



This is the accepted manuscript made available via CHORUS. The article has been published as:

Extraction of Spectra in the Shell Model Monte Carlo Method Using Imaginary-Time Correlation Matrices

Y. Alhassid, M. Bonett-Matiz, C. N. Gilbreth, and S. Vartak

Phys. Rev. Lett. **133**, 182501 — Published 28 October 2024

DOI: [10.1103/PhysRevLett.133.182501](https://doi.org/10.1103/PhysRevLett.133.182501)

Extraction of spectra in the shell model Monte Carlo method using imaginary-time correlation matrices

Y. Alhassid,¹ M. Bonett-Matiz,² C.N. Gilbreth,³ and S. Vartak¹

¹*Center for Theoretical Physics, Sloane Physics Laboratory,
Yale University, New Haven, Connecticut 06520, USA*

²*Physics Department, University of Bridgeport, Bridgeport, Connecticut 06511, USA*

³*Quantinuum, Broomfield, Colorado 80021, USA*

Nuclear energy levels are usually calculated using conventional diagonalization methods in the framework of the configuration-interaction (CI) shell model but these methods are prohibited in very large model spaces. The shell model Monte Carlo (SMMC) method is a powerful technique for calculating thermal and ground-state observables of nuclei in very large model spaces, but it is challenging to extract nuclear spectra in this approach. We present a novel method to extract within SMMC low-lying energy levels for given values of a set of good quantum numbers such as spin and parity. The method is based on imaginary-time correlation matrices (ITCM) of one-body densities that satisfy asymptotically a generalized eigenvalue problem. We validate the method in a light nucleus that allows comparison with exact diagonalization results of the CI shell-model Hamiltonian. The method is broadly applicable to quantum many-body systems in other disciplines.

Introduction.— The extraction of spectra of quantum many-body systems is a central problem in many disciplines, including nuclear, atomic and condensed matter physics and quantum chemistry. In nuclear physics, standard diagonalization methods have been successfully used to extract the spectra of light and mid-mass nuclei in the framework of the configuration-interaction (CI) shell model [1, 2]. However, their applications to heavy open-shell nuclei are prohibited by the combinatorial increase of the dimensionality of the many-particle model space with number of valence orbitals and/or number of nucleons. This difficulty has been overcome in part by the auxiliary-field quantum Monte Carlo (AFMC) approach [3–8], also known in nuclear physics as the shell-model Monte Carlo (SMMC) method. SMMC has enabled CI shell model calculations of thermal and ground-state observables in model spaces that are many orders of magnitude larger than those that can be addressed by conventional CI shell model techniques. The method has been particularly useful in calculating statistical properties of nuclei [9–16]. However, it is a major challenge to extract spectroscopic information about individual excited energy levels.

In principle, spectral information of many-body systems can also be extracted by taking the Fourier transform of real-time response functions. However, in quantum Monte Carlo methods it is only possible to calculate imaginary-time response functions. Here we introduce a method to extract energy levels in SMMC by computing imaginary-time correlation matrices (ITCM) of one-body densities for given values of good quantum numbers. These matrices satisfy a generalized eigenvalue problem (GEVP) in the limit of zero temperature. The generalized eigenvalue (GEV) solutions are then used to extract excitation energies. The ITCM method enables calculation of low-lying excited energy levels for given values of the good quantum numbers such as spin and

parity. GEVPs have been used in lattice QCD to study excited states of baryons [17–20].

We test the ITCM method by applying it to a light *sd*-shell nucleus (^{20}Ne), for which the spectrum can be calculated exactly using conventional diagonalization methods [21], and find close agreement.

While the ITCM method developed here is discussed in the context of atomic nuclei, it is applicable across other disciplines, including atomic and condensed matter physics and quantum chemistry. Here we construct the ITCM for systems with good rotational and reflection symmetries, but similar one-body density ITCM can be constructed for systems with other good symmetries. Our method then enables the calculation of excitation energies for given values of the good quantum numbers that are associated with the symmetries of the system.

Shell model Monte Carlo (SMMC).— The SMMC method is based on the representation of the many-body Gibbs operator $e^{-\beta\hat{H}}$ at inverse temperature $\beta = T^{-1}$ as a coherent superposition of one-body propagators describing non-interacting nucleons moving in external time-dependent auxiliary fields $\sigma = \sigma(\tau)$. In this representation, known as the Hubbard-Stratonovich transformation [22, 23], the Gibbs ensemble has the form

$$e^{-\beta\hat{H}} = \int \mathcal{D}[\sigma] G_{\sigma} \hat{U}_{\sigma}, \quad (1)$$

where $\hat{H}$ is the many-body Hamiltonian (which includes a one-body term and a two-body residual interaction), $\mathcal{D}[\sigma]$ is the metric of the path integral over auxiliary fields, G_{σ} is a Gaussian weight, and $\hat{U}_{\sigma}$ is a one-body propagator in imaginary-time τ between $\tau = 0$ and $\tau = \beta$ for a given configuration of the fields σ .

We define for a σ -dependent quantity, X_{σ} , the following expectation value

$$\overline{X}_{\sigma} = \frac{\int \mathcal{D}[\sigma] W_{\sigma} \Phi_{\sigma} X_{\sigma}}{\int \mathcal{D}[\sigma] W_{\sigma} \Phi_{\sigma}}, \quad (2)$$

where $W_\sigma = G_\sigma |\text{Tr}_\mathcal{A} \hat{U}_\sigma|$ is a positive definite weight function and $\Phi_\sigma = \text{Tr}_\mathcal{A} \hat{U}_\sigma / |\text{Tr}_\mathcal{A} \hat{U}_\sigma|$ is the Monte Carlo sign for a given configuration of the auxiliary fields σ . Here $\text{Tr}_\mathcal{A}$ denotes the trace for a fixed number of particles $\mathcal{A}$ calculated using particle-number projection [10, 24]. We project on both proton number Z and neutron number N , so $\mathcal{A}$ refers to (Z, N) .

The thermal expectation values of an observable $\hat{O}$ in the canonical ensemble for a nucleus with $\mathcal{A}$ nucleons is then given by $\langle \hat{O} \rangle = \langle \hat{O} \rangle_\sigma$, where $\langle \hat{O} \rangle_\sigma = \text{Tr}_\mathcal{A} (\hat{U}_\sigma \hat{O}) / \text{Tr}_\mathcal{A} \hat{U}_\sigma$ is the thermal expectation value for $\hat{O}$ in the canonical system with $\mathcal{A}$ valence nucleons for a given configuration of the σ fields.

The multidimensional path integral in the Hubbard-Stratonovich representation can be calculated stochastically using Monte Carlo techniques. The thermal expectation value of an observable $\hat{O}$ is estimated by $\langle \hat{O} \rangle \approx \sum_k^N \langle \hat{O} \rangle_{\sigma_k} \Phi_{\sigma_k} / \sum_k^N \Phi_{\sigma_k}$, which is obtained from a finite set of uncorrelated configurations of fields σ_k selected according to the distribution W_σ .

Quantum Monte Carlo methods for fermions often encounter a sign problem when the sign Φ_σ can be negative for some of the samples, leading to large statistical errors. There are, however, classes of good-sign interactions for which Φ_σ is positive for all samples [3–8]. In the nuclear case, these include the dominant collective components of effective nuclear interactions, which are attractive [25]. These good-sign interactions have been used to calculate nuclear statistical properties in SMMC [9–14, 16]. Small bad-sign components of effective nuclear interactions can be treated using the extrapolation method of Ref. [5].

Imaginary-time correlation matrices (ITCM).— We denote the single-particle orbitals of the shell model space by $a \equiv (n_a, l_a, j_a)$, where n_a is the radial quantum number, l_a is the orbital angular momentum, and j_a is the total single-particle angular momentum. For each angular momentum K and parity π , we define a correlation (or density response) matrix

$$C_{ac,bd}^{K\pi}(\tau) = \left\langle \sum_M \rho_{KM}^\dagger(ac, \tau) \rho_{KM}(bd) \right\rangle, \quad (3)$$

where the angular-momentum coupled one-body densities are defined by

$$\rho_{KM}(ac) = [a_a^\dagger \times \tilde{a}_c]^{KM} \quad (4)$$

with $\tilde{a}_{j_c m_c} = (-)^{j_c - m_c} a_{j_c - m_c}$. $\rho_{KM}^\dagger(ac, \tau)$ are the imaginary-time evolved densities given by $\rho_{KM}^\dagger(ac, \tau) = e^{\tau \hat{H}} \rho_{KM}^\dagger(ac) e^{-\tau \hat{H}}$. We will refer to the matrix $C_{ac,bd}^{K\pi}(\tau)$ as the imaginary-time correlation matrix (ITCM).

In the limit of large β , the thermal expectation value in (3) reduces to an expectation value in the ground-state $|0\rangle$, which has spin 0 for even-even nuclei. In this limit

the ITCM assumes its spectral decomposition

$$C_{ac,bd}^K(\tau) = \sum_\alpha v_{(ac)\alpha} v_{(bd)\alpha}^* e^{-\tau \Delta E_{\alpha K}}, \quad (5)$$

where $v_{(ac)\alpha}^* = (\alpha K || \rho_K(ac) || 0)$ are reduced one-body transition densities and $\Delta E_{\alpha K} = E_{\alpha K} - E_0$ are excitation energies of states with angular momentum K . The sum over α in Eq. (5) corresponds to a specific parity π , and as a result, the ITCM carries a good parity. For simplicity of the notation, we omit here and in the following the parity label π .

When τ is sufficiently large (but still $\tau \ll \beta$), or the excitation energies with angular momentum K display a large gap, we can truncate the sum over α to N_K terms, where N_K is the dimension of the matrix $C^K(\tau)$

$$C_{ac,bd}^K(\tau) \approx \sum_{\alpha=1}^{N_K} v_{(ac)\alpha} v_{(bd)\alpha}^* e^{-\tau \Delta E_{\alpha K}}. \quad (6)$$

In general, the ITCM (3) and its approximate expression (6) are positive semidefinite matrices [26]. Denoting by v_α the vector with components $v_{(ac)\alpha}$, we assume in the following the N_K vectors v_α ($\alpha = 1, \dots, N_K$) of dimension N_K to be linearly independent, in which case they span a manifold of dimension N_K and $C^K(\tau)$ is a positive-definite matrix (see the Supplemental Material [26]). Defining a biorthogonal basis ψ_γ in this manifold by $(\psi_\gamma, v_\alpha) = \sum_{i=1}^N \psi_{i\gamma}^* v_{i\alpha} = \delta_{\alpha,\gamma}$, the vectors ψ_γ can be shown to be generalized eigenvectors of $C^K(\tau)$ with $C^K(\tau_0)$ as a weight function

$$C^K(\tau) \psi_\gamma = \lambda_\gamma(\tau - \tau_0) C^K(\tau_0) \psi_\gamma, \quad (7)$$

and generalized eigenvalues of $\lambda_\gamma(\tau - \tau_0) = e^{-|\tau - \tau_0| \Delta E_{\gamma K}}$. Thus the many-body excitation energies $\Delta E_{\gamma K}$ of states with angular momentum K (and parity π) can be extracted from the generalized eigenvalue problem (GEVP) in Eq. (7).

In general, the vectors v_α may not be linearly independent and the weight matrix $C(\tau_0)$ is positive semidefinite with several zero eigenvalues. In practice, “zero” eigenvalues also correspond to small eigenvalues that are below the SMMC resolution, and their precise number depends on τ_0 . In these cases the GEVP is ill defined. To remedy this, we transform the matrices $C^K(\tau)$ to the basis of eigenstates of $C^K(\tau_0)$ and truncate them to the positive-definite subspace. We show in the Supplemental Material [26] that an approximate GEVP holds in this subspace with a positive-definite weight matrix and with generalized eigenvalues related to excitation energies as above.

The ITCM $C^K(\tau)$ can be cast in a form suitable for SMMC calculations. For a given field configuration σ , we define

$$\rho_{KM\sigma}(ac, \tau) = [(\mathbf{U}_\sigma^{-T}(\tau) a^\dagger)_i \times (\mathbf{U}_\sigma(\tau) \tilde{a})_k]^{KM}, \quad (8)$$

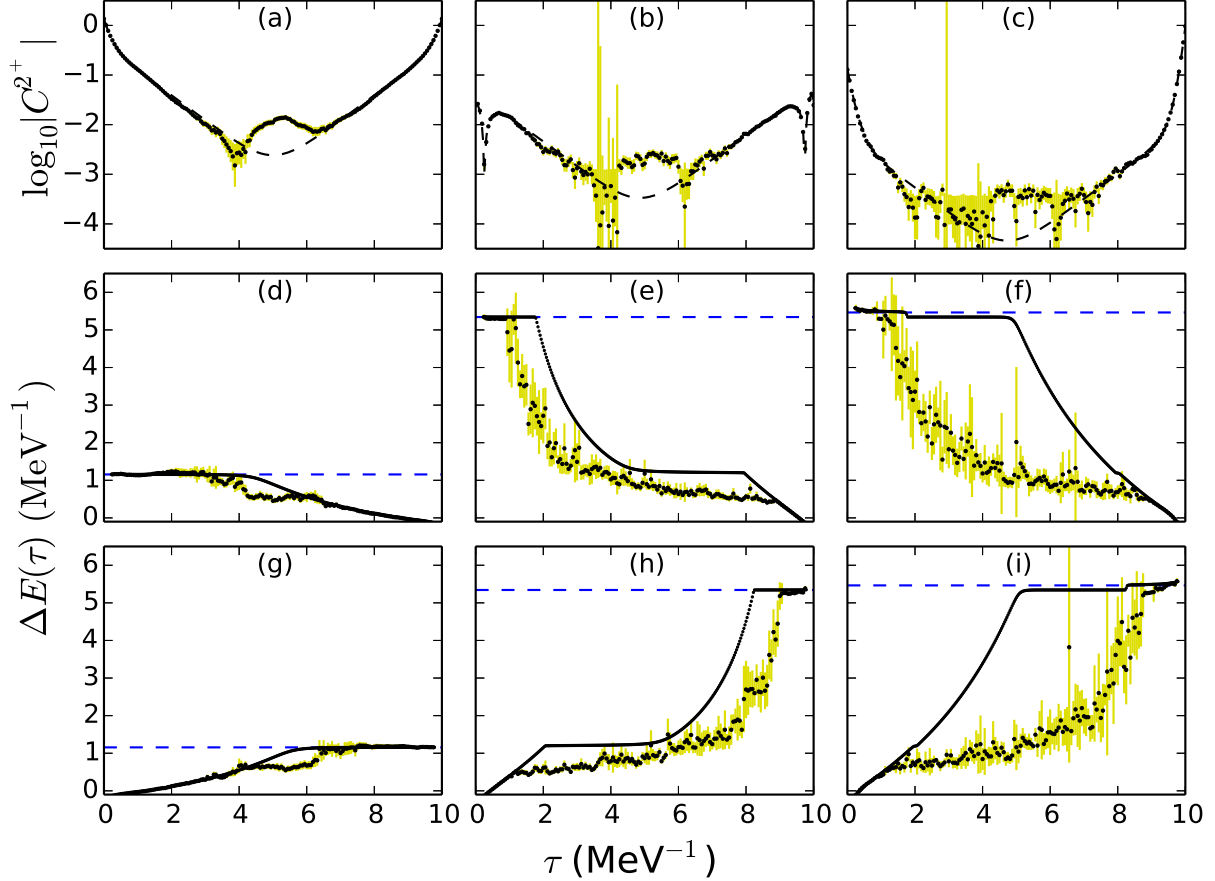


FIG. 1. Panels (a)-(c) in top row: three representative $K^\pi = 2^+$ matrix elements at $\beta = 10 \text{ MeV}^{-1}$ as a function of τ . The exact results obtained from BIGSTICK (dashed black lines) are compared to the SMMC calculations (solid circles with yellow error bars). Panels (d)-(f) in middle row: “excitation energies” $\Delta E_{\gamma K}(\tau) \equiv -\ln \lambda_\gamma / |\tau - \tau_0|$ vs. τ for the first three 2^+ levels using the ITCM method with $\beta = 10 \text{ MeV}^{-1}$ and $\tau_0 = 3/16 \text{ MeV}^{-1}$. The ITCM estimates using BIGSTICK are plotted in black, and the corresponding SMMC results are shown in solid circles with yellow error bars. The constant dashed blue lines are the 2^+ excitation energies calculated by direct diagonalization of the CI shell model Hamiltonian. Panels (g)-(i) in bottom row: same as in panels (d)-(f) but using backward propagation in imaginary time.

where $i = (am_a)$ and $k = (cm_c)$ label single-particle states with good magnetic quantum numbers, $\mathbf{U}_\sigma^{-T}(\tau) = (\mathbf{U}_\sigma^{-1}(\tau))^T$, and $\mathbf{U}_\sigma(\tau)$ describes the matrix representation of $\hat{U}_\sigma(\tau)$ in the single-particle space. In Eq. (8) we have $(\mathbf{U}_\sigma^{-T}(\tau)a^\dagger)_i = \sum_{i'} \mathbf{U}_{\sigma, i' i}^{-1}(\tau) a_{i'}^\dagger$ and $(\mathbf{U}_\sigma(\tau)\tilde{a})_k = \sum_{k'} \mathbf{U}_{\sigma, k k'}(\tau) \tilde{a}_{k'}$. The product $\times$ is defined as in the tensor coupling of Eq. (4). Then, the application of the Hubbard-Stratonovich transformation to the imaginary-time evolved densities results in

$$C_{ac, bd}^K(\tau) = \sum_M \overline{\left\langle \rho_{KM\sigma}^\dagger(ac, \tau) \rho_{KM}(bd) \right\rangle_\sigma}, \quad (9)$$

where we have used the notation of Eq. (2).

Above a certain value of τ , it is necessary to stabilize the SMMC calculation of the ITCM. The method of

stabilization for dynamical observables will be discussed elsewhere.

Validation.— We next validate the ITCM method for an sd -shell nucleus ^{20}Ne for which we can calculate the exact spectrum using the CI shell model code BIGSTICK [21]. The single-particle energies are taken from the USD interaction [27] and the interaction is an attractive quadrupole-quadrupole interaction $-\chi \tilde{Q} \cdot \tilde{Q}$, with $\tilde{Q}_{2\mu} = \sum_i r_i^2 Y_{2\mu}(\hat{r}_i)$ and $\chi = \frac{8\pi}{5} \frac{38.5}{A^{5/3}} \text{ MeV}/b^4$ [28].

We illustrate the ITCM method for $K^\pi = 2^+$. The one-body densities $\rho_{KM}(ac)$ correspond to a specific nucleon type, which leads to the ITCM block structure of the form $(\nu\nu \times \nu'\nu')$, where ν and ν' denote protons or neutrons.

We calculated the ITCM for $\beta = 10 \text{ MeV}^{-1}$ using both SMMC and the CI shell model code BIGSTICK [21]. The original matrix $C^{2^+}(\tau_0)$ has dimension 16 with

two “zero” eigenvalues (within SMMC resolution) for $\Delta\beta = 1/64 \text{ MeV}^{-1}$ and $\tau_0 = 3/16$. We truncated the dimension of the matrices $C^{2^+}(\tau)$ to 14 by restricting to the positive definite subspace of the weight matrix $C^{2^+}(\tau_0)$ [26]. In the case of BIGSTICK, we calculated the exact many-particle spectrum and one-body transition densities $(\alpha J || \rho_K(ac) || \alpha' J')$, and use them to calculate the ITCM via their spectral decomposition.

In the top row of Fig. 1 we show three representative ITCM elements for $K = 2^+$ as a function of imaginary time τ . The dashed black lines are the BIGSTICK calculations and the solid circles with yellow error bars are the SMMC results. The SMMC reproduces the exact results except in the middle regime of $\tau - \tau_0$ where the ITCM elements are below the SMMC resolution of $\sim 10^{-2} - 10^{-3}$.

We find the generalized eigenvalues λ_γ of the GEVP (7) satisfied by the truncated ITCM and calculate the excitation energy estimators $\Delta E_{\gamma K}(\tau) \equiv -\ln \lambda_\gamma / |\tau - \tau_0|$ vs. τ . The panels (d)-(f) in the middle row of Fig. 1 show the results for the first three 2^+ levels using a forward propagation. The curves for $\Delta E_{\gamma K}(\tau)$ calculated from the exact $C^{2^+}(\tau)$ matrices (using BIGSTICK) are shown by solid black lines, and the SMMC results are shown by solid circles with yellow error bars. The excitation energies calculated from direct diagonalization of the CI shell model Hamiltonian are shown by the constant dashed blue lines. We observe that the SMMC result reproduces well the exact excitation energy in a region where there is a plateau in $\Delta E_{\gamma K}(\tau)$ as a function of τ . This plateau tends to be narrower for higher excitation energies. We determine the excitation energy by fitting the energy estimator $\Delta E_{\gamma K}(\tau)$ in the plateau region to a constant. The fitting procedure, described in the Supplemental Material [26], takes into account the correlations between different values of τ .

It is also possible to extract excitation energies by a backward propagation in imaginary time [26]. The bottom panels (g)-(i) in Fig. 1 show the corresponding results for the same three lowest 2^+ levels using a backward propagation.

At higher excitation energies, the SMMC results become more susceptible to statistical noise and contaminations from higher excited states are more pronounced. As a result only eight 2^+ excitation energies are accessible in our example.

For a given level, we extract in SMMC the excitation energies for two values of $\Delta\beta = 1/32, 1/64 \text{ MeV}^{-1}$, and extrapolate to $\Delta\beta = 0$ to find our final estimates. To reduce correlations in the Monte Carlo steps, we average the data over blocks. We then use the jackknife method [29–31] to calculate the ITCM and their generalized eigenvalues. The jackknife method, blocking, and extrapolation to $\Delta\beta = 0$ are described in more details in the Supplemental Material [26].

The allowed spin-parity values in the *sd*-shell model-

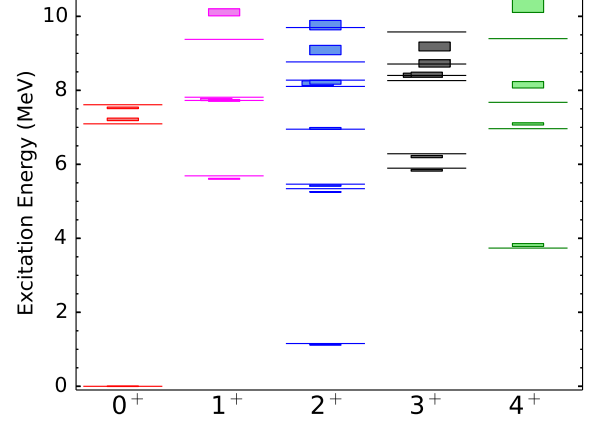


FIG. 2. Excitation energies in ^{20}Ne for different spin-parity values. The SMMC estimates from the ITCM are shown with statistical errors by the boxes centered around the average values. The solid lines are exact excitation energies obtained from diagonalizing the CI shell model Hamiltonian using the code BIGSTICK.

space are $K^\pi = 0^+, 1^+, 2^+, 3^+$ and 4^+ . Results for the excitation energies extracted from the SMMC ITCM for these K^π are shown in Fig. 2. The SMMC values are shown in the colored boxes and are compared to exact excitation energies obtained from direct diagonalization using BIGSTICK (solid horizontal lines). Table I summarizes these results comparing the excitation energies extracted from the SMMC ITCM method with the exact excitation energies. The number of levels extracted from SMMC is less than the dimension of the corresponding

$E_x \text{ (MeV)}$					
K^+	SMMC	Exact	K^+	SMMC	Exact
0^+	0.00	0.0	3^+	5.84(2)	5.9
	7.22(3)	7.1		6.21(3)	6.3
	7.53(2)	7.6		8.41(6)	8.3
1^+	5.61(1)	5.7	4^+	8.42(7)	8.4
	7.73(3)	7.7		8.73(10)	8.7
	7.75(3)	7.8		9.19(12)	9.6
	10.11(10)	9.4		3.82(4)	3.7
2^+	1.12(1)	1.2	4^+	7.09(3)	7.0
	5.25(1)	5.3		8.15(9)	7.7
	5.43(2)	5.5		10.26(25)	9.4
	6.97(3)	7.0			
	8.19(6)	8.1			
	8.22(5)	8.3			
	9.09(13)	8.8			
	9.76(13)	9.7			

TABLE I. SMMC estimates from ITCM of low lying excitation energies in ^{20}Ne in comparison with their exact values (in MeV).

truncated ITCM because statistical errors and contamination prohibit the extraction of the higher excitation energies. With the exception of a few higher energy levels, the energy levels extracted in SMMC are in good agreement with their exact values.

Conclusion.— We have introduced and validated a systematic method within SMMC that employs ITCM to calculate nuclear spectra. The ITCM satisfies a GEVP for which the generalized eigenvalues are simply related to the excitation energies of low-lying levels. We tested the ITCM method for the *sd*-shell nucleus ^{20}Ne , and reproduced the excitation energies of low-lying states calculated by exact diagonalization methods.

It would be interesting to apply the ITCM method to calculate low-lying spectra of heavy nuclei, for which conventional diagonalization methods of the CI shell-model Hamiltonian are prohibited. We have previously identified microscopic signatures of the crossover from vibrational to rotational collectivity in isotopic chains of lanthanides by studying the temperature dependence of $\langle J^2 \rangle$ (where $\mathbf{J}$ is the total angular momentum) [14]. The ITCM method would enable us to identify explicit spectral signatures of collectivity within the CI shell model approach. Of particular interest is the application of the method to heavy neutron-rich nuclei, which are important to *r*-process nucleosynthesis.

More generally, the ITCM method is broadly applicable to quantum many-body systems across other disciplines by constructing one-body density ITCM for given values of the good quantum numbers that characterize the symmetries of the system. The method can then be used to calculate excitation energies for the chosen values of the good quantum numbers.

Acknowledgements.— This work was supported in part by the U.S. DOE grant No. DE-SC0019521. The calculations used resources of the National Energy Research Scientific Computing Center (NERSC), a U.S. Department of Energy Office of Science User Facility operated under Contract No. DE-AC02-05CH11231. We thank the Yale Center for Research Computing for guidance and use of the research computing infrastructure.

[1] B. A. Brown and B. H. Wildenthal, *Annual Review of Nuclear and Particle Science* **38**, 29 (1988).
 [2] E. Caurier, G. Martínez-Pinedo, F. Nowacki, A. Poves, and A. P. Zuker, *Rev. Mod. Phys.* **77**, 427 (2005).
 [3] C. W. Johnson, S. E. Koonin, G. H. Lang, and W. E. Ormand, *Phys. Rev. Lett.* **69**, 3157 (1992).
 [4] G. H. Lang, C. W. Johnson, S. E. Koonin, and W. E. Ormand, *Phys. Rev. C* **48**, 1518 (1993).
 [5] Y. Alhassid, D. J. Dean, S. E. Koonin, G. Lang, and

W. E. Ormand, *Phys. Rev. Lett.* **72**, 613 (1994).
 [6] S. E. Koonin, D. J. Dean, and K. Langanke, *Phys. Rep.* **278**, 1 (1997).
 [7] Y. Alhassid, *Int. J. Mod. Phys. B* **15**, 1447 (2001).
 [8] Y. Alhassid, in *Emergent Phenomena in Atomic Nuclei from Large-Scale Modeling: a Symmetry-Guided Perspective*, edited by K. D. Launey (World Scientific, Singapore, 2017) pp. 267–298.
 [9] H. Nakada and Y. Alhassid, *Phys. Rev. Lett.* **79**, 2939 (1997).
 [10] Y. Alhassid, S. Liu, and H. Nakada, *Phys. Rev. Lett.* **83**, 4265 (1999).
 [11] Y. Alhassid, S. Liu, and H. Nakada, *Phys. Rev. Lett.* **99**, 162504 (2007).
 [12] Y. Alhassid, L. Fang, and H. Nakada, *Phys. Rev. Lett.* **101**, 082501 (2008).
 [13] M. Bonett-Matiz, A. Mukherjee, and Y. Alhassid, *Phys. Rev. C* **88**, 011302(R) (2013).
 [14] C. Özen, Y. Alhassid, and H. Nakada, *Phys. Rev. Lett.* **110**, 042502 (2013).
 [15] Y. Alhassid, C. N. Gilbreth, and G. F. Bertsch, *Phys. Rev. Lett.* **113**, 262503 (2014).
 [16] Y. Alhassid, M. Bonett-Matiz, S. Liu, and H. Nakada, *Phys. Rev. C* **92**, 024307 (2015).
 [17] M. Lüscher and U. Wolff, *Nucl. Phys. B* **339**, 222 (1990).
 [18] S. Beane, W. Detmold, K. Orginos, and M. Savage, *Progress in Particle and Nuclear Physics* **66**, 1 (2011).
 [19] S. R. Beane, W. Detmold, K. Orginos, and M. J. Savage, *Journal of Physics G: Nuclear and Particle Physics* **42**, 034022 (2015).
 [20] R. G. Edwards, N. Mathur, D. G. Richards, and S. J. Wallace (Hadron Spectrum Collaboration), *Phys. Rev. D* **87**, 054506 (2013).
 [21] C. W. Johnson, W. E. Ormand, K. S. McElvain, and H. Shan, “Bigstick: a flexible configuration-interaction shell-model code,” (2018), arXiv:1801.08432 [physics.comp-ph].
 [22] J. Hubbard, *Phys. Rev. Lett.* **3**, 77 (1959).
 [23] R. L. Stratonovich, *Dokl. Akad. Nauk SSSR [Sov. Phys. - Dokl.]* **115**, 1097 (1957).
 [24] W. E. Ormand, D. J. Dean, C. W. Johnson, G. H. Lang, and S. E. Koonin, *Phys. Rev. C* **49**, 1422 (1994).
 [25] M. Dufour and A. P. Zuker, *Phys. Rev. C* **54**, 1641 (1996).
 [26] See the Supplemental Material accompanying this article for additional details regarding the ITCM method. The Supplemental Material includes Refs. [32, 33].
 [27] B. A. Brown and W. A. Richter, *Phys. Rev. C* **74**, 034315 (2006).
 [28] B. Lauritzen and G. Bertsch, *Phys. Rev. C* **39**, 2412 (1989).
 [29] P. Young, *Everything you wanted to know about Data Analysis and Fitting but were afraid to ask*, Springerbrief in Physics (Springer, Cham, 2015).
 [30] M. H. Quenouille, *Ann. Math. Stat.* **20**, 355 (1949).
 [31] J. W. Tukey, *Ann. Math. Stat.* **29**, 614 (1958).
 [32] E. H. Moore, *Bull. Am. Math. Soc.* **26**, 394 (1920).
 [33] R. Penrose, *Mathematical Proceedings of the Cambridge Philosophical Society* **51**, 4067413 (1955).