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On the energy barrier of hypergraph product codes

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A macroscopic energy barrier is a necessary condition for self-correcting quantum memory. In this paper, we prove tight bounds on the energy barrier applicable to any quantum code obtained from the hypergraph product of two classical codes. If the underlying classical codes are low-density parity-check codes (LDPC), the energy barrier of the quantum code is shown to be the minimum energy barrier of the underlying classical codes (and their transposes) up to an additive $O(1)$ constant.

Introduction.- Quantum computers are capable of solving problems that are likely intractable for classical computers, such as factoring of large integers [1] and simulation of quantum systems [2, 3]. However, realistic quantum computers are noisy. In order to build a useful quantum computer capable of carrying out such computational tasks, quantum error correction is likely necessary.

Since the discovery of the first quantum error-correcting code [4], much progress has been made towards finding better codes. While the leading approach to scalable quantum error correction has been the surface code [5] for nearly 20 years, lately, there has been a surge of interest in using quantum low-density parity check (LDPC) codes that are not geometrically local. This recent interest is in part due to Gottesman, who showed that with quantum LDPC codes, one can achieve a constant overhead for fault-tolerant quantum computation [6]. A well-known approach to construct such codes is the hypergraph product construction [7]. More recent studies led to the development of other families of quantum LDPC codes with improved parameters [8–18]. Moreover, recent studies suggest that there are viable fault-tolerant quantum computing architectures that can host such codes [19–21].

One challenge in using these codes lies in the decoding. While there are general methods such as the BP+OSD decoder [22], it is a priori not obvious how well such a general-purpose decoder would work for a given code. One attractive approach is to use codes that have a macroscopic energy barrier, meaning the energy barrier scales with the code's physical qubits number. For such codes, a simple process that iteratively lowers the energy (quantified in terms of the number of stabilizers violated) is a viable decoder candidate [23–33]. Alternatively, such a model can be viewed as a candidate for self-correcting quantum memory, protecting quantum information by the macroscopic energy barrier [34, 35].

The potential for self-correcting quantum memory in three or fewer dimensions could dramatically change our understanding of macroscopic quantum phenomena, as it raises the possibility of having genuinely quantum phenomena like superposition in macroscopic systems at non-zero temperatures. While some no-go theorems exist [36, 37], they do not entirely rule out this possibility. The energy barrier, while not sufficient [38–40], is a necessary condition for self-correcting and serves as a key starting point for further research.

Such approaches work well for four-dimensional (4D) toric code [34] and the quantum expander code [23, 41], both of which have macroscopic energy barriers. Unfortunately, rigorous bounds on the energy barrier are hard to come by and often derived for specific codes (or family of codes) [23, 39, 42–44].

In this paper, we prove a tight bound on the energy barrier for the hypergraph product codes [7], defined in terms of the energy barrier of the underlying classical codes. The hypergraph product is defined in terms of the parity check matrices (or equivalently, the Tanner graphs) of two classical codes. While there are codes with better parameters [11–14], the hypergraph product remains a flexible framework for constructing quantum LDPC codes with many advantages [45], such as the variety of decoders [22, 23, 41, 46, 47], logical gates [48–50], and distance-preserving syndrome extraction circuit [19, 51].

Now, we describe our main result. Without loss of generality, consider a hypergraph product of two classical codes, defined in terms of the parity check matrices H_1 and H_2. We denote the parity check matrix of the resulting quantum code as $H_{(H_1,H_2)}$. For both the quantum and the classical code, we denote the energy barrier as $\Delta(H)$, where H can be a parity check matrix of the quantum or the classical code. We prove that under a modest condition,

$$\Delta(H_{(H_1,H_2)}) = \min(\Delta(H_1), \Delta(H_2), \Delta(H_1^T), \Delta(H_2^T)), \tag{1}$$

where H^T is the transpose of H. Eq. (1) holds if the energy barriers of the classical codes are larger than or equal to a certain sparsity parameter of the code. Importantly, if the underlying codes are LDPC, then the sparsity parameter is $O(1)$. Therefore, Eq. (1) holds if the energy barriers of H_1, H_2, H_1^T, and H_2^T grow as the code size grows. If this condition is not satisfied, the en-

ergy barrier is bounded by a constant, a fact that follows trivially from the definition of the hypergraph product code.

The proof of Eq. (1) is based on two results. First, if two logical operators of a quantum LDPC code are equivalent up to a stabilizer, their energy barriers differ only by a constant related to the code's sparsity parameters w_c, w_q [Theorem 1]. Thanks to this result, proving the energy barrier for a given code reduces to proving the energy barrier for any complete set of logical operators, which can be a much smaller set than the set of all logical operators. We then identify a set of logical operators for which the exact energy barrier can be determined in terms of the energy barriers of the underlying classical codes. Together, these two results imply Eq. (1).

Energy barrier of codes.- We first present a formal definition of the energy barrier for stabilizer codes [36]. Let $\mathcal{C}$ be the code subspace of an $[[n, k, d]]$ stabilizer code, defined in terms of the stabilizer group S. The code subspace can be viewed as the ground state subspace of a Hamiltonian of the form $\hat{H} = \sum_{i=1}^{m}(I - s_i)/2$, where $\{s_1, \ldots, s_m\} \subset S$ is a set of generators. Note that the energy barrier depends on the choice of the generating set. For an operator P in the n-qubit Pauli group $\mathcal{P}$, the energy of the state $P|\psi\rangle$ is given by $\langle\psi| P^\dagger \hat{H} P |\psi\rangle = \epsilon(P)$. Here $|\psi\rangle$ is any ground state with $\langle\psi| \hat{H} |\psi\rangle = 0$, and $\epsilon(P)$ is the energy cost of P. This is the number of s_i that anticommute with P, i.e., $\epsilon(P) = |\{i : s_i P = -P s_i\}|$. Equivalently, one may define the energy barrier in terms of the parity check matrix H of the stabilizer code, where H is presented in symplectic form [52]. Let $v(P)$ be the symplectic binary (bit-string) representation of a Pauli P:

$$\epsilon(P) = \mathrm{wt}(Hv(P)). \tag{2}$$

A sequence $P_0, P_1, \ldots, P_t$ from the Pauli group $\mathcal{P}$ forms a *path* from P_0 to P_t if for each index i, the operators P_i and P_{i+1} differ at no more than one qubit. The notation $w(P_0, P_t)$ represents the collection of all such paths from P_0 to P_t. For $r \in w(P_0, P_t)$, $\epsilon_{\max}(r)$ denotes the highest energy along path r, i.e., $\epsilon_{\max}(r) = \max_{P_i \in r} \epsilon(P_i)$, as the energy barrier along the path r.

The minimum energy associated with a Pauli P is the smallest value of $\epsilon_{\max}$ across all possible paths from 0 to P. This is the energy barrier of P, denoted as $\Delta(P)$:

$$\Delta(P) = \min_{r \in w(I,P)} \epsilon_{\max}(r). \tag{3}$$

The energy barrier of the quantum code is the minimum energy barrier over the set of nontrivial logical operators.

Definition 1. *Let S be a stabilizer group and $L(S)$ be the set of nontrivial logical operators. The energy barrier is*

$$\Delta(H) := \min_{\ell \in L(S)} \Delta(\ell). \tag{4}$$

This is the smallest energy the environment has to overcome to enact a logical operation on the encoded qubit. We can similarly define the energy barrier of classical codes by only considering the path formed by Pauli-Xs. We shall denote this energy barrier also as $\Delta(H)$, where H in this case is the parity check matrix of the classical code.

Quantum LDPC codes and their energy barriers.- A quantum LDPC code is a stabilizer code with a sparse parity-check matrix; see [53, 54] for recent reviews. The sparsity parameters are w_c and w_q, which are the maximum row and column weights of the parity check matrix, respectively. A code is LDPC if $w_c, w_q = O(1)$.

Here, we prove a property that holds for any quantum LDPC code. It states that the energy barrier of two logical operators equivalent under stabilizers are equal, provided that at least one of their energy barriers is larger than or equal to $w_c w_q$. For quantum LDPC codes, $w_c w_q = O(1)$. Therefore, if the energy barrier is $\Omega(1)$ for any given non-trivial logical operator, it must also be the same energy barrier for any equivalent logical operator.

We first prove the following bound.

Lemma 1. *Let $\mathcal{C}$ be a quantum code with sparsity parameters (w_c, w_q). For any stabilizer $s \in S$,*

$$\Delta(s) \leq w_c w_q. \tag{5}$$

Proof. Without loss of generality, any stabilizer s can be expressed as a product $s = s_1 \cdots s_m$, where $s_1, \ldots, s_m$ are the stabilizer generators. Consider a path that applies s_i in sequence, from $i = 1$ to m. For each s_i, we envision applying a (sub)sequence of Pauli operators in the support of s_i. This subsequence has a length of at most w_c, and each Pauli in the sequence affects at most w_q stabilizers. Therefore, the highest energy attained within this subsequence is at most $w_c w_q$. Once s_i is applied, the energy cost becomes zero. Therefore, $\Delta(s) \leq w_c w_q$. □

We note that this lemma applies even to non-LDPC codes, where w_c and w_q may be large. Consequently, the bound we provided has broader applicability. For instance, in a non-LDPC code where the logical operator L has an energy barrier $\Delta(L)$ that grows as $O(n)$, and $w_c w_q < O(n)$, the following theorem remains valid.

Theorem 1. *Let $\mathcal{C}$ be a (w_c, w_q) quantum LDPC code. For any logical operator L and stabilizer $s \in S$,*

$$\Delta(Ls) \leq \max(\Delta(L), w_c w_q). \tag{6}$$

Proof. Without loss of generality, consider a path $r = (\ell_1, \ldots, \ell_N) \in w(0, L)$. We can construct a new path $r' \in w(0, Ls)$ by appending r with a sequence $\delta_r \in w(0, s)$, forming $(\ell_1, \ldots, \ell_N, \ell_N\ell'_1, \ell_N\ell'_2 \ldots, \ell_N\ell'_M)$ where

$\delta_r = (\ell'_1, \ldots, \ell'_M)$. Notice that $\epsilon(\ell_N \ell'_i) = \epsilon(\ell'_i)$ for $i \in \{1, 2, \ldots, M\}$, given that $\ell_N = L$ is a logical operator. Then, by assumption,

$$\epsilon_{\max}(r') = \max(\epsilon_{\max}(r), \Delta(s)). \tag{7}$$

According to Lemma 1, $\Delta(s) \leq w_c w_q$. It implies that for any given path r, there exists a path r' such that $\epsilon_{\max}(r') \leq \max(\epsilon_{\max}(r), w_c w_q)$. Choose r such that $\Delta(L) = \epsilon_{\max}(r)$. Since $\Delta(Ls) \leq \epsilon_{\max}(r')$, we have $\Delta(Ls) \leq \max(\Delta(L), w_c w_q)$. □

We remark that using the same logic, one can deduce $\Delta(L) = \Delta(Lss) \leq \max(\Delta(Ls), w_c w_q)$. Consequently, if $\Delta(L) \geq w_c w_q$ or $\Delta(Ls) \geq w_c w_q$, $\Delta(L) = \Delta(Ls)$. Therefore, to determine the asymptotic scaling of the energy barrier of a quantum LDPC code, it suffices to consider the energy barrier of any *fixed* complete set of logical operators. Once an energy barrier is obtained for such a set, the energy of all the other logical operators is also essentially determined, thanks to Theorem 1.

However, it is a priori not obvious how to choose such a set. Below, we will solve this problem for a large family of quantum codes known as the hypergraph product code [7].

Hypergraph product code and its logical operators.- Hypergraph product codes are CSS codes formed from two classical linear codes. Without loss of generality, let H_1 and H_2 be $r_1 \times n_1$ and $r_2 \times n_2$ parity check matrices, respectively. The parity-check matrix of the hypergraph product code is given by [7]:

$$\begin{aligned} H_X = &\quad \left(H_1 \otimes \mathbf{I}_{n_2} \ \mathbf{I}_{r_1} \otimes H_2^T\right), \\ H_Z = &\quad \left(\mathbf{I}_{n_1} \otimes H_2 \ H_1^T \otimes \mathbf{I}_{r_2}\right). \end{aligned} \tag{8}$$

Because $H_X H_Z^T = 0$, these two parity check matrices define a CSS code, with a quantum parity check matrix

$$H_{(H_1,H_2)} = \begin{pmatrix} H_X & 0 \\ 0 & H_Z \end{pmatrix}. \tag{9}$$

The classical codes defined by H_1, H_2, H_1^T, and H_2^T form the parent classical codes. Without loss of generality, we will assume that H_i and H_i^T define codes with parameters $[n_i, k_i, d_i]$ and $[r_i, k_i^T, d_i^T]$, for $i = 1, 2$. Under this assumption, the quantum code is a $[[n_1 n_2 + r_1 r_2, k_1 k_2 + k_1^T k_2^T, \min(d_1, d_2, d_1^T, d_2^T)]]$ code [7].

We now introduce a particularly useful set of logical operators, which we refer to as the *canonical* logical operators [51, 55]. For the Z-type logical operators, consider the following binary vector (Here, we don't require it to be symplectic, as we're examining CSS codes, where X and Z errors can be handled independently.)

$$\begin{pmatrix} \sum_{k,j} \lambda_{kj} \bar{x}_k \otimes y_j \\ \sum_{\ell,m} \kappa_{\ell m} a_\ell \otimes \bar{b}_m \end{pmatrix}, \tag{10}$$

where (i) $H_1 \bar{x}_i = H_2^T \bar{b}_m = 0$ and (ii) $y_j \notin \mathrm{Im}(H_2^T)$ and $a_\ell \notin \mathrm{Im}(H_1)$ are unit vectors. We note that $j \in \{1, \ldots, k_2\}$ and $\ell \in \{1, \ldots, k_1^T\}$ and that the set of logical operators expressible in this form is complete [51, 55], which means all $k_1 k_2 + k_1^T k_2^T$ logical operators are provided. A canonical logical operator is *elementary* if only one of the coefficients, either λ_{kj} or $\kappa_{\ell m}$, equals one. A similar form of canonical X-type logical operators can also be constructed. Because our discussion below can be applied to such operators with little change, we omit the discussion about the X-type logical operators.

Let $\mathcal{G}_1(V_1, C_1)$ and $\mathcal{G}_2(V_2, C_2)$ be the Tanner graphs of codes defined by H_1 and H_2, respectively. Here, V_1 and V_2 represent the set of bits, and C_1 and C_2 denote the set of checks. We use $\{v_1^i : i \in \{1, 2, \cdots n_1\}\}$ (resp. $\{v_2^j : j \in \{1, 2, \cdots n_2\}\}$) to refer to bit vertices in V_1 (resp. V_2), and also as length n_1 (resp. n_2) unit vectors with the ith (resp. jth) entry as 1. The hypergraph product $\mathcal{G}_1 \times \mathcal{G}_2$ is a bipartite graph with vertex set $V \cup C$, where $V = V_1 \otimes V_2 \cup C_1 \otimes C_2$ is the qubit set and $C = C_1 \otimes V_2 \cup V_1 \otimes C_2$ is the stabilizer set.

The set of qubits can be partitioned into two subsets, $V_1 \otimes V_2$ and $C_1 \times C_2$. For H_X, $H_1 \otimes \mathbf{I}_{n_2}$ acts on $V_1 \otimes V_2$ and $\mathbf{I}_{r_1} \otimes H_2^T$ acts on $C_1 \otimes C_2$. Moreover, the subset $V_1 \otimes V_2$ can be further partitioned into n_2 subsets $V_1 \otimes v_2^1, V_1 \otimes v_2^2, \cdots, V_1 \otimes v_2^{n_2}$, where $V_1 \otimes v_2^k := \{v \otimes v_2^k : v \in V_1\}$ and $V_2 = \{v_2^1, \ldots, v_2^{n_2}\}$.

A Z-type Pauli operator can be expressed as a binary vector $z = \begin{pmatrix} z^{(1)} \\ z^{(2)} \end{pmatrix}$, where $z^{(1)}$ is supported on the qubit subset $V_1 \otimes V_2$ with vector space $\mathbb{F}_2^{n_1} \otimes \mathbb{F}_2^{n_2}$, and $z^{(2)}$ is supported on the qubit set $C_1 \otimes C_2$ with vector space $\mathbb{F}_2^{r_1} \otimes \mathbb{F}_2^{r_2}$. Because $z^{(1)}$ and $z^{(2)}$ act on a tensor product of two vector spaces, one can view them also as a matrix. (For instance, $v_i \otimes u_j$ for unit vectors v_i and u_j can be viewed as a matrix whose entry is 1 on the ith row and the jth column and zero elsewhere.) This procedure is known as vector reshaping, please refer to the supplemental material for the formal definition. After the reshaping, $z^{(1)}$ and $z^{(2)}$ become $Z^{(1)}$ and $Z^{(2)}$ respectively. Here, $Z^{(1)}$ is an $n_1 \times n_2$ matrix, and the entry $Z^{(1)}_{i,j}$ is supported on qubit $v_1^i \otimes v_2^j$. Moreover, the jth column $Z^{(1)}_j$ is supported on qubits $\{v_1^1 \otimes v_2^j, v_1^2 \otimes v_2^j, \cdots, v_1^{n_1} \otimes v_2^j\}$, which we refer to as $V_1 \otimes v_2^j$. Similarly, the subset $C_1 \otimes C_2$ can be partitioned into r_1 rows, and ith row is supported on qubit subset $c_1^i \otimes C_2$ (defined similarly).

An elementary canonical logical-Z operator, which is in the form $\begin{pmatrix} \bar{x}_k \otimes y_j \\ 0_{r_1 r_2} \end{pmatrix}$ (where $0_{r_1 r_2}$ is the $r_1 r_2$-dimensional zero vector), is supported on the subset $V_1 \otimes v_2^j$ if $y_j = v_2^j$. In the matrix form, it is supported on the jth column of $Z^{(1)}$.

It would be instructive to discuss an example of the canonical logical operators. It is well-known that Kitaev's toric code [5] can be viewed as two copies of 1D

repetition code [7]. In this context, the canonical logical operators are the string operators of minimal lengths [Fig. 1].

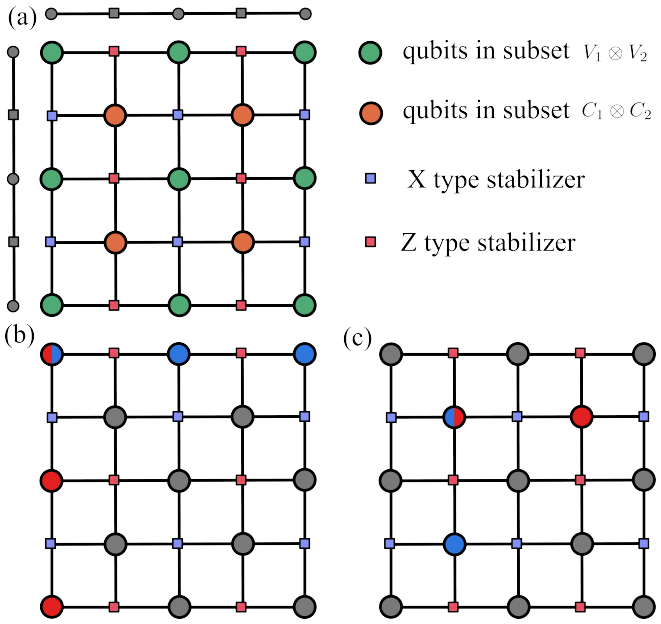

FIG. 1. The hypergraph product construction of the Toric code from two repetition codes (periodic boundary conditions). (a) Delineates the two subsets of qubits, $V_1 \otimes V_2$ and $C_1 \otimes C_2$. (b) Show the logical X operator (blue circles) and the logical-Z operator (red circles) within the subset $V_1 \otimes V_2$. (c) Showcasing the logical X (blue circles) and Z (red circles) operators within the subset $C_1 \otimes C_2$.

Energy barrier of hypergraph product code.- We now study the relationship between the energy barrier of hypergraph product codes and their underlying classical codes. We first upper bound the energy barrier of the hypergraph product code in terms of the energy barrier of the classical codes. We then prove a matching lower bound, establishing their equivalence.

(1) Upper bound.- The upper bound on the energy barrier can be obtained by considering a specific path from the identity to some logical operator. By choosing this logical operator to be a canonical logical-Z operator, we obtain an upper bound. First, consider an elementary canonical logical operator of the form of $\begin{pmatrix} \bar{x}_k \otimes y_j \\ 0_{r_1 r_2} \end{pmatrix}$. Then, consider a path consisting of the operators of the form of $x_i \otimes y_j$ from $i = 1$ to $i = N$, interpolating between $x_1 = (0, \ldots, 0)^T$ to $x_N = \bar{x}_k$. The energy at the ith step is precisely $\mathrm{wt}(H_1 x_i)$, which is the energy of the classical code H_1 evaluated with respect to x_i. Because $x_N = \bar{x}_k$ is a codeword of H_1, the energy barrier along this path is at most $\Delta(H_1)$.

Similarly, we can consider a path of the form of $\begin{pmatrix} 0_{n_1 n_2} \\ a_\ell \otimes b_i \end{pmatrix}$ from $i = 1$ to $i = N$, interpolating between $b_1 = (0, \ldots, 0)^T$ to $b_N = \bar{b}_m$, where $\bar{b}_m$ is a codeword of H_2^T. This yields an energy barrier upper bound of $\Delta(H_2^T)$. Together, we obtain an energy upper bound of $\min(\Delta(H_1), \Delta(H_2^T))$ for the canonical logical-Z operators. A similar argument can be applied to the canonical logical X operators, yielding an upper bound of $\min(\Delta(H_2), \Delta(H_1^T))$.

(2) Lower bound.- We now prove a matching lower bound for the energy barrier of the canonical logical-Z operators.

Proposition 1. *For any nontrivial canonical logical-Z operator L,*

$$\Delta(L) \geq \min(\Delta(H_1), \Delta(H_2^T)). \tag{11}$$

We introduce two lemmas to prove this proposition. We use the following convention in the proof. A path $r = \{P_0, P_1, \ldots, P_F\}$ is said to be supported on U, with $U \subseteq V$, if the support of all P_i in r is included in U.

Lemma 2. *For any elementary canonical logical-Z operator L supported on $V_1 \otimes v_2^\alpha$ (resp. $c_1^\beta \otimes C_2$), its energy barrier is attained by a path supported on $V_1 \otimes v_2^\alpha$ (resp. $c_2^\beta \otimes C_2$).*

Lemma 3. *For any nontrivial canonical logical-Z operator L, the energy barrier $\Delta(L)$ is greater than or equal to the minimum energy barrier of the elementary canonical logical-Z operators.*

Consider an elementary canonical logical-Z operator L supported on $V_1 \otimes v_2^\alpha$. Suppose $\Delta(L)$ is given by a path r. The main idea behind the proof of Lemma 2 is to deform a general path r into a new path r', supported on $V_1 \otimes v_2^\alpha$. Importantly, we show that the energy barrier of r' is no greater than that of the original path r. A similar argument can be applied to prove Lemma 3; see the supplemental material for the proofs.

Let $L = \begin{pmatrix} \bar{x}_k \otimes v_2^j \\ 0_{r_1 r_2} \end{pmatrix}$ be an elementary canonical logical-Z operator, supported on $V_1 \otimes v_2^j$. Lemma 2 implies there is a path r supported on $V_1 \otimes v_2^j$ that attains the energy barrier of $\Delta(L)$.

Using such a path, we can prove a lower bound on the energy barrier of L. Without loss of generality, let $r = (P_0, P_1, \cdots, P_F)$ with $P_0 = I$ and $P_F = L$. Because this path is supported on $V_1 \otimes v_2^j$, in the binary representation, each P_i becomes $u_i \otimes v_2^j$ for some $u_i \in V_1$. In particular, at the ith step, the energy is $\mathrm{wt}(H_1 u_i)$, which is exactly the energy of the classical code with respect to $u_i \in V_1$. Because $u_1 = (0, \ldots, 0)^T$ and $u_F = \bar{x}_k$ for some codeword $\bar{x}_k$ of H_1, the path $(u_0, \ldots, u_F)$ must have an energy barrier of at least $\Delta(H_1)$. Similarly, for any elementary canonical logical-Z operator L on subset $c_1^\beta \otimes C_2$, one can show that $\Delta(L) \geq \Delta(H_2^T)$. Then Lemma 3 immediately implies Proposition 1.

(3) Energy barrier.- Since the lower bound and the upper bound match, the energy barrier of the canonical logical-Z operators is precisely $\min(\Delta(H_1), \Delta(H_2^T))$. Similarly, the energy barrier of the canonical logical-X operators can be shown to be $\min(\Delta(H_2), \Delta(H_1^T))$. Moreover, due to Theorem 1, the energy barrier of any logical operator can be bounded in terms of the canonical logical operators' energy barrier and the code's sparsity parameter. Note that this conclusion applies to the hypergraph product of *any* two classical codes.

If the code is LDPC and the energy barrier is $\Omega(1)$, we conclude that the energy barrier of the quantum code is exactly equal to $\min(\Delta(H_1), \Delta(H_2), \Delta(H_1^T), \Delta(H_2^T))$.

Theorem 2. *Let $\Delta(H)$ be the energy barrier of the hypergraph product code obtained from parity check matrices H_1 and H_2 of $O(1)$ row and column weight. If $\Delta(H_1), \Delta(H_2), \Delta(H_1^T), \Delta(H_2^T) = \Omega(1)$,*

$$\Delta(H) = \min(\Delta(H_1), \Delta(H_2), \Delta(H_1^T), \Delta(H_2^T)). \quad (12)$$

More precisely, for any (w_c, w_q) quantum code derived from a hypergraph product, if the parent classical codes' energy barriers exceed $w_c w_q$, the quantum code's energy barrier matches the minimum energy barrier of these parent codes. This can be used to prove tight bounds on the energy barrier. For instance, a classical code whose Tanner graph is a bipartite expander graph has a macroscopic macroscopic energy barrier. Thus, our result implies that the quantum expander code [23, 41] also has a macroscopic energy barrier. While this is already known [35, 41], note that this argument does not rely on the expansion property of quantum expander code.

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