly, as in Fig. 5. However, look at Fig. 6. We sit, say, at $\ln L_0/L = 2$ and tune bare parameters so that $1/g^2(L) = 0.17$ (the right edge of the dotted line). Now we change the bare parameters by some amount and try to reacquire the same value of g^2 at some other L. When the coupling runs slowly, matching $g^2(L')$ to the fiducial $g^2(L)$ by shifting the bare parameters requires an enormous change in L; the change diverges as we move to the fixed point. Computer resources are finite, and at some point one can no longer support the necessary L'/L ratio in a set of simulations.

One can imagine shrinking the ratio by reducing the size of the shift in couplings. (That is, the lines in Fig. 6 are spaced $\Delta 1/g_0^2 = 0.2$ apart; reduce the shift to 0.1 and try again.) Now the problem is statistics. In my experience (which is limited to the Schrödinger functional, to be described below), the uncertainty in a $1/g^2(L)$ measurement is not too dependent on the slope of the line, or even on N_c or N_f, so the intrinsic *fractional* uncertainty on the slope, from the difference of two $1/g^2(L)$'s, scales inversely with the slope. This is not a favorable result for a slowly running theory, for if the slope cannot be measured, the change in the slope also cannot be measured. Clearly, a less noisy coupling will allow one to take a smaller interval of L by reducing $\Delta(1/g^2)$, but as one approaches the critical coupling, the slope of the line will vanish regardless of definition.

Finally, whatever method is used to measure a running coupling constant, it is important to check it by collecting data in weak coupling, to validate the method against an analytic result. The goal is to see one-loop or two loop running. I, personally, do not know how to evaluate results I see in the literature which do not have such anchor points.

Two methods dominate in lattice calculations of running coupling constants. The Schrödinger functional is the older of the two. More recent calculations tend to use variations on a method called "Wilson flow."

1. Schrödinger functional

The Schrödinger functional (SF) (Della Morte *et al.*, 2005a; Jansen and Sommer, 1998; Luscher *et al.*, 1992, 1994; Sint and Sommer, 1996) is an implementation of the background field method that is especially suited for lattice calculations. It is done by performing simulations in a finite volume of linear dimension L, while imposing fixed boundary conditions on the gauge field (at Euclidean times $t = 0$ and $t = L$). The usual partition function $Z = \mathrm{Tr}\,\exp(-LH)$ is replaced by the "Schródinger functional" $Z(\phi_b, \phi_a) = \langle \phi_b | \exp(-LH) | \phi_a \rangle$. This fixing involves a free parameter η, so call the Schródinger functional $Z(\eta)$. A coupling constant is defined through the variation of the effective action Γ (which in turn is defined as $\Gamma = -\ln Z(\eta)$). The classical field that minimizes the Yang–Mills action subject to these boundary conditions is a background color-electric field. By construction the only distance scale that characterizes the background field is L, so Γ gives the running coupling via

$$\Gamma = g(L)^{-2} S^{\mathrm{cl}}_{\mathrm{YM}}, \tag{67}$$

where $S^{\mathrm{cl}}_{\mathrm{YM}}$ is the classical action of the background field. When Γ is calculated non-perturbatively, Eq. 67 gives a non-perturbative definition of the running coupling at scale L. In a simulation, the coupling constant is determined through differentiation,

$$\left.\frac{\partial \Gamma}{\partial \eta}\right|_{\eta=0} = \left.\left\langle \frac{\partial S_{YM}}{\partial \eta} - \frac{N_f}{2} \mathrm{tr} \left(\frac{1}{D_F D_F^\dagger} \frac{\partial (D_F^\dagger D_F)}{\partial \eta} \right) \right\rangle\right|_{\eta=0} \tag{68}$$

$$\equiv \frac{K}{g^2(L)}. \tag{69}$$

D_F is the lattice Dirac operator. The constant K is chosen to match to a perturbative evaluation of Eq. 68. In words, the expectation value $\langle \dots \rangle$ gives $g^2(L)$.

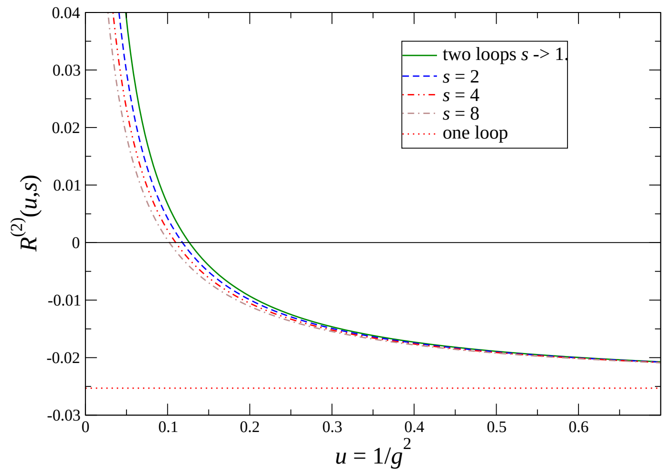

FIG. 10 Rescaled discrete beta function for $SU(2)$ gauge theory with $N_f = 2$ adjoints, calculated in two loops for various scaling factors s. Also shown are the one- and two-loop beta functions; the rescaled DBF approaches the two-loop beta function when $s \to 1$. Top to bottom, the curves are in the order shown in the legend. The figure is taken from DeGrand *et al.* (2011).

By calculating the inverse running coupling on lattices of size L and sL, we obtain the discrete beta function (DBF)

$$B(u, s) = \frac{1}{g^2(sL)} - \frac{1}{g^2(L)}, \qquad u \equiv \frac{1}{g^2(L)} \ . \tag{70}$$

It is necessary to deal with lattice artifacts in $B(u, s)$. This is often done by comparing data from systems at fixed aspect ratio s, for example, $L = 6$ and 12, 8 and 16, 12 and 24.

With the definition of the beta function for the inverse coupling in terms of the usual beta function

$$\tilde{\beta}(1/g^2) \equiv \frac{d(1/g^2)}{d \ln L} = 2\beta(g^2)/g^4 = 2u^2 \beta(1/u), \tag{71}$$

the discrete beta function is

$$\ln s = \int_L^{sL} \frac{dL'}{L'} = \int_u^{u+B(u,s)} \frac{du'}{\tilde{\beta}(u')} \ . \tag{72}$$

The literature is often careful to distinguish between the DBF and the usual beta function. For a quickly running system like QCD, it is necessary to do this. But in a slowly running system the DBF's we can measure are, to high accuracy, just proportional to the beta function itself. This occurs because the coupling runs slowly and because the values of s accessible in a simulation are small. In that case the *rescaled* DBF, defined as

$$R(u, s) = \frac{B(u, s)}{\ln s} \ , \tag{73}$$

will be approximately equal to the beta function $\tilde{\beta}(u)$. The situation for $SU(2)$ with $N_f = 2$ adjoints is illustrated in Fig. 10. The figure shows the two-loop result,

$$R^{(2)}(u,s) = -\frac{2b_1}{16\pi^2} - \frac{b_2}{16\pi^2 b_1} \times \frac{\ln\left[1 + (2b_1/16\pi^2)u^{-1}\ln s\right]}{\ln s}, \quad (74)$$

for the rescaled DBF for scale factor $s = 2, 4, 8$, compared to the one-loop and two-loop beta functions. The rescaled DBF for $s = 2$ is hardly distinguishable from the beta function.

There are two lessons to be drawn from Fig. 10. If the actual DBF resembles the two-loop result, we can combine the rescaled DBF's for many scale factors s onto a single plot to give a good approximation to the actual beta function. Furthermore, since any value of $s \lesssim 2$ is as good as another, we can combine the couplings for all lattice volumes studied to extract the beta function via a fit. Most of the scaling violations will be at the smallest a/L, so we can simply look at the largest L data points. With slow running, one is really asking whether the slope of the $1/g^2(L)$ versus $\ln L$ line varies with L.

An example of a plot of $1/g^2(L)$ versus $\ln L$ is shown in Fig. 11. It is for the case of $SU(2)$ gauge theory coupled to $N_f = 2$ adjoint fermions, from DeGrand *et al.* (2011). The slope changes sign. This is the clearest example of IRFP behavior from a Schrödinger functional analysis, that I know. The picture can also be used to illustrate various ways of dealing with lattice artifacts: different methods amount to computing the slope of each line by taking different mixes of L values. For example, one could compare the slope from L's of fixed ratio, or from the whole line, or by dropping data points at smaller L's.

There are studies of alternative choices of boundary conditions of the Schrödinger functional, with the idea of finding a set with reduced lattice artifacts (Karavirta *et al.*, 2012b; Sint and Vilaseca, 2011, 2012). Typically, this is done using perturbation theory. The issue with using them for slowly-running systems near the bottom of the conformal window is that the place where one really wants to simulate (typically, looking for a zero of a beta function) is at strong coupling. There, perturbation theory is unreliable. Choosing a functional form to extrapolate to zero cutoff that includes lattice artifacts is, at best, phenomenology.

2. "Flow"

The new alternative goes by names such as "gradient flow" or "Wilson flow." It is a smoothing method for gauge fields achieved by diffusion in a fictitious (fifth dimensional) time t. In the continuum version, a smooth gauge field $B_{t,\mu}$ is defined in terms of the original gauge

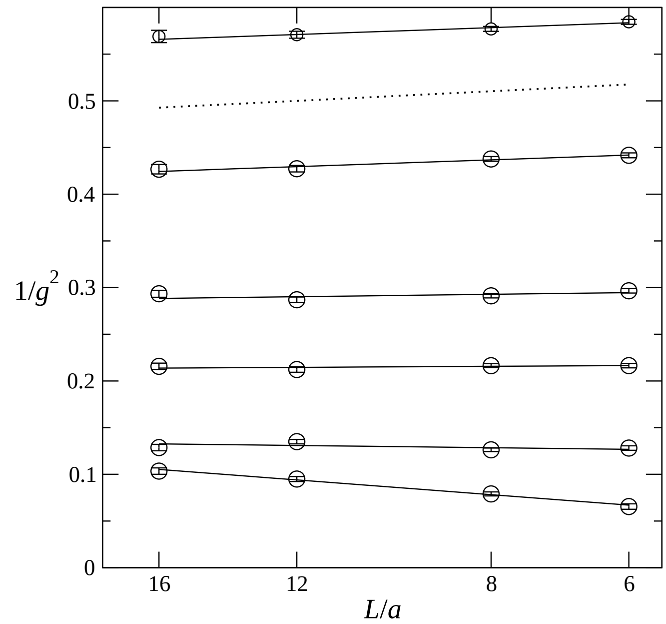

FIG. 11 SF coupling $1/g^2$ *vs.* L/a (plotted on a logarithmic scale), for $SU(2)$ gauge theory coupled to $N_f = 2$ adjoint fermions, from DeGrand *et al.* (2011). Data are at lattice gauge coupling (from the top) $\beta = 3.0$, 2.5, 2.0, 1.75, 1.5, and 1.4. The lines through the data points are fits to the data at each β of the form $1/g^2(L) = a + b\ln(L/a)$. The dotted line has the slope $2b_1/(16\pi^2)$ as given by the lowest-order beta function, Eq. (24).

field A_μ through an iterative process

$$\begin{aligned} \partial_t B_{t,\mu} &= D_{t,\mu} B_{t,\mu\nu} \\ B_{t,\mu\nu} &= \partial_\mu B_{t,\nu} - \partial_\nu B_{t,\mu} + [B_{t,\mu}, B_{t,\nu}], \end{aligned} \quad (75)$$

where the smoothed field begins as the original one,

$$B_{0,\mu}(x) = A_\mu(x). \quad (76)$$

Correlators of the flow field can be used to define a coupling constant (Fodor *et al.*, 2012). For example, one possibility, due to Luscher (2010), is

$$\langle E(t) \rangle = \frac{1}{4} \langle G_{\mu\nu}(t) G_{\mu\nu}(t) \rangle = N_c \frac{g^2}{t^2} + O(g^4). \quad (77)$$

This can be used to define a renormalized coupling at a scale t,

$$g^2_{flow}(t) = \frac{t^2 \langle E(t) \rangle}{N_c}. \quad (78)$$

Simulations in a box of size L set the overall scale, and the second scale, t, is taken to be a fixed fraction of L. The method has many variations. For example, the spatial averaging term in the diffusion equation could be identical to, or different from, the discretized gradient term in

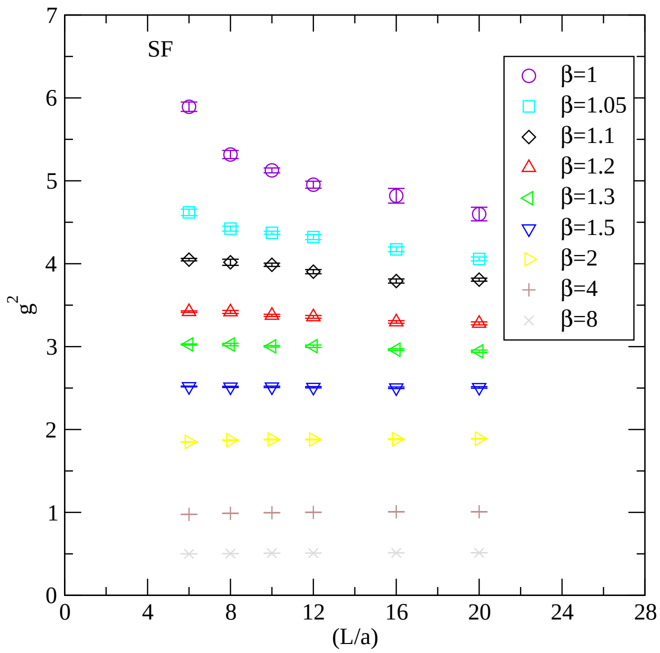

FIG. 12 Continuum extrapolation of the Schrödinger functional step scaling function for $SU(2)$ with $N_f = 2$ adjoints, from Rantaharju (2014).

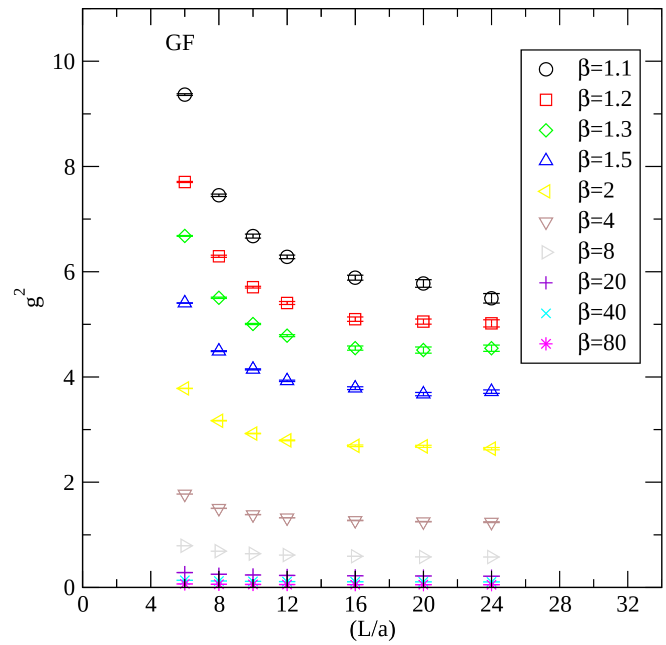

FIG. 13 Continuum extrapolation of the gradient flow step scaling function for $SU(2)$ with $N_f = 2$ adjoints, from Rantaharju (2014).

the action which is simulated. Flow can be combined with the Schrödinger functional (Fritzsch and Ramos, 2013), or can be used by itself to define a coupling constant.

People who have used it report that they can compute a coupling constant with much smaller errors than a Schrödinger functional calculation would give with equivalent statistics. Since the choice of t defines its own coupling, it is possible to collect simultaneous data for different definitions of couplings $g^2_{flow}(t)$ and select the best one (by some criterion) later. Discretization errors must still be removed along the lines previously described. Recently, Rantaharju (2014) compared the Schrödinger functional coupling to gradient flow in $SU(2)$ with $N_f = 2$ adjoints. Here there was an issue with the simplest version of a flow running coupling: discretization errors were observed to be larger than for the Schrödinger functional. I reproduce his figures in Figs. 12-13.

These pictures are only the beginning of a presently ongoing research area, studying how to suppress lattice artifacts in measurements with flow. Tree level improvement is described by Fodor *et al.* (2014b) and (in a preliminary version) by Sint and Ramos (2015). However (as for the Schr̈odinger functional) the theoretical analysis assumes closeness to free field behavior. Fixed points for interesting slowly running systems occur in strong coupling (if at all), and dealing with lattice artifacts in strong coupling will, I think, always be phenomenological.

3. Monte Carlo Renormalization Group

Another approach, called "Monte Carlo Renormalization Group" (MCRG), is an implementation of the real space renormalization group. Take a system defined with a momentum space cutoff Λ (or a lattice spacing a) and some set of dynamical variables U. Introduce some averaging algorithm which replaces the fine grained U's with some coarse grained V's. Now define a system with a smaller Λ' or a bigger lattice spacing, by integrating out the U's, to give a partition function expressed in terms of the coarse-grained variables and their action $S'(V)$:

$$\begin{aligned} Z &= \int DUe^{-S(U)} \\ &= \int dVT(U,V)\int dUe^{-S(U)} \\ &\equiv \int dVe^{-S'(V)}. \end{aligned} \tag{79}$$

Repeat this procedure. The change in the action is encoded in a set of transformation rules for the coupling constants in the action $\{K_1, K_2, K_3 \dots\} \rightarrow \{K'_1, K'_2, K'_3 \dots\}$. In a lattice system, the range of the averaging of U's to V gives the scale change s, so that we can speak of the coupling constants running over a (discrete) scale s. How the couplings run depends on the choice of blocking kernel $T(U,V)$; this is the analog of a renormalization scheme. As the system is repeatedly blocked, the (infrared) irrelevant couplings die away leaving the marginal and relevant ones. These couplings will approach a unique renormalized trajectory emanating from the critical surface. Different bare couplings begin at different places but end upon the renormalized trajectory.

The issue now is, how to measure the couplings. In the "two-lattice matching MCRG method" this is done indirectly, through observables. The idea is that if observables are measured in two different lattice simulations, and if all the observables have identical expectation values, than the systems are identical, so their coupling constants are matched. Now imagine two systems with different K's. Take one system and perform n blocking steps so that the cutoff is reduced by a factor s^n. Measure many observables. Next, suppose that a second system, with its own set of K's, is blocked, and suppose that after $n-1$ steps its observables coincide exactly with those of the first system (and remains identical under further blocking). We would say that when the bare couplings flow from $\{K_1, K_2, K_3 \dots\} \to \{K_1', K_2', K_3' \dots\}$, long distance physics is unchanged under a scale factor s. This is a renormalization group equation for the bare parameters.

So, the rubric is:

1. Generate a first configuration ensemble of size L^d with action $S(K)$. Block each configuration n times and measure a set of expectation values on the resulting $(L/s^n)^d$ set.

2. Generate a second ensemble of configurations of size $(L/s)^d$ with action $S(K')$. Block each configuration $n-1$ times and measure the same expectation values on the resulting $(L/s^n)^d$ set. Compare the results with that obtained in step 1. and tune the coupling K' such that the expectation values agree. A cartoon is shown in Fig. 14.

The method has many good features: one can use smallish lattices and measurements of local operators usually can be done accurately. It has some not so good features: the location of the fixed point, the renormalized trajectory, and the number of steps needed to reach the renormalized trajectory all depend on the choice of action and of blocking kernel. Of course, it is advantageous to be able to tune $T(U, V)$. The analysis is much easier when there is only one relevant variable (for example, in pure gauge theory, the gauge coupling) than when there might be more than one (typically, the mass and perhaps the gauge coupling).

Early references to these methods for spin models are Swendsen (1979, 1984) and for QCD, Bowler *et al.* (1985); Hasenfratz *et al.* (1984a,b). The use of these methods for slowly running theories was revived by Hasenfratz (2009, 2010, 2012). Most of her work was on $SU(3)$ with 8 and 12 flavors of fundamentals. Results will be discussed below.

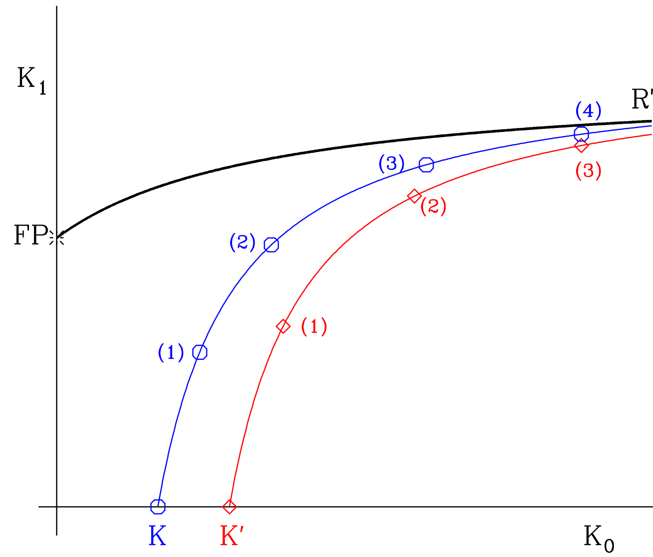

FIG. 14 Sketch (from Hasenfratz (2009)) of the RG flow around a fixed point (FP) with one relevant operator. The coupling pair (K, K') indicates matched couplings whose correlation length differ by a factor of s. The line labeled "RT" is the renormalized trajectory.

A number of other possibilities for renormalized couplings have been proposed; none has a long citation trail. One worth mentioning is a technique (de Divitiis *et al.*, 1994) that defines a coupling through the correlation of Polyakov loops, measured over distances that are a fixed fraction of the lattice size. This has been used by Lin *et al.* (2012) for many-flavor studies in $SU(3)$.

C. Computing the mass anomalous dimension γ_m

1. Schrödinger functional

The Schrödinger functional gives γ_m through the volume dependence of the renormalization factor Z_P of the isovector pseudoscalar density $P^a = \bar\psi\gamma_5(\tau^a/2)\psi$. (The pseudoscalar density is related by a chiral rotation to $\bar\psi\psi$, which is the object of interest.) It is computed from two correlators via (Bursa *et al.*, 2010; Capitani *et al.*, 1999; Della Morte *et al.*, 2005b; Sint and Weisz, 1999)

$$Z_P = \frac{c\sqrt{f_1}}{f_P(L/2)}. \tag{80}$$

f_P is the propagator from the $t=0$ boundary to a point pseudoscalar operator at time x_0,

$$f_P(x_0) = -\frac{1}{3}\sum_a \int d^3y\, d^3z\, \Big\langle \overline{\psi}(x_0)\gamma_5\frac{\tau^a}{2}\psi(x_0) \times \bar\zeta(y)\gamma_5\frac{\tau^a}{2}\zeta(z)\Big\rangle. \tag{81}$$

It is conventional to take $x_0 = L/2$. In the expression, ζ and $\bar\zeta$ are gauge-invariant wall sources at $t = a$, i. e., one lattice layer away from the $t = 0$ boundary. The f_1 factor is the boundary-to-boundary correlator, which cancels the normalization of the wall source. Explicitly, it is

$$f_1 = -\frac{1}{3L^6}\sum_a \int d^3u\, d^3v\, d^3y\, d^3z\, \left\langle \overline{\zeta}'(u)\gamma_5 \frac{\tau^a}{2}\zeta'(v) \right.$$
$$\left. \times\, \overline{\zeta}(y)\gamma_5 \frac{\tau^a}{2}\zeta(z) \right\rangle , \qquad (82)$$

and ζ' and $\bar{\zeta}'$ are wall sources at $t = L - a$.

The (continuum) mass step scaling function (Bursa *et al.*, 2010; Capitani *et al.*, 1999; Della Morte *et al.*, 2005b; Sint and Weisz, 1999) is

$$\sigma_P(v, s) = \left.\frac{Z_P(sL)}{Z_P(L)}\right|_{g^2(L)=v} . \qquad (83)$$

It is related to the mass anomalous dimension via

$$\sigma_P(v, s) = \exp\left[-\int_1^s \frac{dt}{t}\, \gamma_m\left(g^2(tL)\right)\right] . \qquad (84)$$

When the SF coupling $g^2(L)$ runs slowly, Eq. 84 is well approximated by

$$\sigma_P(g^2, s) = s^{-\gamma_m(g^2)}. \qquad (85)$$

We can therefore combine many sL values collected at the same bare parameter values into one fit function giving γ_m,

$$\ln Z_P(L) = -\gamma_m \ln L + \text{const}. \qquad (86)$$

An example of data for Z_P is shown in Fig. 15.

As in the case of the running coupling, the question is whether the slope of the line changes with L, and what its value is at large L. This can be done either by comparing the slope from pairs of points at fixed s, or of the whole line. Again, there are many possibilities.

2. Finite size scaling

Recall that the correlation length ξ of an infrared conformal system would diverge as the fermion mass m_q were taken to zero, but the finite system size L prevents it. If the only large length scales in the problem are ξ and L, then observables can only involve the scales ξ and L, and their ratio. This "finite size scaling" argument says that the correlation length in finite volume ξ_L must scale as

$$\xi_L = LF(\xi/L) \qquad (87)$$

where $F(x)$ is some unknown function of ξ/L. A somewhat more useful version of this relation invokes Eq. 61, to say

$$\xi_L = Lf(L^{y_m} m_q). \qquad (88)$$

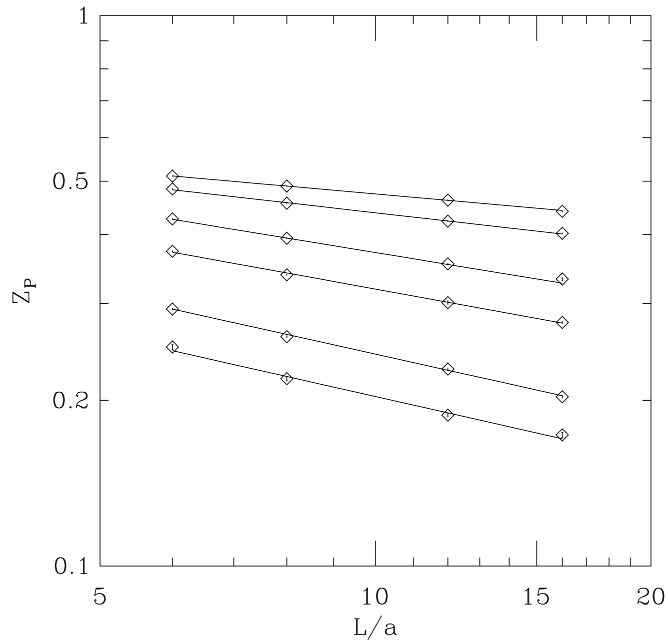

FIG. 15 Pseudoscalar renormalization constant Z_P, from DeGrand *et al.* (2011). From the top, data are from $\beta = 3.0$, 2.5, 2.0, 1.75, 1.5 and 1.4. Lines are fits to $\ln Z_P(L) = -\gamma_m \ln(L/a) + \text{const}$ for each β.

This expression can be used to find the exponent y_m. One can plot ξ_L/L vs $L^{y_m} m_q$ for many L's, and vary y_m. Under this variation, data from different L's will march across the x axis at different rates. The exponent can be determined by tuning y_m to collapse the data onto a single curve. An example of such an analysis is shown in Fig. 16. It is for $SU(3)$ gauge theory and $N_f = 2$ symmetric representation fermions, by DeGrand (2009).

Often, it is unknown whether the system under investigation is infrared conformal, or not. A comparison of its data with Eq. 88 is used to decide the question. This could be misleading: a coupling which runs so slowly that it scarcely changes over the range of available L's would induce effectively conformal behavior.

Many finite size scaling analyses of lattice data replace curve collapse with a fit to some functional form for $F(x)$. The shape of F is known for extreme values of its argument. For example, in Eq. 87, $F(x) \sim x$ for small x and $F(x) \sim 1$ for large x. Fitting to a curve allows one to quote a goodness - of - fit parameter (such as a chi-squared) along with the fit value of y_m. The problem with this is that, generally, the complete functional form of $F(x)$ or $f(x)$ is unknown. A poor fit could occur because the guessed functional dependence of $f(x)$ was incorrect. Sometimes, one can fit the scaling functions to high quality Monte Carlo data from one model which is a member of its universality class, and use those fits to test whether other systems lie in that class. An example of this analysis is that of Engels and Karsch (2012), who

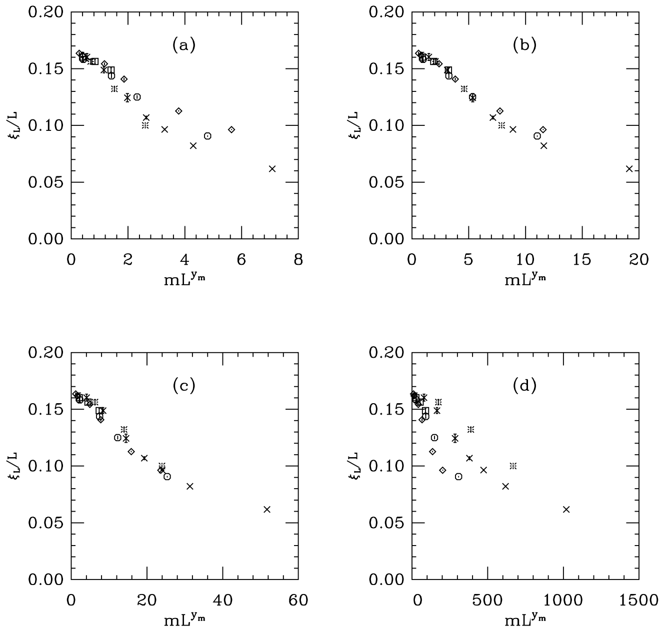

FIG. 16 Curve collapse in $SU(3)$ gauge theory with $N_f = 2$ symmetric-representation fermions, from DeGrand (2009). Plots of ξ_L/L vs $m_q L^{y_m}$ at $\beta = 5.2$ for four choices of y_m: (a) $y_m = 1.0$, (b) $y_m = 1.4$, (c) $y_m = 1.8$ (d) $y_m = 3.0$. Plotting symbols are for different simulation volumes, diamonds, $12^3 \times 6$ ($L = 6$); octagons, $12^3 \times 8$ ($L = 8$); squares, $16^3 \times 8$ ($L = 8$); crosses, 12^4 ($L = 12$); bursts, 16^4 ($L = 16$). Curve collapse seems to be best in panels (b) and (c).

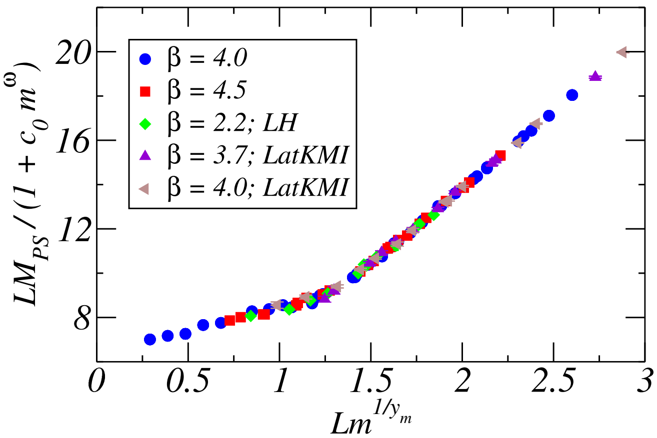

FIG. 17 Cheng *et al.* (2014a)'s best curve collapse fit, combining their data and that of of the LH and LatKMI collaborations (Aoki *et al.*, 2012; Fodor *et al.*, 2011a).

fit scaling functions of the three-dimensional $O(4)$ spin model with the aim of making comparisons with $N_f = 2$ QCD near its chiral transition. One should also keep in mind that different quantities have their own scaling functions. A fit to lattice data for (say) the pseudoscalar mass, the vector mass, and f_π would have to use three different scaling functions, one for each quantity.

In contrast, it is difficult to assign a goodness-of-fit parameter to curve collapse.

An issue with this analysis bedevils many of the systems which have been studied: the gauge coupling g_0 runs very slowly. This means that its exponent $y_g \sim 0$. An analysis that left it out would produce a leading critical exponent y_m that appeared to drift with bare gauge coupling. If the marginal coupling is included in the scaling analysis, Eq. 88 is modified to

$$\xi_L = L f_H\left(x, g_0 m^{\omega}\right), \tag{89}$$

where $\omega \equiv -y_g/y_m$. The scaling function $f_H\left(x, g_0 m^{\omega}\right)$ is analytic even at the fixed point, and can be expanded as

$$\xi_L = L F_H(x)\left\{1 + g_0 m^{\omega} G_H(x) + O\left(g_0^2 m^{2\omega}\right)\right\}. \tag{90}$$

The first term is the usual expression while the second accounts for the leading corrections to scaling.

The first group to go beyond Eq. 88 was Cheng *et al.* (2014a). These authors studied the system with $N_c = 3$ and $N_f = 12$ fundamental fermions. They fit (with $1/\xi_L = M_H$)

$$\frac{L M_H}{1 + c_G g_0 m^{\omega}} = F_H(x). \tag{91}$$

They did fits to several dimensionful parameters (pseudoscalar and vector masses, f_π) over a wide range of volumes and fermion masses. Weaknesses of the calculation are that first, the authors assumed some functional form for the scaling function (to be fair, I do not see how to do curve collapse in a multidimensional space) and second, the confidence levels associated with the chi-squareds of a number of the fits are poor. Nevertheless, I find it quite impressive. y_m is nearly independent of bare gauge coupling over a wide range. Including the non-leading exponent renders all previous studies obsolete.

Figure 17 shows the best curve collapse fit for the pseudoscalar mass from these authors. It uses their data plus results from two other collaborations, with many L's and many β's. Compare the y_m's with and without the correction, Figs. 18-19.

3. Mass anomalous dimension from Dirac eigenvalues

Next, there are a set of related methods extracting y_m from the spectral density of eigenvalues λ of the Dirac operator. The physics seems simple: The Banks-Casher relation (Banks and Casher, 1980) connects the condensate Σ and the density of eigenvalues λ of the Dirac operator $\rho(\lambda)$. At nonzero mass it is

$$\Sigma(m_q) = -\int \rho(\lambda) d\lambda \frac{2 m_q}{\lambda^2 + m_q^2}. \tag{92}$$

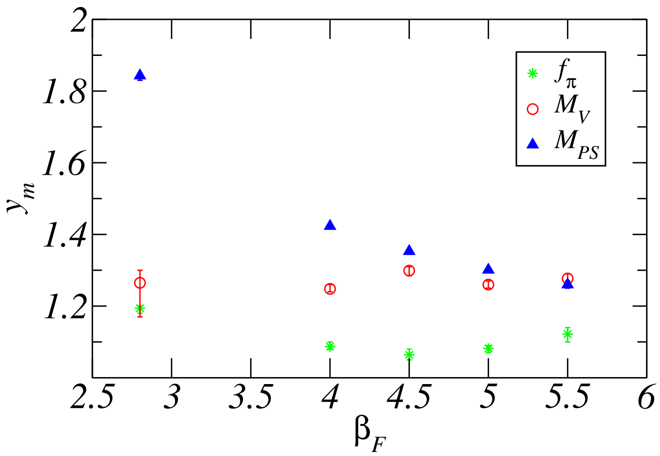

FIG. 18 Exponent y_m from Cheng *et al.* (2014a) for the pseudoscalar mass, vector mass, and pseudoscalar decay constant, not including the non-leading coupling. The abscissa is the bare gauge coupling.

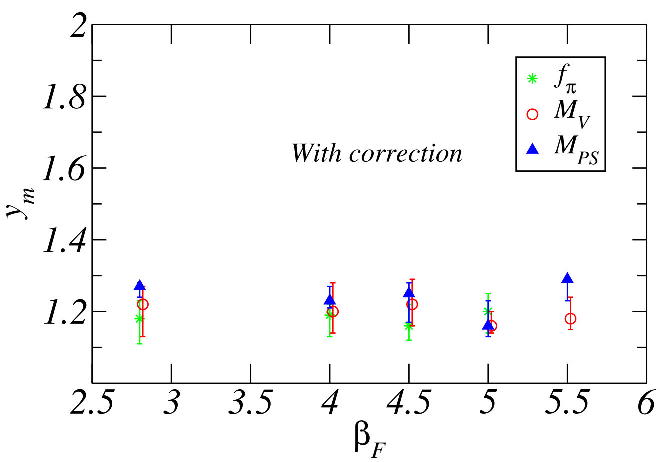

FIG. 19 Exponent y_m from Cheng *et al.* (2014a) for the pseudoscalar mass, vector mass, and pseudoscalar decay constant, including the non-leading coupling.

If the massless theory is conformal, and if the condensate $\Sigma(m_q)$ scales as m_q^α for small mass, then $\rho(\lambda)$ also scales as λ^α.

A finite-size scaling argument (Akemann *et al.*, 1998) relates the scaling for the density ρ to the scaling of the value of individual eigenvalues. If we consider the average value of the ith eigenvalue of the Dirac operator in a box of volume $V = L^D$, and if $\rho(\lambda) \sim \lambda^\alpha$, then we expect

$$\langle \lambda_i \rangle \sim \left(\frac{1}{L}\right)^p \tag{93}$$

where

$$p = \frac{D}{1+\alpha}. \tag{94}$$

For the case of a theory with an IRFP, p is equal to y_m, the leading exponent. Thus $\rho(\lambda) \sim \lambda^{D/y_m - 1}$.

In QCD, or in other chirally broken theories, $\alpha = 0$ and $p = D$. Here the story is rich, and involves an interplay of confinement, chiral symmetry breaking and random matrix theory. [See (Damgaard *et al.*, 1999; Osborn *et al.*, 1999).] Even the probability distribution of individual eigenvalues can be used to determine the condensate. It is a universal function of the product $\lambda\Sigma V$ with $V = L^D$. Results obtained build on ones like Eq. 65.

Most of the beyond Standard Model literature uses the integrated spectral density or mode number. This technique is adapted from its original QCD venue, following the discussion in Giusti and Luscher (2009). Their approach has been applied to near-conformal theories by a number of authors. The most-cited beyond Standard Model study is Patella (2012), who studied the integrated spectral density for $SU(2)$ gauge theory coupled to $N_f = 2$ adjoint fermions.

He split up the eigenvalues into three classes:

- Very small ones, which are sensitive to the simulation volume
- Intermediate ones which show the desired power law scaling behavior
- Large ones which (for an asymptotically free system) go over to free field $\rho(\lambda) \sim \lambda^3$ behavior

He integrated over the intermediate eigenvalues to find an exponent.

An issue with using this method is that the exponent depends on the range of eigenvalues used to measure it. The authors of Cheng *et al.* (2013) combine the intermediate and large eigenvalues to construct a "scale dependent mass anomalous dimension," whose scale in energy space is given by λ itself, and whose extrapolation to small λ gives the actual y_m (assuming, of course, that the system studied is truly conformal). They are able to compare and contrast a confining theory ($SU(3)$ with $N_f = 4$ fundamentals) with a slowly running one ($SU(3)$ with $N_f = 12$ fundamentals), which they identify as conformal. Their prediction for y_m will be quoted below.

When I read these papers, I cannot help thinking: are the smallest eigenvalues, which are the ones most sensitive to the volume, not also the ones that are sensitive to the longest distance physics? And if so, is there not some kind of finite size scaling or curve collapse story that can be told about them? No such story exists in the literature, as far as I know.

There is another issue with the use of eigenvalues, which appears when one thinks about what is actually being measured.

Briefly, the spectral density of the massless Dirac operator

$$\rho(\lambda) = \frac{1}{V}\left\langle \sum_k \delta(\lambda - \lambda_k) \right\rangle \quad (95)$$

is the discontinuity across the imaginary mass axis of the resolvant,

$$\rho(\lambda) = \frac{1}{2\pi} \lim_{\epsilon\to 0} \Sigma_{val}(i\lambda + \epsilon) - \Sigma_{val}(i\lambda - \epsilon) \quad (96)$$

where

$$\Sigma_{val}(m_v) = \frac{1}{V}\sum_k \left\langle \frac{1}{m_v + i\lambda_k} \right\rangle \quad (97)$$

is the expectation value $\langle \psi(0)\bar{\psi}(0) \rangle$ for a fermion of mass m_v. The resolvant cannot be computed in ordinary field theory. The ordinary partition function is simply not a generator for it. We need a generator, and that can be found, but in a partially quenched version of our theory, where the valence fermions have a different mass from the dynamical fermions (and the system has additional bosonic degrees of freedom to remove the valence fermions from the partition function). Only this extended field theory can probe the spectral density.

So, can a partially quenched theory tell us things about an unquenched theory? For a chirally broken and confining theory like QCD, it can, and partial quenching is one of the standard techniques for computing low energy constants. But outside of this framework, I know of no precise statement of the connection. The end result is that if chiral symmetry is unbroken, the physics of the measured spectral density may not be quite what people think it is.

VI. LATTICE RESULTS FOR SYSTEMS WITH SLOWLY RUNNING COUPLINGS – BY SYSTEM

Now we begin a survey of lattice calculations, separated by specific model properties.

A. Early studies (before about 2007)

There is a long history of lattice studies of systems with many fermionic degrees of freedom. Most of the early ones involved thermodynamics. The question was, did the deconfinement temperature T_c scale appropriately (remain a constant ratio with respect to any other massive observable) as the lattice spacing was taken away? The data were ambiguous. An early review, Fleming (2008), contains citations to this work. There were also a number of simulations of Wilson fermions with many flavors of fundamentals by Iwasaki and collaborators [two papers are Iwasaki *et al.* (2004, 1992)]. These studies also searched for the loss of confinement as the number of flavors increased. (They were actually interested in seeing whether a deconfined phase persisted all the way to $\beta = 0$.) Many of the features of later simulations with Wilson fermions are first present in these studies.

Damgaard *et al.* (1997) studied $SU(3)$ gauge theory coupled to sixteen fundamental flavors, and observed that while the system had a strong coupling phase, its weak coupling phase was chirally restored. They argued that they could define a running coupling from the string tension, and that its beta function was positive in the weak coupling phase. Looking back, this was the first appearance of the interior of the conformal window in a simulation. It was followed by Heller (1998) – the first Schrödinger functional measurement of a running coupling in a many-flavor system. Heller also observed a positive beta function. This paper was the inspiration for the later, beyond Standard Model Schrödinger functional work.

The earliest numerical simulation of a system with an explicit place in beyond Standard Model phenomenology was by Catterall and Sannino (2007), who studied what they called "minimal walking technicolor" ($SU(2)$ with $N_f = 2$ adjoint representation fermions). "Minimal" refers to the particle content: with $N_f = 2$ there are three Goldstones to be eaten by the W and Z leaving no technipions behind. "Walking," of course, because the beta function is small in one loop, and the system might be confining according to the analysis of Dietrich and Sannino (2007) and Sannino and Tuominen (2005). They carried out spectroscopic measurements and observed what was at the time very peculiar behavior, that I have already described above – recall Fig. 9. This was later recognized as the spectroscopy of a near conformal system in finite volume.

The field then became very active. To go on, we should separate the discussion of different physical systems into their own sections.

B. Studies of $N_c = 3$ and many flavors of fundamental fermions

I do not think these systems were ever taken seriously as true technicolor candidates. They have have too many ($N_f^2 - 1$) Goldstone bosons. Electroweak symmetry breaking only eats three of them, leaving $N_f^2 - 4$ technipions to be observed in experiment, or somehow explained away. But, all lattice QCD people have computer programs to simulate $SU(3)$ gauge fields and it is easy to modify the code to do many flavors of fermions. With staggered fermions, multiples of 4 are easy, with Wilson fermions, multiples of two. The motivation was just to see whether walking actually occurred, or not. For these reasons, I believe it is still the most-studied lattice beyond Standard Model sector, both in number of papers written and in computer hours consumed.

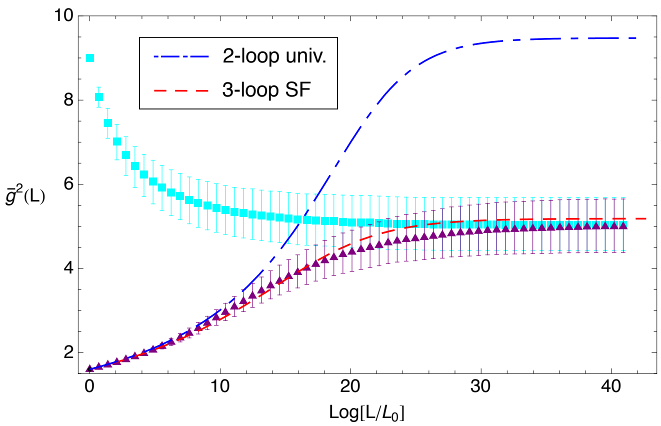

FIG. 20 Continuum running for $SU(3)$ gauge theory coupled to $N_f = 12$ fundamentals, from Appelquist *et al.* (2009). Results shown for running from below the infrared fixed point (purple triangles) are based on $\overline{g}^2(L_0) \equiv 1.6$. Also shown is continuum backwards running from above the fixed point (light blue squares), based on $\overline{g}^2(L_0) \equiv 9.0$.

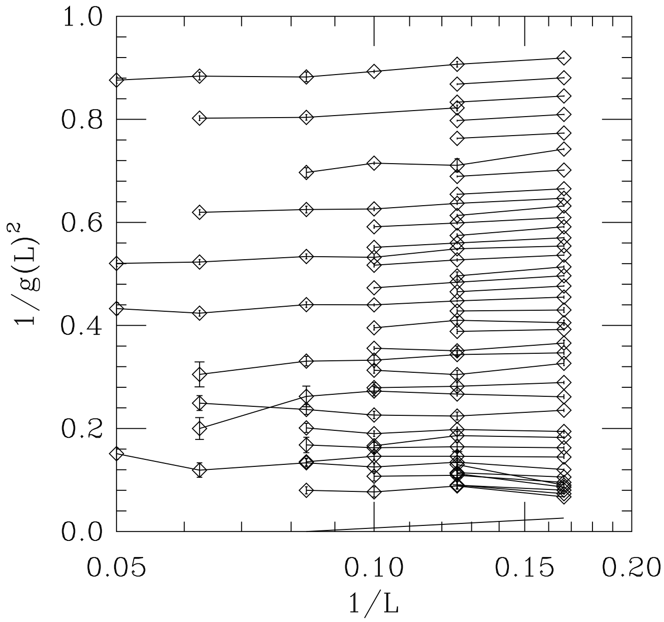

FIG. 21 Raw lattice data for $SU(3)$ gauge theory coupled to $N_f = 12$ fundamentals, from Appelquist *et al.* (2009). Lines connect data with the same bare couplings. The line at the bottom is the slope expected from one-loop running.

The earliest studies in this area were the large scale Schrödinger functional simulations of $N_f = 8$ and 12 by Appelquist *et al.* (2008, 2009). They claimed to observe an IRFP for $N_f = 12$, while the beta function for $N_f = 8$ was everywhere negative. Thus, the boundary for the conformal window was claimed to be somewhere between 8 and 12 flavors. Their results for $N_f = 12$, as they presented them, are shown in Fig. 20.

This figure uses heavily processed lattice data. It comes from a many-parameter fit to all of their data at many bare couplings and many volumes, of the form

$$\frac{1}{\overline{g}^2(\beta, L/a)} = \frac{\beta}{6}\left[1 - \sum_{i=1}^{n} c_{i,L/a}\left(\frac{6}{\beta}\right)^i\right]. \qquad (98)$$

I cannot evaluate results from such global fits. Fortunately, these authors published their data, and it is possible to look at it directly. This is shown in Fig. 21. Lines connect data collected at the same bare gauge couplings. I have drawn a line whose slope is the one-loop beta function result. As the bare gauge coupling moves from weak to strong coupling, the slope of the lines in Fig. 21 flattens slightly. Does it change sign? They said Yes, but the existence of a large literature about this system indicates that others looked at the figure and said Maybe.

The situation with eight flavors seemed to be much more clear cut: the beta function was everywhere negative. This can be seen in a plot of $1/g^2(L)$ versus $\ln L$, Fig. 22. In fact, all the lines show nearly the same slope. This is not surprising from a perturbative viewpoint; b_2 (recall Eq. 24) is nearly zero.

1. $N_f = 12$

The largest set of lattice results concerns $N_f = 12$.

Shortly after Appelquist *et al.* (2009) appeared, Fodor *et al.* (2009) carried out spectroscopic studies with $N_f =$ 4, 8 and 9 flavors and argued that even the larger N_f systems were chirally broken. In a conference proceedings followed by a journal article, Fodor *et al.* (2011a) claimed that $N_f = 12$ was also confining. Their lattice data was taken at many volumes and fermion masses, but one bare gauge coupling. Appelquist *et al.* (2011a) and DeGrand (2011) did finite size scaling studies to the data sets of Fodor *et al.* (2011a) and concluded that they were consistent with infrared conformality. A later large volume study by Aoki *et al.* (2012) contributed data sets at two bare couplings and concluded that the data favored infrared conformality. Finally, Cheng *et al.* (2014a) performed a finite size scaling analysis, with the leading irrelevant operator, on all these data sets, and claimed that all the data were consistent with infrared conformality, strongly affected by a nonleading exponent. Figures were shown above, Figs. 17-19.

Besides Appelquist *et al.* (2009), several groups claimed to observe an IRFP. Lin *et al.* (2012) computed the renormalized coupling from twisted Polyakov loops (de Divitiis *et al.*, 1994) and claimed this. Hasenfratz (2012) used MCRG to observe a positive beta function (in my conventions) for the bare step scaling function, in strong coupling. This is evidence for an IRFP since the beta function is negative in weak coupling. And most re-

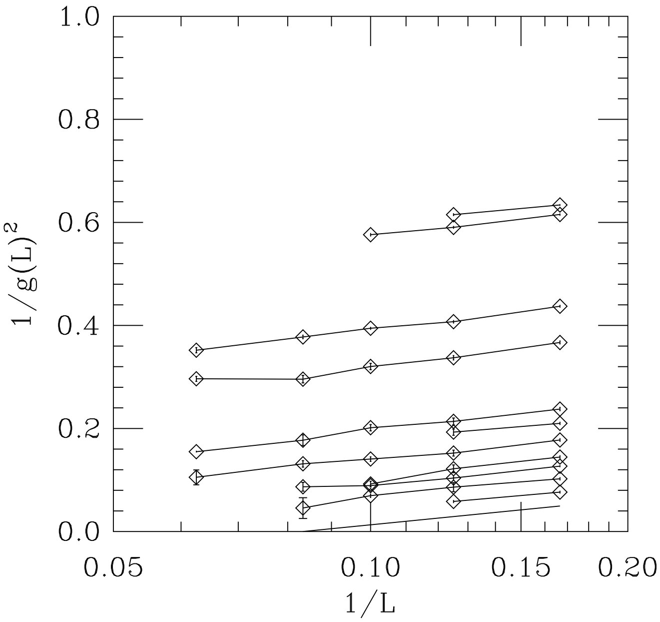

FIG. 22 Raw lattice data for $SU(3)$ gauge theory coupled to $N_f = 8$ fundamentals, from Appelquist *et al.* (2009). Lines connect data with the same bare couplings. The line at the bottom is the slope expected from one-loop running.

Reference	method	result	γ_m
Appelquist *et al.* (2009)	SF	I*	
Fodor *et al.* (2011a)	spectra	C	
Appelquist *et al.* (2011a)	FSS (fit)	I	0.40(1)
DeGrand (2011)	FSS (cc)	I	0.35(23)
Hasenfratz (2012)	MCRG	I*	
Aoki *et al.* (2012)	FSS (cc)	I	0.4-0.5
Lin *et al.* (2012)	other	I*	
Cheng *et al.* (2013)	spectral	I	0.32(3)
Cheng *et al.* (2014a)	FSS (fit)	I	0.235(15)
Cheng *et al.* (2014b)	flow	I*	
Lombardo *et al.* (2014)	FSS (fit)	I	0.235(46))

TABLE I Claims for the phase structure for $SU(3)$, $N_f = 12$ fundamentals. The "result" column is keyed with a C if the authors claim to observe a confined, chirally broken system, I if infrared conformal behavior is claimed or assumed. A "*" indicates that an IRFP was observed. Under "method," "FSS" refers to finite size scaling with "fit" for a fit to a known scaling function and "cc" for curve collapse. "Spectral" refers to use of the spectral density of Dirac eigenvalues. "Spectra" refers to spectroscopy. SF is Schrödinger functional. "Flow" refers to some variant of Wilson flow. Predictions for γ_m are given where available.

cently, Cheng *et al.* (2014b) observed a fixed point using a gradient flow definition of a running coupling.

To summarize: All groups except Fodor *et al.* (2011a) observe behavior consistent with infrared conformality for $N_f = 12$. The mass anomalous dimension γ_m is found to be small by all who report a measurement. There are enough studies of this system that a table is useful. See Tab. I. I think that the evidence in favor of infrared conformality is overwhelming.

Aoki *et al.* (2013a) measure the mass of a scalar resonance in $N_f = 12$. They find it is slightly lighter than the pseudoscalar mass at the nonzero fermion masses where they simulate. This would be the light state consistent with incipient criticality described in Sec. III.F.

This is probably a good place to talk about lattice artifacts in strong coupling. Recall the situation with these slowly - running theories: if the gauge coupling is strong at long distance, than it is also strong at the cutoff scale. This is an invitation for lattice artifacts to appear. Universality should be lost. This is not just a problem of principle. These days, essentially all lattice groups simulate with different lattice actions. In the small lattice spacing limit, all these actions differ by irrelevant operators, and they all should give identical predictions. But in strong coupling, one group might see something which another group does not, just because their actions are different.

However, there are some general features that can be described. I know the situation for Wilson type fermions the best. Recall that the bare Wilson fermion mass is additively renormalized. Any calculation that must be done at $am_q = 0$, such as a Schrödinger functional calculation, is carried out along the the critical kappa line, $\kappa_c(\beta)$ vs β. The generic Wilson fermion artifact is that when the number of fermionic degrees of freedom is large enough, at strong coupling the κ_c line vanishes: there is a line of discontinuity in which the AWI quark mass jumps abruptly from positive to negative. This was seen first in Iwasaki *et al.* (2004, 1992), and nearly every paper with many Wilson fermion degrees of freedom reports it. Nagai *et al.* (2009) is a particularly complete example: the authors studied $SU(2)$ and some $SU(3)$ gauge theories coupled to many flavors of fundamental fermions, at $\beta = 0$. A first order transition appears at around $N_f = 6$ for $SU(2)$.

What is annoying about this transition is that the interesting region for slowly running theories is at strong coupling, but if there is no place where the fermion mass vanishes, one cannot do lattice studies. In particular, running coupling studies typically chase a running coupling into strong coupling, watch it run ever more slowly, and then the transition appears just before (or just after) a zero of the beta function is about to occur.

The precise location of the transition is not universal, and it is possible to design (empirically) actions for which the transition is pushed to stronger coupling. Shamir, Svititsky, and I found it quite useful to do this.

As far as I can tell, there are no other transitions generically observed on the weak coupling side of this transition, so at least the weak coupling phase of a Wilson fermion simulation seems to be analytically connected to the Gaussian fixed point at $g^2 = 0$ or infinite β.

Staggered fermions seem to be more complicated, but maybe that is just because I have no personal experience with them. Cheng *et al.* (2012) has a collection of earlier references and a description of their new strong coupling phase. It is bracketed by jumps in the condensate. It is a phase where lattice translational symmetry is broken: the condensate $\langle\bar\psi\psi\rangle$ is different on even and odd lattice sites. The phase form a pocket extending from small (zero?) quark mass to some maximum value, over a range of strong values of β. This is seen both for $N_f = 12$ and 8 fundamentals. The phase is confining but apparently chirally restored. (Such continuum language may not be appropriate for a strong coupling phase.) Other groups (Deuzeman *et al.*, 2013; Fodor *et al.*, 2011b; Jin and Mawhinney, 2013) have reported similar structure.

Of course, groups are careful to avoid such phases when they see them. But that may not be good enough. One is really interested in physics in the basin of attraction of either the Gaussian fixed point or of an IRFP. A nearby transition may affect what one is seeing, as much as the IRFP or the Gaussian fixed point. This was probably an issue for Wilson fermion Schrödinger functional studies, which were looking for a fixed point very close to a strong coupling transition.

2. $N_f = 10$

Hayakawa *et al.* (2011) computed a running coupling in a Schrödinger functional simulation. Their $s = 2$ discrete beta function is shown in Fig. 23. They certainly observe slower running than the perturbative result. Is there a zero? I am afraid to say Yes, although they have no such fear. This result is significant with respect to $N_f = 12$, because if $N_f = 10$ is infrared conformal, it is hard to see how $N_f = 12$ could not be.

3. $N_f = 8$

$N_f = 8$ is quite curious: The Schrödinger functional beta function of Appelquist *et al.* (2009) is negative. As Fig. 22 shows, the beta function basically runs at its one-loop value over the entire observed range.

Early work by Deuzeman *et al.* (2008) claimed to see a thermal transition that moved to weaker bare coupling as the lattice size increased. So far, so good, for confinement and chiral symmetry breaking. But recently Aoki *et al.* (2013b) studied $N_f = 8$. Most of their data is from three volumes, but one bare gauge coupling. They described observing behavior at small fermion mass consistent with chiral breaking (nonzero pseudoscalar decay constant, nonzero vector meson mass, zero pseudoscalar mass all in the chiral limit). At the same time they found

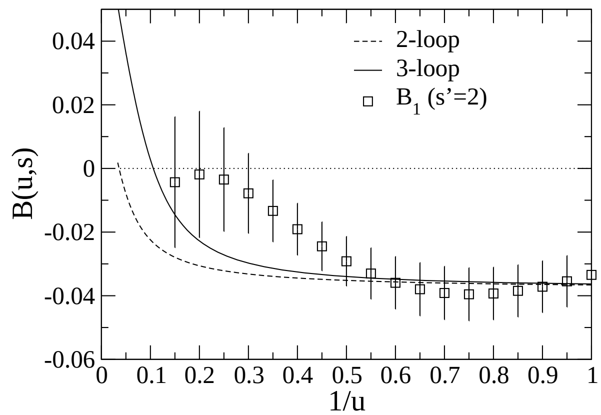

FIG. 23 $N_f = 10$ beta function for $u = g^2$ from Hayakawa *et al.* (2011), along with perturbative expectations.

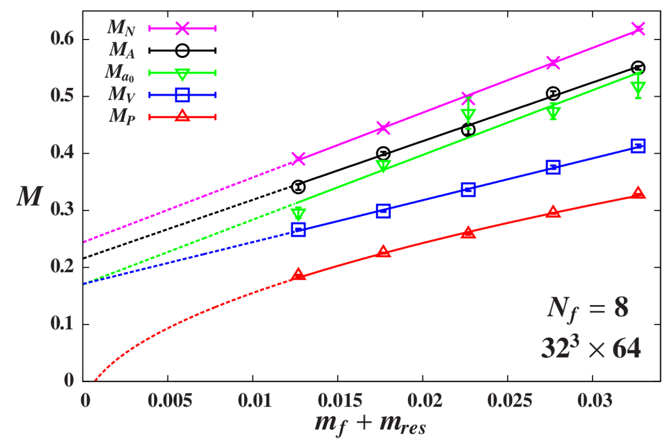

FIG. 24 $N_f = 8$ spectroscopy from Appelquist *et al.* (2014a) with fits motivated by chiral symmetry breaking.

behavior at large fermion mass consistent with power law scaling and a large $\gamma_m \sim 1$. This seems strange; if a data set is going to be infrared conformal, it will be most infrared conformal at the smallest fermion mass, subject to the caveat that finite volume effects are largest there.

Appelquist *et al.* (2014a) also have data at one gauge coupling, two large volumes, and several quark masses. They see separation between the pseudoscalar and vector masses and lack of parity doubling in the vector and axial vector channels, all increasing at their smallest fermion masses. However, simple power law fits (like Eq. 61) also reproduce the data with good quality. (Their two volumes did not have the overlapping region needed for a real finite size scaling analysis.) Compare Figs. 24 and 25.

The Dirac eigenvalue study of Cheng *et al.* (2013) reported a large $\gamma_m \sim 1$ from the integrated spectral density. (Recall Eq. 94.) They did not observe good quality chiral behavior a la Banks-Casher.

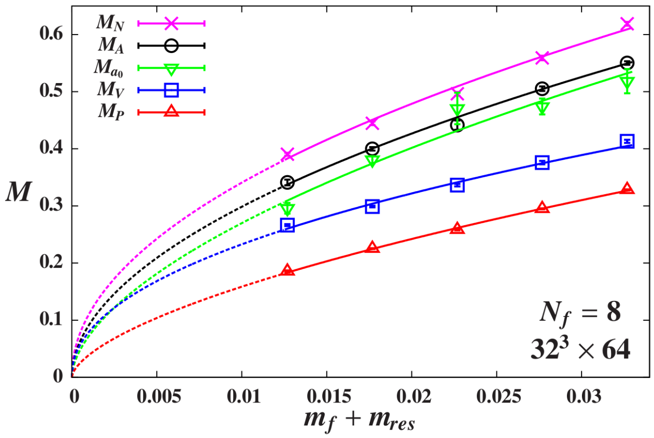

FIG. 25 $N_f = 8$ spectroscopy from Appelquist *et al.* (2014a) with fits to power laws, Eq. 61.

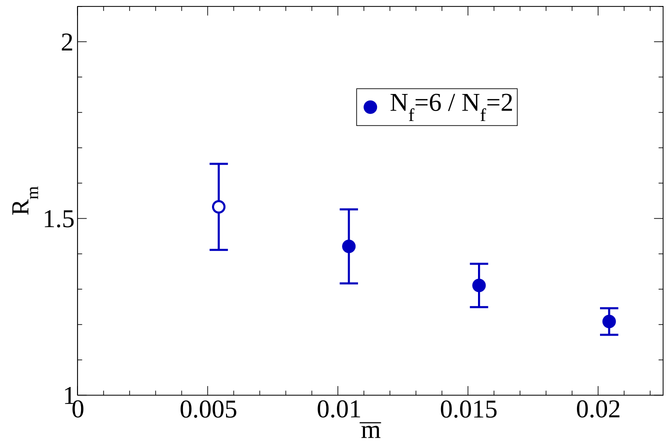

FIG. 26 The ratio $\langle \bar\psi\psi \rangle / F^3$ of $N_f = 6$ to $N_f = 2$, $R_m \equiv [M_m^2/2mF_m]_{6f}/[M_m^2/2mF_m]_{2f}$, versus average fermion mass $\overline{m}$, from Appelquist *et al.* (2010).

Finally, two recent groups, Fodor *et al.* (2015a) and Hasenfratz *et al.* (2015), report calculations of a gradient flow running coupling. The beta function is everywhere negative, smaller than (Hasenfratz *et al.*, 2015) or consistent with (Fodor *et al.*, 2015a) its small two loop value.

I think that lattice calculations of running couplings provide strong evidence that $N_f = 8$ is not inside the conformal window. I am not sure what can be done with spectroscopy to support this claim. Simulations at several bare gauge couplings, along with data at enough volumes for a real finite size scaling analysis, might help. But it might just be that the coupling is running so slowly, that it can never grow across any imaginable simulation volume; then the range of accessible volumes makes the system effectively conformal. One would have to simulate deep in strong coupling to see signals of confinement or chiral symmetry breaking. But then, the system would be strongly interacting at its shortest lattice distances. Where would be a connection to asymptotic freedom?

Aoki *et al.* (2014) report a light isoscalar scalar state, whose mass is roughly equal to that of the pseudoscalar at the nonzero fermion masses where they collected data. They argue that its mass extrapolates to a nonzero value at zero fermion mass, and thus it is a candidate for a dilatonic Higgs. I am not prepared to believe this claim since where they have data, the mass of the scalar is degenerate with the mass of the pseudoscalar state.

4. $N_f \le 6$

With $N_f = 6$ and below, we are back on more familiar ground. These systems are confining and chirally broken. At $N_t = 6$, there is a finite temperature transition that moves with lattice size in a reasonable way (Miura *et al.*, 2012). The LSD collaboration has done several studies comparing observables with some relation to electroweak physics at $N_f = 2$ and 6. $N_f = 6$ has a running, not a walking, coupling, but the collaboration was hoping to see trends that might become stronger closer to the edge of the conformal window. Their data is from one β value per N_f, with several fermion masses.

Appelquist *et al.* (2010) shows that the ratio of the condensate to the cube of the pseudoscalar decay constant F, $\langle \bar\psi\psi \rangle / F^3$, increases with increasing N_f. (They actually compute $\langle \bar\psi\psi \rangle$ from $M_\pi^2 F^2/(2m_q)$.) Their plot is shown in Fig. 26. The larger condensate allows for larger quark masses while keeping flavor changing neutral currents small. (Recall the discussion around Eq. 31.) They tell us that their $N_f = 6$ data sets are not at small enough quark mass and big enough volume to be reliably deep into the chiral limit, so their results are tentative.

Next, Appelquist *et al.* (2011b) compute the S-parameter. It is proportional to the limiting value of $d(q^2\Pi_T^{LR}(q^2))/dq^2$ (recall Eq. 38) at small q^2, after Goldstone boson effects are subtracted.

At heavier quark masses, their S parameter scales roughly linearly with N_f (or more simply, the $N_f = 6$ value is three times the $N_f = 2$ value). This is expected behavior, just counting degrees of freedom. However, at their smallest $N_f = 6$ fermion mass their S parameter plunges to become nearly equal to the $N_f = 2$ value. This is shown in Fig. 27.

This is only one point, but they argue it is a real effect, with the following cause: If the correlator can be saturated by a sum of resonances, it can be written as

$$\Pi_T^{LR}(q^2) = \sum_V \frac{f_V^2 M_V^2}{q^2 + M_V^2} - \sum_A \frac{f_A^2 M_A^2}{q^2 + M_A^2} - \frac{f_\pi^2}{q^2} \qquad (99)$$

and the S-parameter is dominated by the difference of vector and axial vector contributions of this expression. The masses of the lightest vector and axial vector mesons

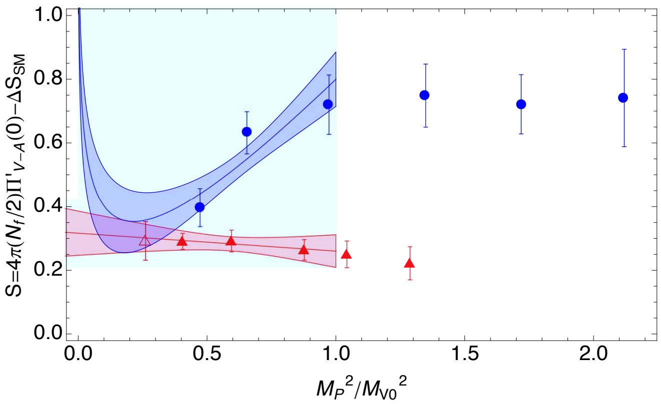

FIG. 27 S parameter for $N_f = 2$ (red diamonds and band) and $N_f = 6$ (blue circles and band), from Appelquist *et al.* (2011b). For each of the solid points, $M_P L > 4$.

are relatively easy to extract from lattice data. The authors observed that these states became more degenerate at $N_f = 6$ than they were at $N_f = 2$ and at the same time the S parameter decreased. They took the decrease to be a favorable generic result for a technicolor solution to beyond Standard Model physics.

Their third calculation is of WW scattering parameters (Appelquist *et al.*, 2012). This is done using the Goldstone equivalence theorem; longitudinal W scattering amplitudes can be computed in terms of Goldstone boson scattering amplitudes. The scattering amplitude (more precisely, the low energy scattering phase shift) can be computed from the shift in energy of the two-pion state in finite volume. This is not an easy calculation even in QCD, and LSD only had one volume. They could measure a scattering length for the maximal isospin channel in $N_f = 2$ and 6. It is consistent with the lowest order chiral perturbative result.

C. $N_c = 2$ and many fundamental flavors

The cost of a simulation scales as N_c^3, so these systems are cheaper than $N_c = 3$. This means that, in principle, one can study a larger range of volumes for an equivalent use of resources. However, they are less studied than $N_c = 3$.

A Schródinger functional analysis by Karavirta *et al.* (2012a) claims that $N_f = 10$ has an IRFP and $N_f = 4$ has a negative beta function. A conference proceedings by Ohki *et al.* (2010) argues that $N_f = 8$ has an IRFP. Rantaharju *et al.* (2014) recently presented a conference proceedings with preliminary results of a gradient flow coupling for $N_f = 8$. They observe perturbative running in weak coupling with no direct evidence for a fixed point.

$N_f = 6$ is the most controversial point. The two-loop beta function has a zero deep in strong coupling. Bursa *et al.* (2011b) claimed slow running, but could not tell if there is a fixed point. Hayakawa *et al.* (2013) claimed to see an IRFP, with a small γ_m, though with large errors ($0.26 \leq \gamma_m^* \leq 0.74$). Karavirta *et al.* (2012a) have inconclusive results for $N_f = 6$. Their γ_m for $N_f = 6$ ranges from 0.1-0.25 over observed g^2 range, smaller than the one loop perturbative value. The largest statistics study to date of $N_f = 6$ is the Schrödinger functional study of Appelquist *et al.* (2014c). They found no evidence for an IRFP.

Reference	method	γ_m
Del Debbio *et al.* (2010)	scaling	0.22(6))
DeGrand *et al.* (2011)	SF	0.31(6)
Patella (2012)	spectral	0.371(20)
Giedt and Weinberg (2012)	FSS	0.50(26)
Del Debbio *et al.* (2013)	spectral	0.38(2)

TABLE II Mass anomalous dimension γ_m in $SU(2)$ with $N_f = 2$ adjoint fermions, from publications with reasonably small uncertainties. Under "method," FSS refers to finite size scaling, "scaling" to a fit to Eq 61. "Spectral" refers to use of the spectral density of Dirac eigenvalues. SF is Schrödinger functional.

D. Fermions in higher dimensional representations

An alternative way to achieve slow running is to bundle the many fermion degrees of freedom into a small number of higher dimensional representations. This could be phenomenologically attractive: with $N_f = 2$ there are no un-eaten Goldstones to become technipions. On the other hand, this could be phenomenologically unattractive: technifermions are in different color representations from Standard Model fermions, so they cannot be members of the same multiplet.

The most studied of these systems is $SU(2)$ with $N_f = 2$ adjoints ("minimal walking technicolor"). Every technique I have mentioned – and probably more – has been applied to this system. I have already shown examples of its spectroscopy, by Hietanen *et al.* (2009a). Interesting Schrödinger functional studies include Bursa *et al.* (2010); DeGrand *et al.* (2011); and Hietanen *et al.* (2009b). It is the clearest example of an IRFP system.

I will pause and show a few more pictures from DeGrand *et al.* (2011). Figure 11 and Fig. 15 showed raw lattice data for the running coupling and Schr̆odinger functional $Z_P(L)$. These data can be turned into plots of the beta function and coupling-dependent mass anomalous dimension. These are shown in Figs. 28 and 29.

Many groups have observed that γ_m is small. Some numbers are given in Table II.

Recently Athenodorou *et al.* (2015) report that $SU(2)$ with one adjoint flavor is near conformal with a mass anomalous dimension near unity. This comes from finite size scaling of the spectrum and the integrated spectral

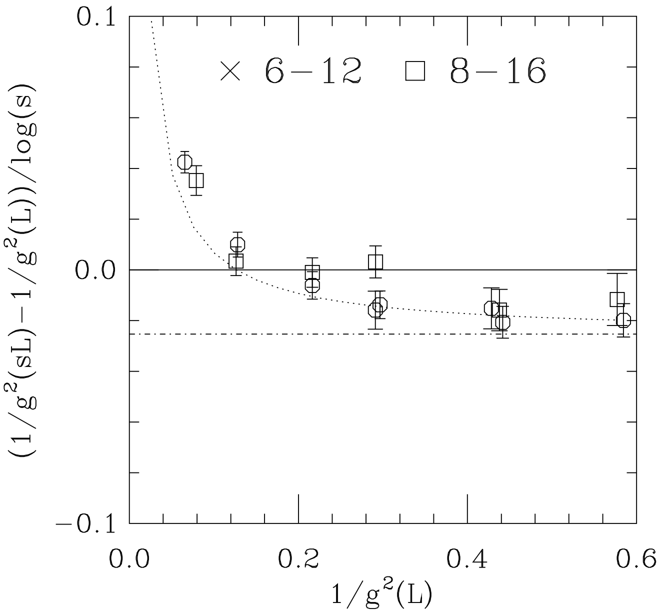

FIG. 28 Beta function for $SU(2)$ with $N_f = 2$ adjoints, from DeGrand *et al.* (2011). This is based on comparisons of two L values related by a scale factor $s = 2$. The straight line is the one-loop beta function, and the curved line is the two loop beta function.

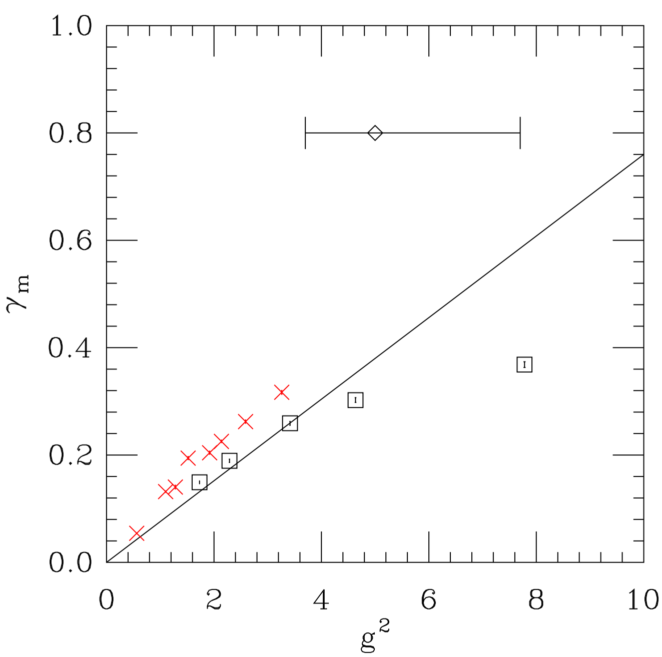

FIG. 29 Mass anomalous dimension $\gamma_m(g^2)$ from DeGrand *et al.* (2011) The horizontal bar at the top marks our location (with its uncertainty) for the critical coupling. The crosses are the data of Bursa *et al.* (2010), analyzed with the same fit. The diagonal line is the lowest order perturbative result.

density. Perturbatively, this system is like $SU(3)$ with eight fundamentals; b_1 is small and b_2 is even smaller.

The other work in this area I know of is mostly by me and my collaborators, all with Wilson fermions, mostly using Schr̆odinger functional:

- $SU(3)$ with $N_f = 2$ two-index symmetric (S2) representation fermions, (DeGrand *et al.*, 2009, 2010, 2013a), also with spectroscopy (DeGrand *et al.*, 2009), and finite size scaling (DeGrand, 2009)
- $SU(4)$ with $N_f = 2$ S2 representation fermions (DeGrand *et al.*, 2012)
- $SU(3)$ with $N_f = 2$ adjoints (DeGrand *et al.*, 2013b)
- $SU(4)$ with $N_f = 6$ AS2 fermions (DeGrand *et al.*, 2013b)

We could not tell if the beta function had a zero for any of these systems – it either followed, or ran more slowly than, the two loop formula, deep into strong coupling. At this point we lost control of the calculation: either we hit the Wilson fermion transition where zero quark mass was lost, or the calculation simply became too expensive. Spectral data for $SU(3)$ S2 at several bare gauge couplings shows curve collapse consistent with near conformal behavior distorted by the finite volume. (That was shown in Fig. 16.) All these systems are claimed to have small mass anomalous dimension at values of the coupling constant where the observed beta function is small. This region dominates the integral in the formula for the running condensate, Eq. 33, so most of the evolution is at small γ_m. This renders these systems uninteresting for technicolor, we said, even if the beta function were to become large and negative at even stronger coupling.

Large-N_c scaling is a nice way to present these systems. Figure 30 shows γ_m from our S2 studies and Fig. 31 shows the beta function with two adjoint flavors.

There is a controversy about $SU(3)$ with $N_f = 2$ S2 fermions: Fodor *et al.* (2012a) claim that the system is confining and chirally broken. Their results have, so far, mostly only been presented in a long series of conference proceedings. It is difficult to evaluate such works in progress, so my description of their results might be incomplete. Let me try:

Their calculations use staggered fermions with $N_f = 2$ flavors achieved by rooting the fermion determinant. The bulk of their published simulations are almost all at one gauge coupling, deep in strong coupling, although data at four couplings is said to exist. Data are collected at many bare fermion masses and lattice volumes, and the pseudoscalar mass and chiral condensate are presented after an extrapolation to infinite volume using chiral perturbation theory. The chiral condensate extrapolates to a nonzero value in the zero mass limit. The vector and axial vector mesons do not appear to be degenerate (so that

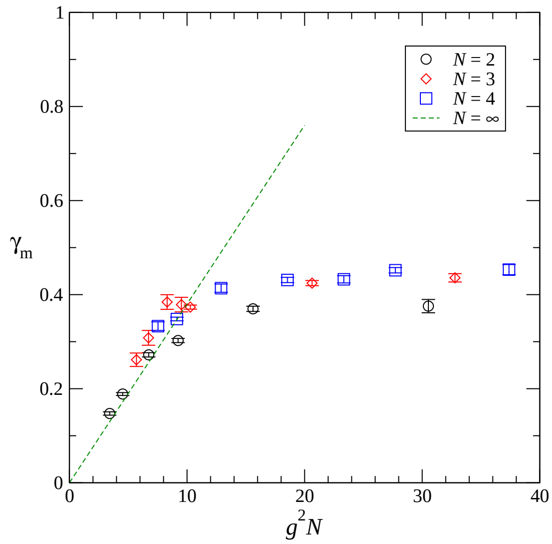

FIG. 30 γ_m from simulations with two flavors of symmetric representation fermions, displayed as a function of $\lambda = g^2N$ where g^2 is the Schrödinger functional coupling. Data is from DeGrand *et al.* (2012). The line is the lowest order large-N prediction.

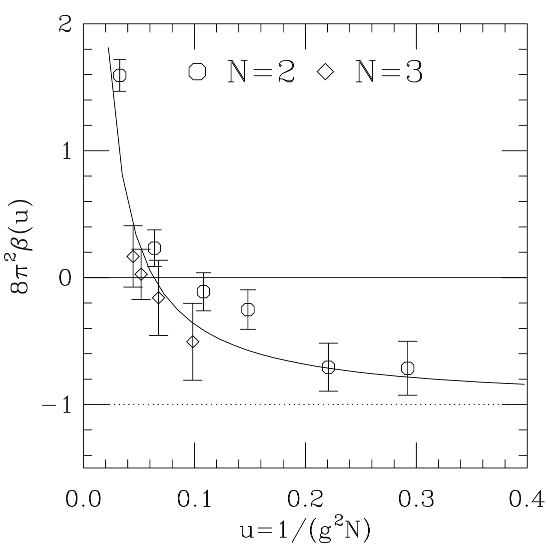

FIG. 31 The beta function from from simulations with two flavors of adjoint fermions, Data is from DeGrand *et al.* (2011, 2013b). The lines are the beta functions at one and two loops.

the parity doubling which would indicate chiral restoration is absent). The pseudoscalar and vector masses appear to separate; looking at the figures in Fodor *et al.* (2012a), at the lightest fermion mass, the pseudoscalar's mass is about half the vector's. They observe $m_\rho/f_\pi \sim 7$ in the chiral limit. Fits of different (infinite-volume extrapolated) quantities to the naive Eq. 61 done in Fodor *et al.* (2012a) do not give y_m's which are consistent with each other. Finally, Fodor *et al.* (2015b, 2014a) claim evidence for a light isoscalar scalar state. When I compare figures in their two papers, it appears to be lighter than their pseudoscalar mass at the lightest recorded quark mass.

Observable related to the potential and a running coupling present contrasting pictures. Fodor *et al.* (2012b) shows a plot of the static force versus distance r. At $r/a = 4-5$, it looks Coulombic to the eye, and by $r/a = 10$ the force is constant. This says that the potential changes qualitatively over a scale factor of distance of about two. With their vector meson masses am_ρ to give a dimensionless number, the crossover is at a distance where $m_\rho r \sim 2$. In QCD, the inflection point is at about $r \sim 0.3$ fm, so $m_\rho r \sim 1$. This comparison plus the spectroscopy reported in the previous paragraph argues for a QCD-like system with a rapidly varying coupling constant.

However, Fodor *et al.* (2015b) presents a calculation of a "flow" coupling constant. They show a figure overlaying their result on the one and two-loop perturbative beta function. It shows a coupling which increases over the observed range, without a fixed point. The coupling appears to be running at much lower rate than that of the one-loop beta function, and at their largest coupling, it runs more slowly than the two-loop formula. (Recall that the one loop $b_1 = 13/3$ for this system, as opposed to $b_1 = 9$ for $SU(3)$ with three fundamental flavors or $b_1 = 3$ for twelve fundamentals.) I do not know how to reconcile the results of this paragraph with those of the previous one. There is insufficient published data from these authors to allow further conclusions to be drawn.

Over the last few years, Kogut and Sinclair (2010, 2011, 2014) have investigated this system, and the $N_f = 3$ S2 system, using the motion of finite temperature phase transition as a potential indicator of confining versus infrared conformal behavior. Their results are ambiguous: with small lattices in the temporal direction ($N_t = 4$ and 6) the transitions, which are located at strong coupling, move quickly, while at larger N_t the transition continues to move, but more slowly. "However, further simulations at larger $N_t(s)$ are needed," write Kogut and Sinclair (2014).

E. An attempt to sum up

I think that nearly all lattice results from systems with many fermion degrees of freedom show behavior which is consistent with expectations from the one-loop beta function, as described around Eq. 57. (The one exception is the work by one group on $SU(3)$ with $N_f = 2$ S2 fermions, noted immediately above.) Nearly all systems studied have both spectroscopic data and coupling constant measurements, and more-or-less power law behavior for spectra (Eqs. 61 and 87) is correlated with the presence of slowly running couplings. What is still unknown is the precise boundary of the conformal window. This is not surprising: slow running is not that different from no running.

The plot of Dietrich and Sannino (2007), Fig. 2, has been frequently mentioned. Let us try to build our own picture, using lattice data. Figure 2 shows the boundary of the conformal window for various representations as lines, as if one could imagine systems with fractional flavor number. The situation for small N_f and N_c is more discrete, of course. How to present things? The relevant variable is something like the number of fermionic degrees of freedom, versus the number of gluonic ones. Looking at the two-loop beta function, a reasonable choice is $N_f T(R)/N_c$. (Of course, b_2 includes $C_2(R)$, but it basically scales as N_c, and it has a small coefficient compared to 20/3.)

Figure 32 is my best guess at the status. The labels are "C" for confined and chirally broken, "D" for deconfined, chirally restored, and probably conformal in the massless limit, and "?" for unknown. Colors are black for fundamental representation, red for S2, blue for AS2 and purple for adjoint $N \neq 2$. If you are viewing this figure in black and white, the rightmost symbols for $N_c = 2$ and 3 are S2 (equivalent to adjoint for $N_c = 2$) and the leftmost ones are for fundamentals. The two $N_c = 4$ entries are S2 and AS2.

I assigned the following systems question marks: for $N_c = 2$, $N_f = 6$ fundamentals and $N_f = 1$ adjoints; for $N_c = 3$, $N_f = 10$ fundamentals. I listed all of the higher-representation systems with $N_c = 3$ and 4 as "unknown." Yes, after seven years of work, there are still question marks. But, people did not study systems where they knew the answer. These are all difficult systems.

And did anyone ever publish a plot with the 'technicolor dream" beta function, Fig. 3? Not in a simulation of a four dimensional system of gauge fields and fermions.

To say once more why slow running was difficult: A dip in the beta function was not the issue, the problem was the extremely small value of the one loop beta function as the number of fermion degrees of freedom increased. Contrast Fig. 3 with real two-loop running, Fig. 33.

I think that generally, when they began studying systems with slowly running couplings, people did not appreciate how different they were from QCD. Much of the context QCD studies used to evaluate results was absent. For example, dealing with rooted staggered fermions requires knowledge that the system is chirally broken, plus access to the Gaussian fixed point, the ability to make the system weakly interacting at short distance while maintaining strong interactions at long distance.

Several approaches worked poorly. I do not think that simulations at finite temperature have been too useful. Large scale spectroscopic simulations at a single value of the bare gauge coupling generally proved inadequate for determining whether a system was confining and chirally

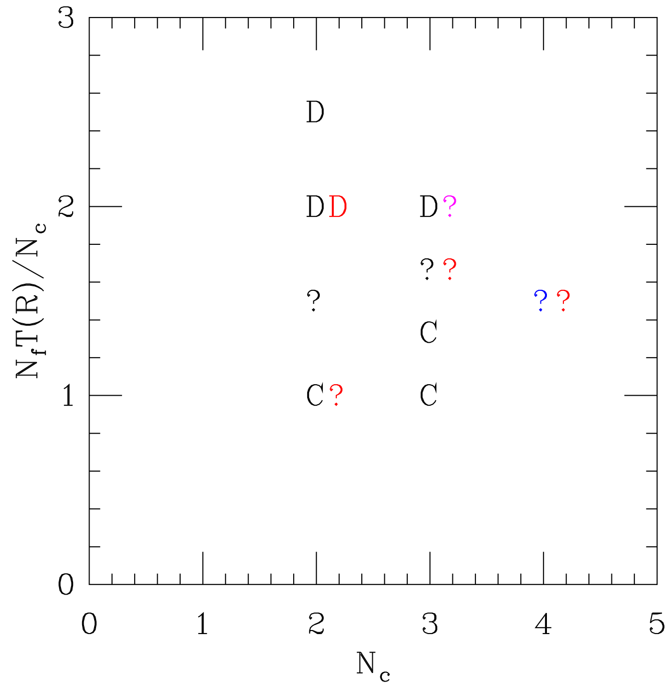

FIG. 32 (Color online) My attempt at a synthesis of lattice results for the vacuum structure of various systems. The labels are "C" for confined and chirally broken, "D" for deconfined, chirally restored, and probably conformal in the massless limit, and "?"for unknown. Colors are black for fundamental representation, red for S2, blue for AS2 and purple for adjoint $N \neq 2$ (close in color to red because they are the same for $SU(2)$).

broken, or infrared conformal. Simulations at many volumes were useful. Simulations at one large volume, hoping to approximate infinite volume, were less so. Remember Eq. 57. When the coupling constant runs slowly, no volume is large enough. I think that the case of $N_f = 12$ fundamentals clearly illustrates this conclusion. Fig. 17 is a smoking gun for infrared conformality, and the authors needed data from many bare parameters and volumes to build it.

The situation for the mass anomalous dimension is harder to summarize. The best-determined numbers are for $SU(2)$ with $N_f = 2$ adjoints, and $SU(3)$ with $N_f = 12$ fundamentals. In both cases γ_m is small. Only for the systems $SU(3)$ with $N_f = 8$ fundamentals, $SU(2)$ with $N_f = 6$ fundamentals, and $SU(2)$ with $N_f = 1$ adjoint are there claims of large γ_m's, and the claims are presented very cautiously. All other systems appear to have small mass anomalous dimensions. Apparently, a large mass anomalous dimension can only occur for a theory which is right on the sill of the conformal window (if at all).

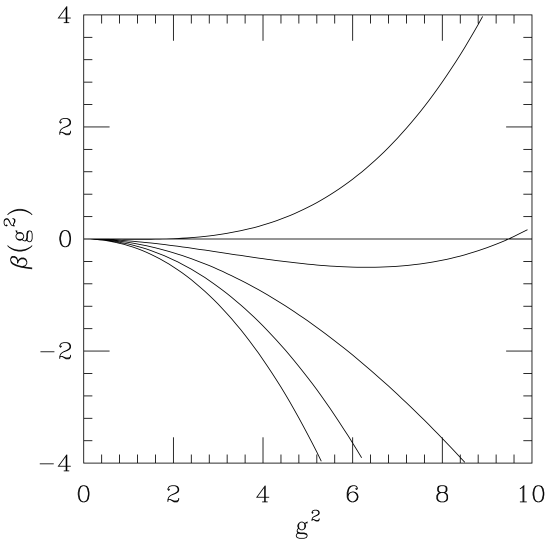

FIG. 33 Two-loop beta function for $SU(3)$ with (from the bottom up) $N_f = 3$, 6, 9, 12, and 15 flavors of fundamental representation fermions. ($N_f = 9$ has a zero at $g^2 = 66$, far to the right off the plot.)

VII. CONCLUSIONS

At the time I am writing, there is no evidence for any particular beyond Standard Model scenario in data. The Higgs exists with near Standard Model couplings, and evidence for beyond Standard Model physics (neutrino masses and mixing, dark matter, and so on) remains a set of disconnected observations. As an outsider in the beyond Standard Model field, it seems to me that the dominant theoretical motivation for new physics is not so much that there is physics beyond the Standard Model, but that there is a theoretical issue with the Standard Model itself – the hierarchy problem. This takes us back to a nonperturbative resolution of the hierarchy problem as a possibly attractive choice, Eq. 21 and beyond.

This is a niche for the lattice. But it requires an ultraviolet completion, before any lattice calculation can be envisioned. The lattice is not about symmetries, it is about low energy constants. Phenomenologists who have a favorite beyond Standard Model scenario and want lattice people to study it have to give them a concrete ultraviolet completion.

Most lattice work focused on one particular corner of beyond Standard Model dynamics, technicolor, and on one small area of technicolor, mostly $SU(3)$ and many fundamentals. This certainly seems peculiar, given the wide set of continuum beyond Standard Model possibilities in the literature. Why did this happen? I am not sure. It might be because lattice simulations have to begin with some ultraviolet completion, and because the framework of candidate ultraviolet completions was naturally present in the technicolor literature, in a lattice-friendly way: non-Abelian gauge theories with fermions in four dimensions.

Studies of near-conformal systems tell us that when N_c is small, there are actually only a small number of confining and chirally broken systems. Most of them are not appropriate for beyond Standard Model physics associated with the Higgs: the coupling constants probably run too fast for technicolor, and the flavor symmetry groups are often too small for composite Higgs scenarios. However, some of them are composite Higgs candidates, and some of them could be composite Dark Matter candidates. Most of them are also interesting as analog systems for QCD. Little is known about their mass (and other) anomalous dimensions. All of them could be explored with today's available software and computer power. In particular, much of the technology for computing QCD matrix elements can be straightforwardly applied to these systems. This could be an interesting thing to do. It would check large N_c counting of matrix elements, and the variation of observables on N_f could be probed. Recall how, in Sec. III.C, I said that if technicolor was like QCD, it would be ruled out by experiment, but that technicolor might not be QCD-like? A larger version of this question is to ask how much like QCD are theories which are nearby it in the space of N_c, N_f and fermion representation.

Of course, no theoretical calculation by itself is going to reveal the existence of some particular beyond Standard Model scenario. Without new experimental data, all theory can do is suggest possibilities. In the long term, whether or not the words "Lattice" and "beyond Standard Model" should – or will – appear again in the same title is a question only experiment can decide.

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