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Failure of Topological Invariants in Strongly Correlated Matter

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We show exactly that standard 'invariants' advocated to define topology for non-interacting systems deviate strongly from the Hall conductance whenever the excitation spectrum contains zeros of the single-particle Green function, G, as in general strongly correlated systems. Namely, we show that if the chemical potential sits atop the valence band, the 'invariant' changes without even accessing the conduction band but by simply traversing the band of zeros that might lie between the two bands. Since such a process does not change the many-body ground state, the Hall conductance remains fixed. This disconnect with the Hall conductance arises from the replacement of the Hamiltonian, $h(\mathbf{k})$, with G^{-1} in the current operator, thereby laying plain why perturbative arguments fail.

The stability of a gapped ground state against smooth deformations of the Hamiltonian that do not close a spectral gap is the cornerstone of topology. Such stability is captured by quantized invariants. Key invariants that arise in topological systems are the Chern numbers. While they appear as coefficients of the Chern-Simons Lagrangian, they have physical import as well. For example, the first Chern number, C_1, is the coefficient,

$$\sigma_{\rm H} = C_1 \frac{e^2}{h}, \tag{1}$$

of the Hall conductance [1, 2]. As a topological invariant, C_1 can only change if the chemical potential crosses a band or more generally, if there are zero-energy excitations, measured with respect to the chemical potential. Any movement of the chemical potential within a spectral gap amounts to an adiabatic change of the system Hamiltonian, and so cannot change C_1.We will take such a change to be the paradigmatic definition of an infinitesimal deformation.

For computational purposes, it has become common to formulate Chern numbers in terms of single-particle Green functions. Consider the commonly conceived invariant N_3[3, 4] for the two-dimensional quantum anomalous Hall (QAH) insulator (also named as N_2 in Ref. [5, 6])

$$N_3 = \frac{\epsilon_{\alpha\beta\gamma}}{6} \\ {\rm tr}\int_{-\infty}^{\infty} d\omega \int \frac{d^2\mathbf{k}}{(2\pi)^2} G^{-1}\partial_{k_\alpha} G G^{-1}\partial_{k_\beta} G G^{-1}\partial_{k_\gamma} G, \tag{2}$$

where $G(\omega, \mathbf{k})$ is the zero temperature (single-particle) Green function in momentum space, α, β, and γ take values 0, 1 and 2, such that $k_0 = \omega$, and k_1, k_2 are components of the crystal momentum, and tr denotes the trace over the fermionic degrees of freedom of G. For non-interacting electrons, N_3 reduces to the first Chern number C_1, or equivalently the Thouless-Kohmoto-Nightingale-den Nijs (TKNN) [1] invariant. That N_3 is invariant to small deformations of the Hamiltonian follows from substituting the infinitesimal,

$$\begin{aligned}\delta(G\partial_{k_\alpha}G^{-1}) &= \delta G\partial_{k_\alpha}G^{-1} - G\partial_{k_\alpha}(G^{-1}\delta G G^{-1}) \\ &= -G(\partial_{k_\alpha}G^{-1})\delta G G^{-1} - \partial_{k_\alpha}(\delta G)G^{-1},\end{aligned} \tag{3}$$

into the variation of N_3 which leads to a recasting of the resultant integrand as a total derivative. As the integral of a total derivative, δN_3 will naturally vanish for δG continuously connected to zero (i.e. for small deformations). Consequently, N_3 is invariant to infinitesimal changes in the underlying Hamiltonian provided periodic boundary conditions are imposed.

The utility of Eq. (2) is that only the Green function is required to evaluate N_3, rather than the full spectrum of the eigenstates as is typically needed to compute the Berry curvature or the TKNN invariant[1]. Consequently, one may hope that Eq. (2) naturally applies to interacting systems. However, when interactions are present, the Green function can vanish[7] along a connected surface in momentum space for frequencies within the gap. This defines the Luttinger surface, which is a Mott fixed point under local perturbations[8, 9]. What happens to N_3 when the chemical potential crosses such a surface? If the ground state evolves continuously and the gap does not close, then the topological invariants of the ground state cannot change. That is, C_1 should remain fixed. However, it is known[3, 4] that N_3 is sensitive to a zero or an edge-state (pole in the propagator) crossing the chemical potential. It is this sensitivity that underlies a recent claim that zeros are topological in the context of doped Mott insulators[10]. In particular for models of fractional quantum Hall effect (FQHE), it has been shown that N_3 is in general not equal to the C_1[11]. Even more, pairs of fractional quantum Hall states with different Chern numbers (and hence different ground state topology) can be shown to have equal values of N_3. However, to our knowledge, the precise relationship between N_3 and C_1 as a function of chemical potential has not been established for an interacting system.

It is this loophole that we address in this paper. For the Hatsugai-Kohmoto model[12, 13] with a topological nontrivial ground state, we use the exact Green function to show that even without closing the gap, N_3 changes when a band of zeros crosses the chemical potential. By definition, such a change constitutes an infinitesimal variation that does not close an energy gap, and hence there should be no change in topological invariants characterizing the ground state. Consequently, we demonstrate explicitly that N_3 in Eq. (2) and C_1 are disconnected should zeros appear in the Green function.

In general for interacting systems, although N_3 is a topological property of the single-particle Green function, it does not necessarily encode a topological invariant of the ground state in contrast to previous claims[3, 4, 10].

The computation of N_3 requires knowledge of the full single-particle Green function. To this end, we adopt a model that affords an exact treatment of interaction and topology for the QAH effect[14]. For a square lattice with the orbitals positioned at lattice sites, the non-interacting part of a two-fold (spinful) Chern insulator can be written as,

$$H_0 = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} h(\mathbf{k}) c_{\mathbf{k}} = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} \begin{pmatrix} h_{\mathrm{QAH}}(\mathbf{k}) & 0 \\ 0 & h_{\mathrm{QAH}}(\mathbf{k}) \end{pmatrix} c_{\mathbf{k}}, \quad (4)$$

where $c^\dagger = \{c^\dagger_{O_1,\uparrow} c^\dagger_{O_2,\uparrow}, c^\dagger_{O_1,\downarrow} c^\dagger_{O_2,\downarrow}\}$ is a four-component spinor, and $O_{1/2}$ stands for different orbitals or sub-lattices, respectively. $h_{\mathrm{QAH}}(\mathbf{k}) = \vec{d}(\mathbf{k}) \cdot \vec{\sigma}$ describes a 2×2 QAH Hamiltonian for each spin, e.g. $\vec{d}(\mathbf{k}) = (\sin k_x, \sin k_y, m - \cos k_x - \cos k_y)$. This non-interacting Hamiltonian can be diagonalized under a unitary transformation into $h(\mathbf{k}) = V(\mathbf{k})\mathrm{diag}(\varepsilon_{-,\mathbf{k}}, \varepsilon_{-,\mathbf{k}}, \varepsilon_{+,\mathbf{k}}, \varepsilon_{+,\mathbf{k}}) V^\dagger(\mathbf{k})$ where upper $(+)$ and lower $(-)$ bands are given by

$$\varepsilon_{\pm,\mathbf{k}} = \pm|\vec{d}(\mathbf{k})| = \pm\sqrt{d_x^2(\mathbf{k}) + d_y^2(\mathbf{k}) + d_z^2(\mathbf{k})}. \quad (5)$$

Electrons with opposite spin have the same dispersion and chirality. This momentum space basis is not destroyed under the local-in-momentum Hatsugai-Kohmoto (HK) interaction that includes Mottness[8, 12, 14–16]

$$H_{\mathrm{QAH-HK}} = \sum_{\mathbf{k},\sigma} \left[(\varepsilon_{+,\mathbf{k}} - \mu) n_{+,\mathbf{k},\sigma} + (\varepsilon_{-,\mathbf{k}} - \mu) n_{-,\mathbf{k},\sigma}\right] + U \sum_{\mathbf{k}} (n_{+,\mathbf{k},\uparrow} + n_{-,\mathbf{k},\uparrow})(n_{+,\mathbf{k},\downarrow} + n_{-,\mathbf{k},\downarrow}). \quad (6)$$

The interaction term is rotational symmetric under the unitary transform $V(\mathbf{k})$ since $n_{+,\mathbf{k},\sigma} + n_{-,\mathbf{k},\sigma}$ is a trace in either the orbital or band basis. The exact Green function in the band basis,

$$G_{\pm,\mathbf{k},\sigma}(\omega) = \frac{\langle(1 - n_{+,\mathbf{k},\bar{\sigma}})(1 - n_{-,\mathbf{k},\bar{\sigma}})\rangle}{\omega + \mu - \varepsilon_{\pm,\mathbf{k}}} + \frac{\langle n_{+,\mathbf{k},\bar{\sigma}} + n_{-,\mathbf{k},\bar{\sigma}} - 2n_{+,\mathbf{k},\bar{\sigma}} n_{-,\mathbf{k},\bar{\sigma}}\rangle}{\omega + \mu - (\varepsilon_{\pm,\mathbf{k}} + U)} + \frac{\langle n_{+,\mathbf{k},\bar{\sigma}} n_{-,\mathbf{k},\bar{\sigma}}\rangle}{\omega + \mu - (\varepsilon_{\pm,\mathbf{k}} + 2U)}, \quad (7)$$

has 6 poles at any given momentum. However, only some of them have a non-vanishing weight in the insulating state for $U \gg W$, where $W = 2\max_{\mathbf{k}} |\vec{d}(\mathbf{k})|$ is the bandwidth. At quarter-filling, the degenerate ε_- band is singly occupied, thus $\langle n_{-,\mathbf{k},\uparrow}\rangle = \langle n_{-,\mathbf{k},\downarrow}\rangle = 1/2$. The ε_+ band remains empty for both spin, $\langle n_{+,\mathbf{k},\sigma}\rangle = 0$ and $\langle n_{+,\mathbf{k},\bar{\sigma}} n_{-,\mathbf{k},\bar{\sigma}}\rangle = 0$. Thus, the poles at $\varepsilon_{\pm,\mathbf{k}} - \mu + 2U$ have zero weight while the poles at $\varepsilon_{\pm,\mathbf{k}} - \mu + U$ and $\varepsilon_{\pm,\mathbf{k}} - \mu$ both have $1/2$ weight.

At half-filling and $U \gg W$, the ground state always occupies both $\varepsilon_\pm$ with the same spin, $\langle n_{-,\mathbf{k},\sigma}\rangle = \langle n_{+,\mathbf{k},\sigma}\rangle = 1/2$ and $\langle n_{+,\mathbf{k},\sigma} n_{-,\mathbf{k},\sigma}\rangle = \frac{1}{2}$. Thus the poles at $\varepsilon_{\pm,\mathbf{k}} - \mu + U$ have zero weight. The remaining 4 poles all have the same weight of $1/2$, generating the zero branches located at the poles of the self-energy,

$$\Sigma_{\pm,\mathbf{k},\sigma}(\omega) = U + \frac{U^2}{\omega + \mu - \varepsilon_{\pm,\mathbf{k}} - U}. \quad (8)$$

The position of the 4 poles relative to the chemical potential defines the electron filling. In the case of half-filling, the lower two poles located at $\varepsilon_{\pm,\mathbf{k}} - \mu$ lie below the chemical potential, while $\varepsilon_{\pm,\mathbf{k}} - \mu + 2U$ lies above, thereby maintaining the gapped Mott state.

According to a previous analysis[14] on the topology of this model, we know that both the QAH-HK and QAH-Hubbard models predict a topologically trivial phase at half-filling when the interactions dominate. There is a topological phase transition from half-filling to quarter-filling, leading to a topological Mott insulator at quarter-filling with $C_1 = 1$, which is half the Chern number of the non-interacting insulator at half filling .

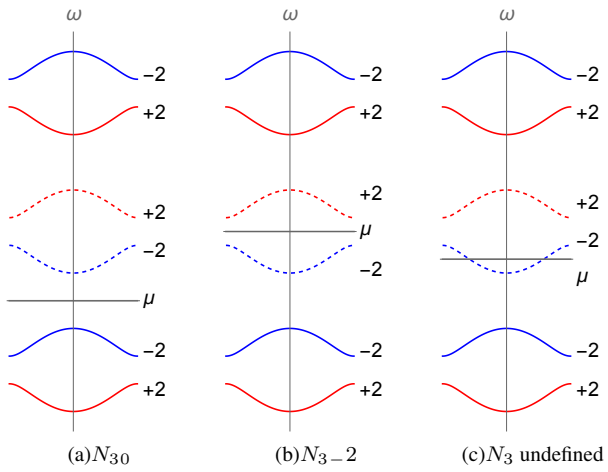

FIG. 1: The pole structure for the Green function from Eq. (7) at half-filling with $U \gg W$. The solid lines represent the poles, the dashed line represents the zeroes. The numbers next to the curves are the corresponding contribution to N_3 of that particular band. Note that all three of these configurations represent the same gapped half-filling ground state, while the N_3 for each setup is $N_3 = 0, -2$, or undefined. Here we use the Haldane model[17] as an example for Eq. (6) with an HK interaction to construct the band dispersion.

At any filling with a gap, such as half-filling where U sets the gap scale, we can shift the chemical potential μ inside this gap without affecting the many-body ground state. As this constitutes an infinitesimal variation of the Hamiltonian, there should be no change in the topology. However, this shift of μ drastically changes the value of N_3 due to the location of zeroes, as shown in Fig. 1. At half-filling, $\langle n_{\pm,\mathbf{k},\bar{\sigma}}\rangle = 1/2$ for both of the spin as well as the upper$(+)$ and lower$(-)$ topological bands. All the spinful bands of zeroes or poles below the

chemical potential contribute a ± 2 to N_3 as labeled in Fig. (1). The zero bands locate at $\varepsilon_{\pm,\mathbf{k}} - \mu + U$. If the branches of the zeros are located on the same side of the chemical potential (Fig. 1(a)), $N_3 = 0$. In the vicinity of the symmetry point, $\mu = U$, the chemical potential is located between the two zero branches (Fig. 1(b)), giving rise to a non-zero $N_3 = -2$. When the chemical potential passes through the zeroes band (Fig. 1(c)), N_3 diverges as if the system is in a metallic state.

This seems to give rise to a contradiction if we expect N_3 to be proportional to the Chern number (equivalently, the Hall conductance). That is, there seems to be a change in the topological invariant without changing the many-body ground state. A similar change in the Luttinger count has been noted previously[18, 19], because moving the chemical potential in the gap changes the positions of the zeros but ultimately cannot change the filling. It is for this reason that it has been correctly argued that the Luttinger count, which counts zeros and poles, does not enumerate the charge density in generic interacting systems. Similarly, we have shown here explicitly that N_3 is counting both zeros and poles of the Green function and hence does not enumerate the Chern number in general. This derivation could also apply to the quantum spin Hall (QSH) system[20–22] with strong interactions where a similar interaction-induced topological phase is observed[13].

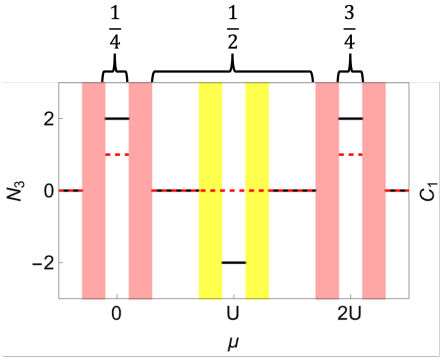

FIG. 2: The change of N_3 (Solid Black line) according to Eq. (2) and C_1 (Dashed Red line) according to Eq. (12) for a QAH-HK model with $U \gg W$ as a function of the chemical potential μ. The fillings are labeled at the top. Inside the red regions, both N_3 or C_1 are undefined due to the crossing of poles (metallic state); inside the yellow regions, N_3 is undefined due to the crossing of zeros.

To address this conundrum, we compute the Hall conductance directly and establish when it is permissible for it to be recast as N_3. The advantage of the HK model is that the interactions preserve the center of mass and U does not have any dependence on momentum. Thus, the current operator in the orbital basis

$$\mathbf{J}(\mathbf{q}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}-\mathbf{q}/2} \frac{\partial h(\mathbf{k})}{\partial \mathbf{k}} c_{\mathbf{k}+\mathbf{q}/2}, \tag{9}$$

can be taken to be unchanged from its non-interacting form (See Appendix), where $h(\mathbf{k})$ is the 4×4 non-interacting Hamiltonian defined in Eq. (4).We substitute this current operator into the Kubo formula[23] and obtain the current-current response function at finite temperature

$$\begin{aligned} R_{\alpha\beta}(q,\tau) &= \langle T[J_\alpha(q,\tau) J_\beta(-q,0)] \rangle \\ &= \frac{1}{V} \sum_{k,k'} \frac{\partial h^{ab}(\mathbf{k})}{\partial k_\alpha} \frac{\partial h^{cd}(\mathbf{k}')}{\partial k'_\beta} \\ &\quad \langle T[c^{\dagger}_{k-q/2,a}(\tau) c_{k+q/2,b}(\tau) c^{\dagger}_{k'+q/2,c} c_{k'-q/2,d}] \rangle \,, \end{aligned} \tag{10}$$

where α and β represent real-space directions and a, b, c, d are orbital and spin indices. Since the HK interaction does not mix momentum, the 4-fermion correlation function can be calculated according to Wick's theorem[9]. We find that

$$\begin{aligned} &\langle T[c^{\dagger}_{k-q/2,a}(\tau) c_{k+q/2,b}(\tau) c^{\dagger}_{k'+q/2,c} c_{k'-q/2,d}] \rangle \\ &= \langle c^{\dagger}_{k-q/2,a}(\tau) c_{k'-q/2,d} \rangle \langle c_{k+q/2,b}(\tau) c^{\dagger}_{k'+q/2,c} \rangle \,. \end{aligned} \tag{11}$$

The Fourier transform of the current-current response function gives $j_\alpha(q,\omega) = R_{\alpha\beta}(q,\omega) A_\beta(q,\omega)$. The conductivity is thus given via analytical continuation $\sigma_{\alpha\beta}(\omega) = \lim_{q\to 0} \frac{1}{i\omega} R_{\alpha\beta}(q, i\nu_r \to \omega + i\eta)$ with

$$\begin{aligned} R_{\alpha\beta}(q, i\nu_r) &= \frac{k_B T}{V} \sum_{k,n} \\ &\mathrm{Tr}\left[\frac{\partial h(\mathbf{k})}{\partial k_\alpha} G(k+q/2, \omega_n) \frac{\partial h(\mathbf{k})}{\partial k_\beta} G(k-q/2, \omega_n - \nu_r) \right]. \end{aligned} \tag{12}$$

For a non-interacting system, $h(\mathbf{k})$ in Eq. (10) can be replaced by G^{-1} which will bring the Hall conductance into the form of N_3. However, for an interacting system, no such correspondence can be made; in general

$$\frac{\partial G^{-1}(\mathbf{k})}{\partial k_\alpha} = \frac{\partial h(\mathbf{k})}{\partial k_\alpha} + \frac{\partial \Sigma(\mathbf{k})}{\partial k_\alpha}, \tag{13}$$

because the presence of the self-energy in the Green function introduces added momentum dependence. A non-trivial $\Sigma(\mathbf{k})$ with a band of poles (yielding the band of zeros in the Green function as shown in Fig. 1(b)) gives rise to the non-zero contribution to N_3. Also, $\Sigma(\mathbf{k})$ diverges at the Luttinger surface, accounting for the undefined N_3 in Fig. 1(c). Hence, for any interacting model with a pole in its self-energy, replacing $h(\mathbf{k})$ with G^{-1} fails. As a consequence, there will be a general disconnect between N_3 with the Hall conductance whenever zeros exist. A recent derivation of the Hall conductance using diagrammatic perturbation theory (which inherently assumes adiabatic continuity with the non-interacting limit) purports to derive an equivalence between N_3 and C_1 [24]. As zeros in the Green function indicate that the self-energy diverges, no such adiabatic continuity exists and hence any correspondence between N_3 and C_1 fails based on perturbative arguments. This is consistent with two prior results. First, the breakdown of Luttinger's theorem for interacting systems stems been tied to the non-existence of the Luttinger-Ward

functional on account of poles in the self energy[19]. Second, the disconnect between N_3 and C_1 for fractional quantum Hall states—which are not perturbatively connected to non-interacting topological phases—was pointed out in Ref. [11] and also in the context of Weyl-Mott insulators[25]. Consequently, anytime there is a breakdown of perturbation theory, N_3 and C_1 cannot be directly related.

In all such cases, the Hall conductance must be computed directly from the Kubo formula, or equivalently by integrating the Berry curvature as a function of twisted boundary conditions [2]. We illustrate this here with a computation of the Hall conductance directly from Eq.(7). The full details are provided in the Appendix. We define the Chern number of the ground state according to Eq. (1), where the ground state is taken to be the zero-temperature limit of a thermal state to account for the spin degeneracy. At quarter-filling, the Hall conductance $C_1 = 1$ is halved compared with the non-interacting two-fold QAH result $C_1 = 2$. At half-filling, the Hall conductance remains zero as long as no pole of the Green function crosses the chemical potential. To illustrate the deviation of N_3 from the Hall conductance C_1, we plot their values as a function of the chemical potential μ in Fig. 2. Besides the conflict between a non-zero N_3 and a vanishing C_1 at half-filling, we observe an additional difference by a factor of 2 at quarter-filling between them. This difference of factor is similar to the deviation between N_3 and C_1 in FQHE[2, 11] caused by the ground state degeneracy. Thus, neither the trivial phase at half-filling nor the topological phase at quarter-filling could be captured accurately by N_3. This invariant fails to capture properties of the ground state that are robust to perturbations of the Hamiltonian, both qualitatively and quantitatively. We have thus shown that the deviation of N_3 from C_1 stems from poles in the self-energy or equivalently zeros of the single-particle Green function. A similar problem occurs for the Luttinger count,

$$n = 2\int_{\mathrm{ReG(p},\omega=0)>0} \frac{d^d p}{(2\pi)^d}, \tag{14}$$

which makes no distinction between the mechanisms for $\mathrm{ReG(p},\omega)$ crossing the real axis. There is now ample evidence[18, 19, 26] that it is zeros that disconnect the Luttinger count from the physical particle density. At play here is a similar trend: any movement of the chemical potential within the gap changes the Luttinger count but ultimately should not change the physical charge density. This is not surprising as the Luttinger count is reducible to the analogous expression for N_3 with just a single product $\partial G^{-1} G$, thereby defining N_1[4]. It was shown in Ref. [27] that two $1+1$-dimensional interacting systems with unequal N_1 could nevertheless possess topologically equivalent ground states. Taken together, we see that all generalized invariants of the form, N_ℓ are disconnected from the physics of the many-body ground state because of the zeros of the single-particle Green function. For both N_1 and N_3, this discrepancy arises precisely when the *single-particle* Green function fails to accurately capture properties of the *many-body* ground state; the emergence of Green function zeros signifies the importance of multi-particle spectral weight. The charge density and Hall conductance, being properties of the ground state and *not* properties of single-particle excitation, encode physics beyond the single-particle Green function. Finally, we note that Refs. [4, 10] showed that at the interface between two systems across which N_3 jumps by ΔN_3 with no other differing topological invariant, there will be ΔN_3 zeros in the boundary Green function. While this result is certainly correct and encodes topological properties of the single-particle Green function, our work here calls into question the significance of this result for ground-state topological properties. In particular, we have shown here that N_3 can jump at an interface where the chemical potential changes smoothly while remaining in the bulk gap. Although the single-particle Green function will develop boundary zeros, we have shown that robust observables computed from the many-body ground state cannot change across the interface. In order to reconcile these observations, what is needed is an analysis of higher-order correlation functions to reinstate the connection between ground state topology and robust observables [28].

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Appendix: Current operator of HK model

Here we provide a detailed derivation of the current operator of the HK model. We start with a generalized form of the HK Hamiltonian in a band basis,

$$\begin{aligned} H =& H_0 + H_I \\ =& \sum_{\mathbf{k},\sigma} c^\dagger_{\mathbf{k},a\sigma} h^{ab}_\sigma(\mathbf{k}) c_{\mathbf{k},b\sigma} \\ &+ \sum_{\mathbf{k}} A^{abcd}(\mathbf{k}) c^\dagger_{\mathbf{k},a\uparrow} c_{\mathbf{k},b\uparrow} c^\dagger_{\mathbf{k},c\downarrow} c_{\mathbf{k},d\downarrow}, \end{aligned} \tag{15}$$

where $h_\sigma(\mathbf{k})$ is the 2×2 QAH Hamiltonian for each spin, a, b, c, d are orbital indices which take value from O_1 or O_2. The current satisfies the continuity equation,

$$\frac{\partial \rho(x)}{\partial t} + \nabla \cdot \mathbf{J}(x) = 0. \tag{16}$$

The density operator for origin-localized orbitals can be Fourier transformed into

$$\rho(q) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},a,\sigma} c^\dagger_{\mathbf{k},a\sigma} c_{\mathbf{k}+\mathbf{q},a\sigma}. \tag{17}$$

The continuity equation then yields

$$\mathbf{q}\cdot\mathbf{J}(q) = [\rho(q), H] = [\rho(q), H_0] + [\rho(q), H_I], \tag{18}$$

where the first term is the non-interacting current operator Eq. (9). We focus on the contribution from the second term. According to the general Fermion commutation relation

$$\begin{aligned}
&\sum_{\mathbf{k}',e}[c^\dagger_{\mathbf{k}',e\sigma}c_{\mathbf{k}'+\mathbf{q},e\sigma}, c^\dagger_{\mathbf{k},a\sigma}c_{\mathbf{k},b\sigma}] \\
&= c^\dagger_{\mathbf{k}-\mathbf{q},a\sigma}c_{\mathbf{k},b\sigma} - c^\dagger_{\mathbf{k},a\sigma}c_{\mathbf{k}+\mathbf{q},b\sigma} \\
&= -\mathbf{q}\cdot\nabla_{\mathbf{k}}c^\dagger_{\mathbf{k},a\sigma}c_{\mathbf{k},b\sigma} - c^\dagger_{\mathbf{k},a\sigma}\mathbf{q}\cdot\nabla_{\mathbf{k}}c_{\mathbf{k},b\sigma} \\
&= -\mathbf{q}\cdot\nabla_{\mathbf{k}}(c^\dagger_{\mathbf{k},a\sigma}c_{\mathbf{k},b\sigma}),
\end{aligned} \tag{19}$$

where we have expanded the operator to linear order in q and neglected higher order terms $O(q^2)$ since we will take the $q \to 0$ limit in the Kubo formula. The commutator between the density operator and the interaction is

$$\begin{aligned}
&\sum_{\mathbf{k}} A^{abcd}(\mathbf{k}) \sum_{\mathbf{k}',e,\sigma}[c^\dagger_{\mathbf{k}',e\sigma}c_{\mathbf{k}'+\mathbf{q},e\sigma}, c^\dagger_{\mathbf{k},a\uparrow}c_{\mathbf{k},b\uparrow}c^\dagger_{\mathbf{k},c\downarrow}c_{\mathbf{k},d\downarrow}] \\
&= -\sum_{\mathbf{k}} A^{abcd}(\mathbf{k})\Big(\mathbf{q}\cdot\nabla_{\mathbf{k}}(c^\dagger_{\mathbf{k},a\uparrow}c_{\mathbf{k},b\uparrow})c^\dagger_{\mathbf{k},c\downarrow}c_{\mathbf{k},d\downarrow} \\
&\qquad + c^\dagger_{\mathbf{k},a\uparrow}c_{\mathbf{k},b\uparrow}\mathbf{q}\cdot\nabla_{\mathbf{k}}(c^\dagger_{\mathbf{k},c\downarrow}c_{\mathbf{k},d\downarrow})\Big) \\
&= -\sum_{\mathbf{k}} A^{abcd}(\mathbf{k})\mathbf{q}\cdot\nabla_{\mathbf{k}}(c^\dagger_{\mathbf{k},a\uparrow}c_{\mathbf{k},b\uparrow}c^\dagger_{\mathbf{k},c\downarrow}c_{\mathbf{k},d\downarrow}) \\
&= \sum_{\mathbf{k}} \mathbf{q}\cdot(\nabla_{\mathbf{k}}A^{abcd}(\mathbf{k}))c^\dagger_{\mathbf{k},a\uparrow}c_{\mathbf{k},b\uparrow}c^\dagger_{\mathbf{k},c\downarrow}c_{\mathbf{k},d\downarrow},
\end{aligned} \tag{20}$$

where at the last step we integrated by part using the fact that the Brillouin zone is compact. Thus there is no contribution to $J(q \to 0)$ as long as $\nabla_{\mathbf{k}}A^{abcd}(\mathbf{k}) = 0$. This is true for the interaction term we used in Eq. (6) where $A^{abcd}(\mathbf{k}) = U\delta_{ab}\delta_{cd}$. Thus, we can use Eq. (9) for the current operator for the purposes of computing response functions in the $q \to 0$ limit.

Appendix: Hall conductance of QAH-HK model

Eq.(12) can be directly used to calculate the Hall conductance with the exact Green function Eq.(7) using numerical integration techniques. Here we will follow the derivation by Bernevig[29] by introducing a flat-band limit in order to analytically compute the Hall conductivity for the QAH-HK model. We place all the occupied energy at $\mu - U < \varepsilon_G < \mu$, whereas all the unoccupied states at energy $\varepsilon_E > \mu$, while keeping the eigenstates of the system unmodified. The Hall conductance (Eq. (12)) is invariant under this deformation since the deformation of the dispersion is smooth. We also take the $\beta \to \infty$ limit to achieve the zero temperature result.

The non-interacting Hamiltonian for each spin can be written as the sum over the projectors $P_G(\mathbf{k}) = \sum_{i\in G}|i,k\rangle\langle i,k|$ onto the occupied states and the projectors $P_E(\mathbf{k}) = \sum_{j\in E}|j,k\rangle\langle j,k|$ the empty states,

$$h_\sigma = \varepsilon_G P_G + \varepsilon_E P_E. \tag{21}$$

We emphasize that the filled bands are defined according to the interacting system. Thus the projection operators depend on the filling. The current is thus

$$\frac{\partial h_\sigma(\mathbf{k})}{\partial k_\alpha} = \varepsilon_G\frac{\partial P_G}{\partial k_\alpha} + \varepsilon_E\frac{\partial P_E}{\partial k_\alpha} = (\varepsilon_G - \varepsilon_E)\frac{\partial P_G}{\partial k_\alpha}. \tag{22}$$

In the flat band limit, the form of the Green function depends on the filling. We enumerate the possibilities here:

1. At quarter-filling, as analyzed above, either spin of the two degenerate ε_- bands is occupied, thus $\dim P_G = 2$ and it projects onto the ε_- bands

$$\begin{aligned}
P_G h &= P_G V\,\mathrm{diag}(\varepsilon_G,\varepsilon_G,\varepsilon_E,\varepsilon_E)V^\dagger \\
&= P_G V\,\mathrm{diag}(\varepsilon_G,\varepsilon_G,0,0)V^\dagger.
\end{aligned} \tag{23}$$

   The ε_+ bands remain empty for both spin, $\dim P_E = 2$ and it projects onto the ε_+ bands.

$$\begin{aligned}
P_E h &= P_E V\,\mathrm{diag}(\varepsilon_G,\varepsilon_G,\varepsilon_E,\varepsilon_E)V^\dagger \\
&= P_E V\,\mathrm{diag}(0,0,\varepsilon_E,\varepsilon_E)V^\dagger.
\end{aligned} \tag{24}$$

   The exact Green function in the original basis is obtained by performing the unitary transform on Eq. (7)

$$G(\mathbf{k},\omega) = V\,\mathrm{diag}(G_{-,\mathbf{k},\uparrow}, G_{-,\mathbf{k},\downarrow}, G_{+,\mathbf{k},\uparrow}, G_{+,\mathbf{k},\downarrow})V^\dagger. \tag{25}$$

   The projection operator P_G (P_E) thus projects onto the first two (last two) elements of the diagonal matrix, leaving

$$\begin{aligned}
P_G G(\mathbf{k},\omega) &= V\,\mathrm{diag}(G_{-,\mathbf{k},\uparrow}, G_{-,\mathbf{k},\downarrow}, 0, 0)V^\dagger \\
&= G_{-,\mathbf{k},\sigma}P_G,
\end{aligned} \tag{26}$$

$$\begin{aligned}
P_E G(\mathbf{k},\omega) &= V\,\mathrm{diag}(0, 0, G_{+,\mathbf{k},\uparrow}, G_{+,\mathbf{k},\downarrow})V^\dagger \\
&= G_{+,\mathbf{k},\sigma}P_E.
\end{aligned} \tag{27}$$

   Thus the exact Green function at quarter-filling in the orbital basis could be written as

$$\begin{aligned}
G(\mathbf{k},\omega) = &\left(\frac{\frac{1}{2}}{\omega+\mu-\varepsilon_G} + \frac{\frac{1}{2}}{\omega+\mu-U-\varepsilon_G}\right)P_G \\
&+ \frac{1}{\omega+\mu-\varepsilon_E}P_E.
\end{aligned} \tag{28}$$

2. At half-filling, since U is much greater than the bandwidth, the ground state always singly occupies both $\varepsilon_\pm$ bands, this means that both ε_+ and ε_- are flattened to ε_G. This flattening process does not affect the Hall conductance even if zero bands are crossing the chemical potential. P_G projects onto all four bands and $P_E = 0$ vanishes. The exact Green function in the original basis reads

$$G(\mathbf{k},\omega) = \left(\frac{\frac{1}{2}}{\omega+\mu-\varepsilon_G} + \frac{\frac{1}{2}}{\omega+\mu-2U-\varepsilon_G}\right)P_G. \tag{29}$$

With these projection operators, the current-current correlator becomes

$$R_{\alpha\beta}(q, i\nu_r) = \frac{1}{V\beta}\sum_{k,n}(\varepsilon_G - \varepsilon_E)^2 \\ \mathrm{Tr}\left[\frac{\partial P_G}{\partial k_\alpha} G(k+q/2, \omega_n)\frac{\partial P_G}{\partial k_\beta} G(k-q/2, \omega_n - \nu_r)\right]. \tag{30}$$

One quick inspection of this expression tells us that at half-filling, this response function has to be 0. The derivative on P_G always vanishes as $P_G = I$. Tuning the chemical potential without crossing the poles does not break this 0 value. The Hall conductance at half-filling configuration shall always vanish.

At quarter filling, we adopt the projection operator identities

$$(\partial_\alpha P_G)P_G(\partial_\beta P_G)P_G = (\partial_\alpha P_G)P_E(\partial_\beta P_G)P_E = 0, \tag{31}$$
$$(\partial_\alpha P_G)P_G(\partial_\beta P_G)P_E = (\partial_\alpha P_G)(\partial_\beta P_G)P_E, \tag{32}$$
$$(\partial_\alpha P_G)P_E(\partial_\beta P_G)P_G = (\partial_\alpha P_G)(\partial_\beta P_G)P_G. \tag{33}$$

Together with the contour integration method to perform the summation over the Matsubara frequencies ω_n, we find that

$$R_{\alpha\beta}(i\nu_r) = \frac{1}{2V}\sum_{k}(\varepsilon_G - \varepsilon_E)^2 \\ \mathrm{Tr}\left[\frac{(\partial_\alpha P_G)(\partial_\beta P_G)P_E}{\varepsilon_G - \varepsilon_E - i\nu_r} + \frac{(\partial_\alpha P_G)(\partial_\beta P_G)P_G}{\varepsilon_G - \varepsilon_E + i\nu_r}\right]. \tag{34}$$

The dependence on U is fully removed since the poles at $\varepsilon_G + U$ always lie on the same side as ε_E relative to the chemical potential. We may complete the integral in the lower half plane without enclosing any poles. Taking the antisymmetric part between α and β and performing the analytic continuation on Matsubara frequencies $i\nu_r \to \omega$ leads to

$$R_{\alpha\beta}(\omega \to 0) = \frac{1}{2V}\sum_{k,n}(\varepsilon_G - \varepsilon_E)^2 \mathrm{Tr}\left[\frac{2\omega(\partial_\alpha P_G)(\partial_\beta P_G)P_G}{(\varepsilon_G - \varepsilon_E)^2 - \omega^2}\right] \\ = \frac{\omega}{V}\sum_{k,n}\mathrm{Tr}\left[(\partial_\alpha P_G)(\partial_\beta P_G)P_G\right]. \tag{35}$$

This response function is exactly half of the non-interacting value due to the halved weight in the non-interacting Green function. By substituting the wave function formalism for the projection operator $P_G = \sum_{i\in G}|i,k\rangle\langle i,k|$, we conclude that the Hall conductance, in units of $\frac{e^2}{h}$, is half of the Berry curvature of the filled non-interacting band, which means $C_1 = \frac{1}{2}C_1^{\text{non-interacting}} = 1$.

Appendix: Ferromagnetic ground state

The ground state of the HK model is known to possess a large degeneracy due to the spin degrees of freedom[12, 15]. This degeneracy can be removed by applying an infinitesimal Zeeman field that picks one certain direction for the ferromagnetic ground state[9, 13, 14]. By applying an infinitesimal magnetic field along the z-direction, the Green function is modified at all energies. The locations of the poles do not move, but the weight of the Hubbard band poles is now unity, and the bands are fully spin-polarized. At quarter- and half-filling, the system remains insulating, and the exact Green function in the band basis [30]

$$G_{\pm,\mathbf{k},\uparrow}(\omega) = \frac{1}{\omega + \mu - \varepsilon_\pm}, \tag{36}$$
$$G_{\pm,\mathbf{k},\downarrow}(\omega) = \frac{1}{\omega + \mu - \langle n_{\mathbf{k}\uparrow}\rangle U - \varepsilon_\pm}, \tag{37}$$

where $n_{\mathbf{k}\uparrow} = n_{+,\mathbf{k}\uparrow} + n_{-,\mathbf{k}\uparrow}$ is the total number of filled electrons in the spin-up bands. The system is now equivalent to two separate QAH systems. In the limit of $U \gg W$, only the spin-up bands will be occupied. Thus the ground state is a spin-polarized QAH state. The Hall conductance is unchanged since the Zeeman field is an infinitesimal perturbation. However, since the Zeeman field removes the zeros, it drastically changes N_3 even for values of the chemical potential far from the zero bands. At quarter filling, N_3 goes from 2 to 1, while at half filling $N_3 = 0$ for all values of the chemical potential. This shows that N_3 is not a property of the ground state manifold but instead depends on the properties of the Green function at all energies. This is in contrast to the Hall conductance which is related to a zero frequency response function.

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