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Role of Topology in Relaxation of One-Dimensional Stochastic Processes

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Stochastic processes are commonly used models to describe dynamics of a wide variety of nonequilibrium phenomena ranging from electrical transport to biological motion. The transition matrix describing a stochastic process can be regarded as a non-Hermitian Hamiltonian. Unlike general non-Hermitian systems, the conservation of probability imposes additional constraints on the transition matrix, which can induce unique topological phenomena. Here, we reveal the role of topology in relaxation phenomena of classical stochastic processes. Specifically, we define a winding number that is related to topology of stochastic processes and show that it predicts the existence of a spectral gap that characterizes the relaxation time. Then, we numerically confirm that the winding number corresponds to the system-size dependence of the relaxation time and the characteristic transient behavior. One can experimentally realize such topological phenomena in magnetotactic bacteria and cell adhesions.

Introduction.— Since the discovery of the quantum Hall effect [1, 2], band topology has played an important role in condensed matter physics. Topology guarantees invariance against continuous deformations, which is the origin of the robustness of topological phenomena [3, 4]. Recently, the notion of topology has been extended to non-Hermitian systems [5–9], and unique phenomena such as the localization of bulk eigenvectors known as the non-Hermitian skin effect (NHSE) [10–21] have broadened the range of applications.

On another front, stochastic processes are commonly used as models of nonequilibrium systems [22–25], which have attracted growing interest thanks to the recent progress in experimental techniques [26, 27]. One of the typical theoretical approaches to stochastic processes is based on Markov jump processes, which can model various nonequilibrium phenomena including adaptation [28, 29] and ultrasensitivity [30–32] in biological systems. The dynamics of a Markov jump process is described by the master equation, which can be regarded as a non-Hermitian Schrödinger equation because of its linearity. In recent years, there have been studies that apply topological methods to the master equations [33–39]. These attempts have argued localized stationary states [35] and Hall-effect-like chiral edge modes [37], as analogues of conventional topological edge modes. Yet it is still unclear whether or not there are topological phenomena genuinely unique to stochastic processes.

In this Letter, we propose a unique topological feature of one-dimensional classical stochastic processes, characterized by the correspondence between bulk topology and relaxation phenomena. This does not fall into the class of conventional topological phenomena described by general non-Hermitian band structures. Specifically, we define a winding number that characterizes topology of the system under the periodic boundary condition (PBC) where the system forms a cycle without boundaries [40] and show the corresponding qualitative changes in the relaxation behavior. Figure 1 shows the schematic illustration of these changes in terms of the time evolution of the probability distribution. Our key theoretical observation is that nonzero winding numbers imply the finite spectral gap below the zero spectrum of transition-rate matrices under the open boundary condition (OBC) where the system has two boundaries at both ends [40]. This is a manifestation of the role of topology in relaxation phenomena, as the first spectral gap determines the relaxation time of the transition-rate matrix. Quantitatively, we will observe that the finite spectral gap implies $O(N)$ relaxation time with N being the system size, while the vanishing spectral gap implies $O(N^2)$. In addition, we show that nonzero winding numbers accompany the so-called cutoff phenomena [41–44], where the relaxation does not occur until a certain time and then rapidly proceeds. These relaxation features are characterized by the winding number, as summarized in Table I. We discuss the possible realization of the proposed topological relaxation phenomena in magnetotactic bacteria [45, 46] and cell adhesions [47–49].

TABLE I. Summary of the correspondence between the winding number and the relaxation phenomena.

winding number	spectral gap	relaxation time	cutoff phenomena
nonzero	nonzero	$O(N)$	exist
zero	gapless	$O(N^2)$	not exist

Setup.— We consider a general stochastic process described by a master equation $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = W\mathbf{p}(t)$, where $\mathbf{p}(t)$ is a vector representing the probability distribution of the system and W is a transition-rate matrix. We can formally regard the master equation as a non-Hermitian Schrodinger equation by considering the transformation $H = i\hbar W$. Unlike general non-Hermitian Hamiltonians, the off-diagonal components of W

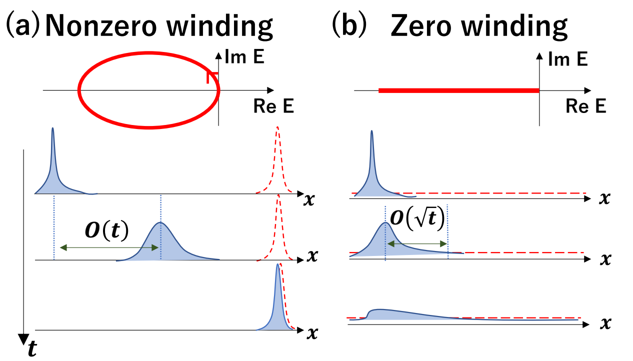

FIG. 1. Schematic of the correspondence between winding numbers and relaxation phenomena for (a) nonzero and (b) zero winding numbers. The upper figures show the spectra under the PBC. The lower figures show the time evolutions of the probability distribution (blue solid curves) and the steady state (red dotted curves) under the OBC. In systems with nonzero winding numbers (a), the displacement is proportional to t because of the drifted motion. Meanwhile, in systems with zero winding numbers (b), the width of the distribution is proportional to $\sqrt{t}$ because of the diffusion. Furthermore, in the topological case, the overlaps between the distribution and the steady state suddenly increase at a certain period, which is observed as a cutoff phenomenon.

should be nonnegative real numbers and the sum of each column of W always gives zero. These constraints can lead to unique properties beyond general non-Hermitian systems as detailed below.

In addition, we assume that the process is ergodic. The Perron-Frobenius theorem states that every ergodic stochastic process has a unique stationary state with zero spectrum, and the real parts of any other spectra of W are less than zero [50]. Then, we may naively estimate the relaxation time τ of the system from the real spectral gap below the zero spectrum $\Delta\lambda \geq 0$ as

$$\tau \sim \frac{1}{\Delta\lambda}. \tag{1}$$

While this holds true in zero-dimensional systems, the relaxation times can diverge in one-dimensional systems even when the $\Delta\lambda$ is finite. This unexpected divergence is known as the gap discrepancy problem [51–53], leading to the $O(N)$ behavior in Table I.

We further assume the spatial translation invariance in the bulk and locality of the master equation. Denoting the components of the matrix W by $W_{nm;\sigma\nu}$, where $n, m = 1, \ldots, N$ being the indices of sites and σ, ν being those of internal degrees of freedom, one can describe the translation invariance in the bulk as $W_{nm;\sigma\nu} = W_{(n-m)0;\sigma\nu}$ for n, m, σ, ν, such that $(n, \sigma) \neq (m, \nu)$ where $W_{(n-m)0;\sigma\nu}$ is a function of $n-m$, σ, ν. The locality of W implies that there exists some integer l_0 such that $W_{nm;\sigma\nu} = 0$ for any σ, ν whenever $|n-m| > l_0$.

Winding numbers in classical stochastic processes.— We first remark that general non-Hermitian topology is characterized by the winding number $w := (2\pi i)^{-1} \int_0^{2\pi} \frac{d}{dk} \log(\det(W(k) - E_0))\mathrm{d}k$ [9], which is defined with a reference point $E_0 \in \mathbb{C}$ and the matrix $W(k)_{\sigma\nu} := \sum_n W_{n0;\sigma\nu} e^{ikn}$. Intuitively, w counts how many times each eigenvalue of $W(k)$ goes around E_0 in the complex plane while k changes from 0 to 2π. We note that E_0 is arbitrarily chosen by a physical motivation and fixed throughout the argument. Since we focus on the relaxation to the steady state corresponding to the zero spectrum, the winding number around $E_0 = 0$ should describe its topological feature. However, since $W(k = 0)$ always has a zero spectrum, it is impossible to directly apply the conventional definition of the winding number to stochastic processes.

Here, to introduce a matrix without a zero spectrum, we consider a matrix with the scale-transformation

$$\left(W^\lambda\right)_{nm;\sigma\nu} := W_{nm;\sigma\nu} e^{\lambda(n-m)}. \tag{2}$$

Then, we define w_+ and w_- as the winding numbers around $E_0 = 0$ calculated at sufficiently small positive and negative λ, respectively:

$$w_\pm := \lim_{\lambda\to\pm 0} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}k} \log\left(\det W^\lambda(k)\right) \mathrm{d}k. \tag{3}$$

Finally, we define the winding number w as $w := w_+ + w_-$ [54]. Figure 2(a) shows the schematics of the case $w \neq 0$. By counting how many times the blue solid curve ($\lambda = +0$) and the red solid curve ($\lambda = -0$) go around $E_0 = 0$, we obtain $w_+ = +1$ and $w_- = 0$. Thus, the winding number is $w = 1$ in Fig. 2(a). Meanwhile, in a symmetric random walk, we obtain $w_\pm = \pm 1$ and thus $w = 0$. This calculation is also valid for the multi-band system where multiple bands can necessarily form a single loop. In general, a nonzero winding number indicates the emergence of the NHSE [10–21], which is detected as the drastic change between the PBC and OBC spectra.

Spectral gap.— We theoretically show that the nonzero gap in the OBC spectrum of a stochastic process corresponds to the nonzero winding number. The spectral gap is defined as the difference between the largest and the second-largest real parts of spectra. Specifically, we will prove the following Main Claim:

Main Claim: In any translationally invariant ergodic classical stochastic process, the OBC spectrum in the thermodynamic limit $N \to \infty$ has a nonzero gap below zero spectrum when the winding number is nonzero, and is gapless when the winding number is zero.

Before going to the proof, we shall first illustrate the basic concepts of this theorem and show the related numerical results. Figure 2(b) shows the illustration of the Main Claim. It asserts that a gap represented by the green dotted arrow opens between the zero spectrum and the continuum spectra in the OBC. We can intuitively understand this gap opening as the combination of shrinkage of the OBC continuum spectra and the fixed zero mode due to the Perron-Frobenius theorem. We note that the spectral gap opens even when the OBC spectrum can have complex spectra.

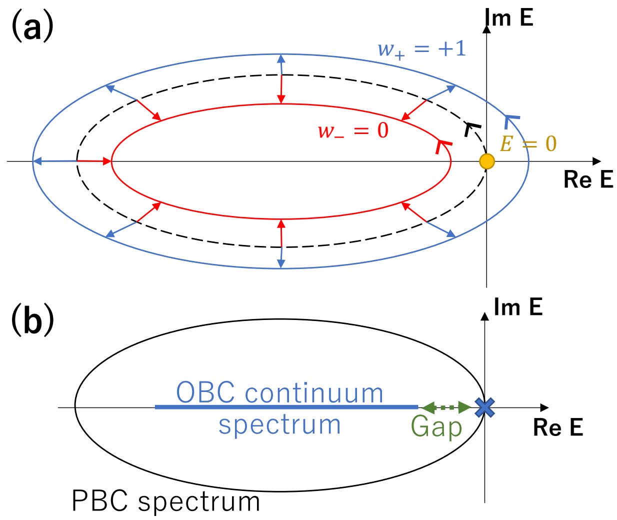

FIG. 2. (a) Schematic of the definition of the winding number in stochastic processes. The black dashed ellipse represents the spectrum of the original stochastic process, the outer blue (inner red) ellipse represents the spectrum obtained by a scale transformation with $\lambda > 0$ ($\lambda < 0$). The zero spectrum $E = 0$ is represented by a yellow circle, and each direction of the derivative by wavenumber k of spectra is indicated by arrows. (b) Schematic of the Main Claim. The black ellipse (blue line) represents the PBC spectrum (OBC continuum spectra). The blue cross represents the zero mode. The Main Claim asserts that the gap indicated by the green dotted arrow opens in the OBC.

We check the correspondence between the winding number and the spectral gap in simple models, the non-Hermitian Su-Schrieffer-Heeger (NHSSH) model [55],

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = \begin{pmatrix} 0 & a_{2+} \\ 0 & 0 \end{pmatrix} p_{n-1}(t) + \begin{pmatrix} 0 & 0 \\ a_{2-} & 0 \end{pmatrix} p_{n+1}(t) + \begin{pmatrix} -(a_{1+}+a_{2-}) & a_{1-} \\ a_{1+} & -(a_{1-}+a_{2+}) \end{pmatrix} p_n(t). \quad (4)$$

and the 2-random walk,

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = a_1 p_{n-1}(t) + b_1 p_{n+1}(t) + a_2 p_{n-2}(t) + b_2 p_{n+2}(t) - (a_1 + b_1 + a_2 + b_2) p_n(t). \quad (5)$$

Analyzing these models, we can confirm that both the internal degrees of freedom and the hopping range have no effects on the Main Claim. We note that the parameters of the models and the time have arbitrary units.

We fit the OBC spectral gaps in the system size N of the NHSSH model and the 2-random walk with a function $\alpha_0 + \alpha_1(1/N) + \alpha_2(1/N)^2$ to estimate the value of the spectral gap at $N \to \infty$ (Fig. 3). We confirm that a gap remains if and only if the winding number is nonzero. This indicates the correspondence between the winding number and the spectral gap. We also show this correspondence analytically in the NHSSH model and numerically in the other models with a larger number of internal degrees of freedom and longer-range hoppings [40].

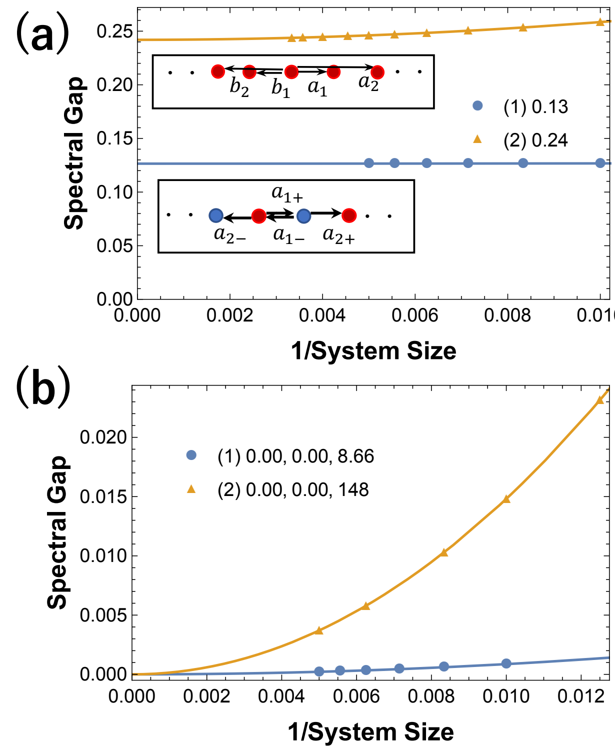

FIG. 3. (a) Numerical results of the spectral gaps with the nonzero winding numbers in (1) NHSSH model and (2) 2-random walk. The inset shows the transition diagram of each model. The circles represent the state of the system and the arrows represent hoppings. The legends indicate the values of fitting parameters α_0. The parameters used are (1) $(a_{1+}, a_{1-}, a_{2+}, a_{2-}) = (1.35, 0.65, 1.35, 0.65)$, (2) $(a_1, a_2, b_1, b_2) = (10, 2, 5, 2.5)$. (b) Numerical results of the spectral gaps with the zero winding number in (1) NHSSH model and (2) 2-random walk. The legends show fitting parameters α_0, α_1, and α_2, respectively. The parameters used are (1) $(a_{1+}, a_{1-}, a_{2+}, a_{2-}) = (1.35, 0.65, 0.65, 1.35)$, (2) $(a_1, a_2, b_1, b_2) = (7, 2, 5, 3)$.

We also check the robustness of the spectral gap in an asymmetric random walk: $W(k) = ae^{ik} + be^{-ik} - (a + b)$. Here, disorders are introduced as $W_{ij} \mapsto \tilde{W}_{ij} := W_{ij} + \Delta_{ij}$, $|\Delta_{ij}| \leq W_{ij} (i \neq j)$ so that $\tilde{W}_{ij}$ is also a transition rate matrix. The OBC spectral gaps remain nonzero under the existence of these disorders [40].

Sketch of proof.— We separately prove the statements in the case of the nonzero winding numbers and the zero winding numbers. To show the nonzero winding number part, we first slightly generalize the claim from stochastic processes to general non-Hermitian systems. Note that unlike conventional non-Hermitian systems, the on-site terms at the open boundaries in stochastic processes are modified from those in the bulk to satisfy the conservation of probability. However, a previous study [12] has shown that the OBC continuum spectra are independent of the boundary terms. Based on the above argument, we generalize the Main Claim to the following Lemma.

Lemma: Suppose that the point E_0 in the PBC spectrum is on the outer edge of the PBC spectrum and is not a self-intersection point. Then, if the first-order derivative by wavenumber of the PBC spectrum at E_0 is nonzero, E_0 is not included in the continuum spectra of the OBC.

Intuitively, this Lemma asserts the shrinkage of the spectrum from the PBC to the OBC. If we focus on zero spectrum in the PBC spectrum of a stochastic process, the condition of the nonzero first-order derivative is equivalent to that of the nonzero winding number [40]. Since a stochastic process always has a zero mode, this Lemma leads to the existence of the OBC spectral gap in a stochastic process with a nonzero winding number.

To prove the Lemma, we utilize the representation of the OBC continuum spectra using scale transformations [13]. The key idea is to examine the small deformation of the PBC spectrum with respect to the scale transformation via the Cauchy-Riemann equation. The remaining zero winding number part of the Main Claim is proved by checking that $E = 0$ satisfies the generalized-Brillouin-zone condition [12]. We describe the details of the proof in Supplementary Material [40].

Gap discrepancy problem.— We next discuss that the relaxation time diverges in one-dimensional stochastic processes under the OBC. Indeed, we rigorously show the divergence of the relaxation time $\tau(N)$ [40], using a similar Lieb-Robinson-like speed limit discussed in Ref. [56]. Combining this with the Main Claim, we conclude that the discrepancy between the nonzero spectral gap and the divergent relaxation time necessarily occurs in general one-dimensional stochastic systems with a nonzero winding number.

We also numerically confirm that the winding number corresponds to the system-size dependence of the relaxation time and a characteristic transient behavior called the cutoff phenomenon. By using the Euler method, we evolve the initial state with one-site excitation at the opposite end to the direction of localization of the steady state. We calculate the 1-norm distance $d(t) := \sum_{n\sigma} |p_{n\sigma}(t) - p^{\mathrm{ss}}_{n\sigma}|$ between the probability distribution of the system $\mathbf{p}(t)$ at time t and the steady state $\mathbf{p}^{\mathrm{ss}}$. Then, we define the relaxation time $\tau(N)$ in the system size N as the smallest time t at which $d(t) < 0.02$ holds. To investigate the system-size dependence, we fit $\{\tau(N)\}$ with the power function $\tau(N) = aN^p$.

Numerical results in the NHSSH model and the 2-random walk are shown in Fig. 4. The relaxation time scales $O(N^2)$ ($O(N)$) in the zero (nonzero) winding number region. Moreover, our rigorous speed limit indeed provides a lower bound on the relaxation times in nonzero winding number systems. The $O(N^2)$ scaling in the gapless OBC systems is reminiscent of the Brownian motion [22] where the standard deviation of the position is proportional to $\sqrt{t}$. We note that while the transition from $O(N)$ to $O(N^2)$ is sharp, there is a finite-size effect on the scaling [40]. Furthermore, in the transient dynamics of the system with the nonzero winding number, the relaxation does not occur until $t \simeq 200$, from which the relaxation proceeds rapidly. This indicates the emergence of a cutoff phenomenon. These results also suggest a correspondence between the nonzero winding number and characteristic relaxation phenomena. We note that the cutoff phenomena depend on the initial state [51, 52] and do not occur from a uniform or random initial condition. The scaling $O(N)$ of the relaxation time remains unchanged under the existence of disorders [40]. This indicates the topological protection, i.e., the robustness originated from the topology, of the scaling behavior of the relaxation time.

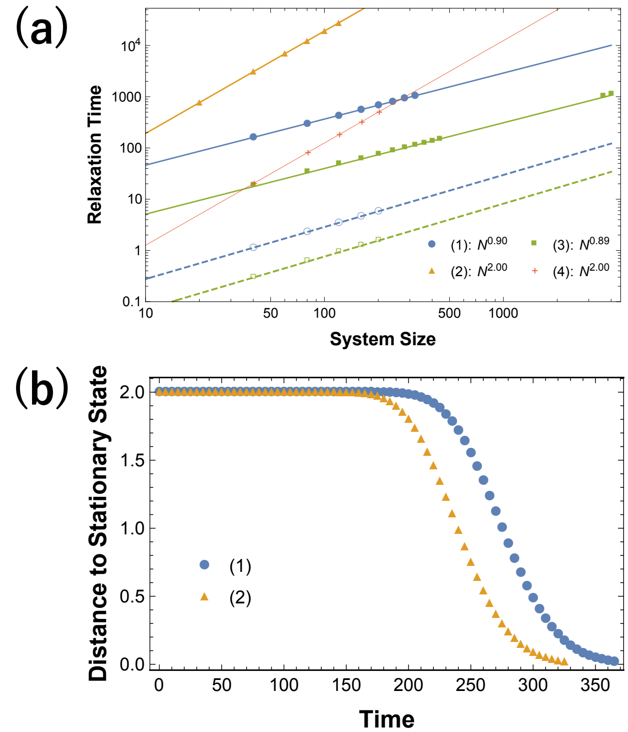

FIG. 4. (a) System-size dependence of relaxation times in (1,2) NHSSH model and (3,4) 2-random walk. The dashed lines show the $O(N)$-lower bound for (1) and (3). The parameters used are (1) $(a_{1+}, a_{1-}, a_{2+}, a_{2-})$ = (1.35, 0.65, 1.35, 0.65), (2) $(a_{1+}, a_{1-}, a_{2+}, a_{2-})$ = (1.35, 0.65, 0.65, 1.35), (3) (a_1, a_2, b_1, b_2) = (10, 2, 5, 2.5), (4) (a_1, a_2, b_1, b_2) = (7, 2, 5, 3). (b) Time evolution of the distance to the steady state in the NHSSH model with nonzero winding number. The parameters used are (1) $(a_{1+}, a_{1-}, a_{2+}, a_{2-})$ = (1.35, 0.65, 1.35, 0.65), (2) $(a_{1+}, a_{1-}, a_{2+}, a_{2-})$ = (1.3, 0.7, 2.6, 1.4).

The exponent of the relaxation time and the emergence of a cutoff phenomenon can be theoretically inferred from the expansion coefficients of the initial state with a one-site excitation [51, 52]. If the winding number is nonzero, because of the NHSE, the expansion coefficients $c_j(0)$ can be $\exp(O(N))$ in the right-eigenvector expansion of a localized initial state $p(0) = \sum_j c_j(0)|\psi_j\rangle$ with $|\psi_j\rangle$ being the jth right eigenvector. Meanwhile, if there is a nonzero spectral gap $\Delta\lambda \in \mathbb{R}$, it takes $O\left(\log[\max|c_j|]/\Delta\lambda + 1/\Delta\lambda\right)$ time until the expansion coefficients $c_j(t)$ become small compared to that of the steady state. These two observations indicate the $O(N)$ dependence of the relaxation time [40]. Furthermore, since the relaxation does not proceed much until the expansion coefficient becomes less than one, we can also expect the presence of the cutoff phenomena. We note that spectral gaps are directly related to the scaling of relaxation times through the NHSE in topological systems. We also discuss the $O(N^2)$ scaling of the relaxation time in a system with a zero winding number by a similar eigenvector expansion [40].

Discussion.— We showed that the nonzero PBC winding number corresponds to the nonzero spectral gap under the OBC in translationally invariant and ergodic one-dimensional classical stochastic processes. Furthermore, the nonzero winding number also corresponds to the system-size dependence of the relaxation time and the presence of a cutoff phenomenon in the transient regime.

The finite spectral gap apparently implies the finite relaxation time (1) in the thermodynamic limit of topological OBC systems. However, in topological OBC systems, we obtained the divergent relaxation time with the unusual system-size dependence instead. Such discrepancy between the divergent relaxation time and the nonzero spectral gap is termed the gap discrepancy problem, and we clarified the conditions for gap discrepancy problems from a topological perspective. While we analyzed lattice models, the same correspondence should hold for a Langevin system since we can describe such a Langevin dynamics by using a Fokker-Planck equation [22], which is a continuous counterpart of a master equation.

The characteristic relaxation phenomena with a nonzero winding number can be experimentally confirmed using Brownian particles under a steady flow and active matter [20, 45–49, 57], since their directed motions can be modeled by the nonreciprocal hopping. In fact, the nonreciprocal lattice models are obtained from discretization of the Fokker-Planck equations under a steady flow [58]. One can also realize discrete systems with nonzero winding numbers by utilizing systems of cell adhesions [48, 49] with a lattice-patterned substrate. We note that the qualitative difference also emerges in the transient behavior of the bulk local current [40]. Finally, it is intriguing to extend our results to higher-dimensional systems. Recent progress on the non-Bloch band theory [12, 13] in higher-dimensional systems [59–61] can provide insights to this end, while the boundary-geometry-dependence of bulk spectra prevents a straightforward extension of our results. It also seems to be hard to construct topological higher-dimensional stochastic systems because the existing non-Hermitian topological systems [13, 62–65] utilize complex hopping terms. Overcoming these points is a challenging future issue.

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[1] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[2] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[3] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[4] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[5] T. Kato, *Perturbation theory for linear operators* (Springer, 1966).
[6] V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, Phys. Rev. B **97**, 121401(R) (2018).
[7] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).
[8] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. **120**, 146402 (2018).
[9] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018).
[10] S. Yao and Z. Wang, Phys. Rev. Lett. **121**, 086803 (2018).
[11] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, Phys. Rev. Lett. **124**, 056802 (2020).
[12] K. Yokomizo and S. Murakami, Phys. Rev. Lett. **123**, 066404 (2019).
[13] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Phys. Rev. Lett. **124**, 086801 (2020).
[14] K. Zhang, Z. Yang, and C. Fang, Phys. Rev. Lett. **125**, 126402 (2020).
[15] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, Nat. Phys. **16**, 747 (2020).
[16] T. Hofmann, T. Helbig, F. Schindler, N. Salgo, M. Brzezińska, M. Greiter, T. Kiessling, D. Wolf, A. Vollhardt, A. Kabaši, C. H. Lee, A. Bilušić, R. Thomale, and T. Neupert, Phys. Rev. Res. **2**, 023265 (2020).
[17] A. Ghatak, M. Brandenbourger, J. van Wezel, and C. Coulais, Proc. Natl. Acad. Sci. U.S.A. **117**, 29561 (2020).
[18] L. Xiao, T. Deng, K. Wang, G. Zhu, W. Y. Zhong Wang, and P. Xue, Nat. Phys. **16**, 761 (2020).
[19] S. Weidemann, M. Kremer, T. Helbig, T. Hofmann, A. Stegmaier, M. Greiter, R. Thomale, and A. Szameit, Science **368**, 311 (2020).
[20] L. S. Palacios, S. Tchoumakov, M. Guix, I. Pagonabarraga, S. Sánchez, and A. G. Grushin, Nat. Commun. **12**, 4691 (2021).
[21] D. Wu, J. Xie, Y. Zhou, and J. An, Phys. Rev. B **105**, 045422 (2022).
[22] N. G. van Kampen, *Stochastic processes in physics and chemistry* (Elsevier, 1992).
[23] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[24] J. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).
[25] D. A. Bagrets and Y. V. Nazarov, Phys. Rev. B **67**, 085316 (2003).
[26] S. Ciliberto, Phys. Rev. X **7**, 021051 (2017).
[27] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, Nat. Phys. **6**, 988 (2010).
[28] N. Barkai and S. Leibler, Nature **387**, 913 (1997).

[29] G. Lan, P. Sartori, S. Neumann, V. Sourjik, and Y. Tu, Nat. Phys. **8**, 422 (2012).
[30] J. E. Segall, S. M. Block, and H. C. Berg, Proc. Natl. Acad. Sci. U.S.A. **83**, 8987 (1986).
[31] Y. Tu, Proc. Natl. Acad. Sci. U.S.A. **105**, 11737 (2008).
[32] Y. Tu, Annu. Rev. Biophys. **42**, 337 (2013).
[33] N. A. Sinitsyn and I. Nemenman, EPL **77**, 58001 (2007).
[34] J. Ren and N. A. Sinitsyn, Phys. Rev. E **87**, 050101(R) (2013).
[35] A. Murugan and S. Vaikuntanathan, Nat. Commun. **8**, 13881 (2017).
[36] K. Dasbiswas, K. K. Mandadapu, and S. Vaikuntanathan, Proc. Natl. Acad. Sci. U.S.A. **115**, E9301 (2018).
[37] E. Tang, J. Agudo-Canalejo, and R. Golestanian, Phys. Rev. X **11**, 031015 (2021).
[38] B. Mahault, E. Tang, and R. Golestanian, Nat. Commun. **13**, 3036 (2022).
[39] P. Mehta and J. Rocks, arXiv:2206.07761 (2022).
[40] See Supplementary Material at "link" for the precise definitions of boundary conditions, the detailed discussions and the additional numerical results.
[41] P. Diaconis, Proc. Natl. Acad. Sci. U.S.A. **93**, 1659 (1996).
[42] D. A. Levin and Y. Peres, *Markov Chains and Mixing Times* (American Mathematical Society, 2009).
[43] M. J. Kastoryano, D. Reeb, and M. M. Wolf, J. Phys. A: Math. Theor. **45**, 075307 (2012).
[44] E. Vernier, SciPost Phys. **9**, 049 (2020).
[45] J. F. Rupprecht, N. Waisbord, C. Ybert, C. Cottin-Bizonne, and L. Bocquet, Phys. Rev. Lett. **116**, 168101 (2016).
[46] S. Klumpp and D. Faivre, Eur. Phys. J. Spec. Top. **225**, 2173 (2016).
[47] J. T. Parsons, A. R. Horwitz, and M. A. Schwartz, Nat. Rev. Mol. Cell Biol. **11**, 633 (2010).
[48] J. d'Alessandro, A. Barbier-Chebbah, O. B. Victor Cellerin, R. M. Mège, R. Voituriez, and B. Ladoux, Nat. Commun. **12**, 4118 (2021).
[49] L. Yamauchi, T. Hayata, M. Uwamichi, T. Ozawa, and K. Kawaguchi, arXiv:2008.10852 (2020).
[50] R. Bhatia, *Matrix Analysis* (Springer, 1997).
[51] T. Haga, M. Nakagawa, R. Hamazaki, and M. Ueda, Phys. Rev. Lett. **127**, 070402 (2021).
[52] T. Mori and T. Shirai, Phys. Rev. Lett. **125**, 230604 (2020).
[53] T. Mori, Phys. Rev. Res. **3**, 043137 (2021).
[54] The definition of winding number in the previous study [35] also uses scale transformations. However, it differs from our definition because the previous study [35] considered only w_+ or w_-.
[55] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).
[56] R. Hamazaki, PRX Quantum **3**, 020319 (2022).
[57] S. Gangwal, O. J. Cayre, M. Z. Bazant, and O. D. Velev, Phys. Rev. Lett. **100**, 058302 (2008).
[58] K. Yokomizo, T. Yoda, and S. Murakami, Phys. Rev. Res. **4**, 023089 (2022).
[59] K. Yokomizo and S. Murakami, Phys. Rev. B **107**, 195112 (2023).
[60] K. Zhang, Z. Yang, and C. Fang, Nat. Commun. **13**, 2496 (2022).
[61] H.-Y. Wang, F. Song, and Z. Wang, arXiv:2212.11743 (2022).
[62] C. H. Lee, L. Li, and J. Gong, Phys. Rev. Lett. **123**, 016805 (2019).
[63] E. Edvardsson, F. K. Kunst, and E. J. Bergholtz, Phys. Rev. B **99**, 081302(R) (2019).
[64] K. Kawabata, M. Sato, and K. Shiozaki, Phys. Rev. B **102**, 205118 (2020).
[65] R. Okugawa, R. Takahashi, and K. Yokomizo, Phys. Rev. B **102**, 241202(R) (2020).