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Kondo Lattice Model of Magic-Angle Twisted-Bilayer Graphene: Hund's Rule, Local-Moment Fluctuations, and Low-Energy Effective Theory

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We apply a generalized Schrieffer–Wolff transformation to the extended Anderson-like topological heavy fermion (THF) model for the magic-angle ($\theta = 1.05^\circ$) twisted bilayer graphene (MATBLG) (Phys. Rev. Lett. **129**, 047601 (2022)), to obtain its Kondo Lattice limit. In this limit localized f-electrons on a triangular lattice interact with topological conduction c-electrons. By solving the exact limit of the THF model, we show that the integer fillings $\nu = 0, \pm1, \pm2$ are controlled by the heavy f-electrons, while $\nu = \pm3$ is at the border of a phase transition between two f-electron fillings. For $\nu = 0, \pm1, \pm2$, we then calculate the RKKY interactions between the f-moments in the full model and analytically prove the SU(4) Hund's rule for the ground state which maintains that two f-electrons fill the same valley-spin flavor. Our (ferromagnetic interactions in the) spin model dramatically differ from the usual Heisenberg antiferromagnetic interactions expected at strong coupling. We show the ground state in some limits can be found exactly by employing a positive semidefinite "bond-operators" method. We then compute the excitation spectrum of the f-moments in the ordered ground state, prove the stability of the ground state favored by RKKY interactions, and discuss the properties of the Goldstone modes, the (reason for the accidental) degeneracy of (some of) the excitation modes, and the physics of their phase stiffness. We develop a low-energy effective theory for the f-moments and obtain analytic expressions for the dispersion of the collective modes. We discuss the relevance of our results to the spin-entropy experiments in TBG.

Introduction— The discovery of the correlated insulating phase [1] and superconductivity [2] in the MATBLG [3] has driven considerable theoretical [4–22] and experimental efforts [23–46] to understand its topology [47–63] and correlation physics [47, 62–78]. Theoretically, correlated insulators [79–90], ferromagnetic order [87, 91–94], superconductivity [83, 95–110], and other exotic quantum phases [111–118] have been identified and systematically studied which all point to rich physics [62] of the MATBLG. The recent experiments [30, 66, 119, 120] have provided evidence for fluctuating local moments and Hubbard-like physics. Meanwhile the theoretical understanding is challenging since the stable topology [54, 59] of the flat bands obstructs the symmetric real-space description. A real-space extended Hubbard model [70, 72, 121–123] can still be constructed, but a certain symmetry ($C_{2z}T$ or P) becomes non-local. To address this problem the authors of Ref. [124] have introduced an exact mapping of MATBG to a THF model. THF is a version of the extended Anderson lattice model describing localized f-electrons interacting with topological conduction c-electrons. The f-electrons have zero kinetic energy and strong Hubbard interactions; they admit a description in terms of localized Wannier orbitals centered at the AA-stacking region. The topological flat bands can be recovered from the hybridization between the f- and the c-electrons [124].

In this letter, we map the THF model to a Kondo lattice model using a generalized Schrieffer-Wolff (SW) transformation which takes into account the density-density interaction term between f- and c-electrons. In this limit, the dynamics of the localized orbitals becomes the one of the f-moments. By solving exactly a particular limit of the THF model, we show that the integer fillings $\nu = 0, \pm1, \pm2$ are controlled by the f-electrons, while the situation is different for $\nu = \pm3$ which sits at the phase transition between two f-electrons fillings. In the Kondo lattice model of MATBLG, the local moments formed by the localized f-electrons interact with topological conduction electron bands via Kondo and direct ferromagnetic exchange interaction. These two types of interactions induce an RKKY interaction. At $\nu = 0, -1, -2$, the RKKY interaction dominates the physics and stabilizes the ferromagnetic ground states obeying Hund's rule. This result provides an analytic derivation of the Hund's rule found in the Hartree-Fock calculations [124]. We proceed to investigate fluctuations of the f-moments in the symmetry broken ground states by developing the low-energy effective theory and calculating the excitation spectrum.

Schrieffer–Wolff transformation and Kondo lattice model— The single-particle Hamiltonian of the THF model contains the kinetic term $\hat{H}_c$ describing the topological conduction c-electron bands (SM [125], Sec. I), and the hybridization between the f- and the c-electrons $\hat{H}_{fc}$ [124, 125]:

$$\hat{H}_{fc} = \sum_{\substack{|\mathbf{k}|<\Lambda_c,\mathbf{R} \\ i,\xi,\xi'}} \left(\frac{e^{i\mathbf{k}\cdot\mathbf{R} - \frac{|\mathbf{k}|^2\lambda^2}{2}} \tilde{H}^{(fc)}_{\xi\xi'}(\mathbf{k})}{\sqrt{N_M}} \psi^{f,\xi,\dagger}_{\mathbf{R},i} \psi^{c',\xi'}_{\mathbf{k},i} + \text{h.c.} \right)$$

$$\tilde{H}^{(fc)}(\mathbf{k}) = \begin{bmatrix} \gamma & v'_\star(k_x - ik_y) \\ v'_\star(k_x + ik_y) & \gamma \end{bmatrix}. \quad (1)$$

where $\psi^{c',\xi,\dagger}_{\mathbf{k},i}$ creates Γ_3 "conduction" c-electron with momentum $\mathbf{k}$, valley-spin flavor $i \in \{1,2,3,4\}$ (with $(1,2,3,4)$ corresponding to $(+\uparrow, -\uparrow, +\downarrow, -\downarrow)$), and "orbital" index $\xi = (-1)^{a+1}\eta$ (with $a = 1, 2$ are original orbital indices and

$\eta = \pm$ are valley indices defined in Ref. [124]). $\psi^{f,\xi,\dagger}_{\mathbf{R},i}$ creates f-electron at moiré unit cell $\mathbf{R}$ with valley-spin flavor i, and orbital index $\xi = (-1)^{a+1}\eta$, where $a = 1, 2, \eta = \pm$ [124]. Λ_c is the momentum cutoff, N_M is the total number of moiré unit cells and λ is the damping factor [124]. In the hybridization matrix $\tilde{H}^{(fc)}(\mathbf{k})$, we keep the first two terms (γ and $v'_\star$) in the expansion in powers of $\mathbf{k}$. The flat band limit is realized at $M = 0$, where M is taken as a parameter of $\hat{H}_c$.

The interaction Hamiltonian of the THF model is $\hat{H}_I = \hat{H}_U + \hat{H}_J + \hat{H}_V + \hat{H}_W$, where $\hat{H}_U, \hat{H}_J$ describe respectively the on-site Hubbard interaction of the f-electrons ($U = 57.95$meV), and the ferromagnetic exchange between the f- and the c-electrons ($J = 16.38$meV), $\hat{H}_V, \hat{H}_W$ describe respectively the repulsion between the c-electrons (~ 48meV) and the repulsion between the f- and the c-electrons ($W = 47$meV) [124, 126]. The full Hamiltonian is $\hat{H}_c + \hat{H}_{fc} + \hat{H}_I$. The parameter values are taken from Ref. [124]. The model possesses the $U(4) \times U(4)$ symmetry in the chiral-flat limit ($M = 0$, $v'_\star = 0$), a flat $U(4)$ symmetry in the nonchiral-flat limit ($M = 0, v'_\star \neq 0$), a chiral $U(4)$ symmetry in the chiral-nonflat limit ($M \neq 0, v'_\star = 0$) and a $U(2) \times U(2)$ symmetry in the nonchiral-nonflat limit ($M \neq 0, v'_\star \neq 0$) [4, 87, 121, 124, 125, 127, 128].

At sufficiently strong on-site Coulomb interaction U, the f-electrons are fully localized and give rise to local f-moments which are defined as

$$\hat{\Sigma}^{(f,\xi\xi')}_{\mu\nu}(\mathbf{R}) = \sum_{i,j} \frac{1}{2} T^{\mu\nu}_{ij} \psi^{f,\xi,\dagger}_{\mathbf{R},i} \psi^{f,\xi'}_{\mathbf{R},j}\,. \tag{2}$$

$\{T^{\mu\nu}_{ij}\}$ with $\mu, \nu \in \{0, x, y, z\}$ are given in SM [125], Sec. I. Eq. 2 are the generators of the $U(8)$ group ($8 = 2$(orbital) $\times$ 2(valley) $\times$ 2(spin)) where the $U(1)$ charge component can be gauged away for the fully-localized f-electrons.

We first analyze the zero-hybridization limit of the model ($\gamma = 0, v'_\star = 0$). We treat $\hat{H}_V$ in the mean-field approximation ($\hat{H}^{MF}_V$) and drop the $\hat{H}_J$ which is relatively weak. We can then solve the model exactly under the assumption that each site is filled with $\nu_f + 4$ f-electrons with an integer ν_f (SM. [125], Sec. II). We use ν_c to denote the filling of the c-electrons and use $\nu = \nu_f + \nu_c$ to denote the total fillings. In Fig. 1 (a), we plot ν_f and ν_c of the ground state as a function of ν. At $\nu = 0, -1, -2, -3$, the ground state has $\nu_f = \nu$ and $\nu_c = 0$. $\nu = -3$ is close to the transition point between $\nu_f = -3$ state and $\nu_f = -2$ states. Thus, at $\nu = -3$, our assumption of uniform charge distribution may be violated [117], and a Kondo model description fails. This is consistent with the special place that $\nu = -3$ has in the TBG physics [117].

At $\nu = 0, -1, -2$, we fix the filling of the f-electrons (according to Fig. 1 (a)) and perform a generalized SW transformation (SM. [125], Sec. IV), which leads to the following Kondo lattice Hamiltonian:

$$\hat{H}_{Kondo} = \hat{H}_c + \hat{H}_{cc} + \hat{H}_K + \hat{H}_J + \hat{H}^{MF}_V + \hat{H}_W \tag{3}$$

where $\hat{H}_K$ and $\hat{H}_{cc}$ are the Kondo interaction and one-body

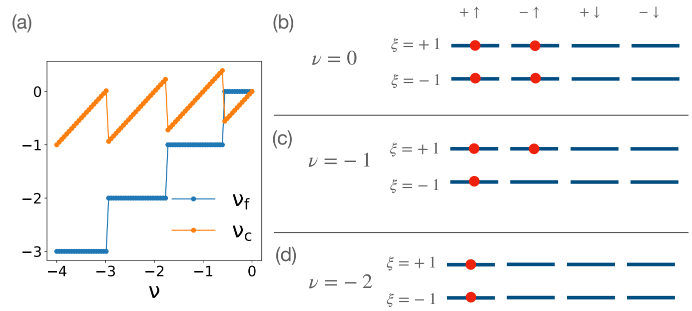

FIG. 1. (a) Filling of f electrons(ν_f) and c-electrons(ν_c) as a function of total filling(ν) in the zero hybridization model. Three transition points are $\nu \sim -0.59, -1.73, -2.95$ (b), (c), (d) Illustrations of ground states at $\nu = 0, -1, -2$. The red dot means the filling of one f-electron.

scattering term generated by the SW transformation respectively [125, 129]. The Kondo interaction $\hat{H}_K$ takes the form of

$$\hat{H}_K = \sum_{\mathbf{R}} \sum_{|\mathbf{k}|<\Lambda_c, |\mathbf{k}+\mathbf{q}|<\Lambda_c} \sum_{\mu\nu\xi\xi'} \frac{e^{-i\mathbf{q}\cdot\mathbf{R}} e^{-\frac{|\mathbf{k}|^2+|\mathbf{k}+\mathbf{q}|^2}{2}\lambda^2}}{N_M}$$
$$\Bigg[\frac{\gamma^2}{D_{\nu_c,\nu_f}} : \hat{\Sigma}^{(f,\xi\xi')}_{\mu\nu}(\mathbf{R}) :: \hat{\Sigma}^{(c',\xi'\xi)}_{\mu\nu}(\mathbf{k},\mathbf{q}) : + \tag{4}$$
$$\left(\frac{v'_\star\gamma(k_x - i\xi k_y)}{D_{\nu_c,\nu_f}} : \hat{\Sigma}^{(f,\xi\xi')}_{\mu\nu}(\mathbf{R}) :: \hat{\Sigma}^{(c',\xi'-\xi)}_{\mu\nu}(\mathbf{k},\mathbf{q}) : +\text{h.c.}\right)\Bigg]$$

where the colon symbols represent the normal ordering and $\Sigma^{(c',\xi'\xi)}_{\mu\nu}(\mathbf{k},\mathbf{q})$ (SM [125], Sec. I) is the $U(8)$ moment of c-electrons defined in the manner similar to $\Sigma^{(f,\xi'\xi)}_{\mu\nu}(\mathbf{R})$ in Eq. 2. The parameter D_{ν_c,ν_f} at $\nu = \nu_f = 0, -1, -2$ is defined as

$$\frac{1}{D_{\nu_c=0,\nu_f}} = -\frac{1}{(U-W)\nu_f - \frac{U}{2}} + \frac{1}{(U-W)\nu_f + \frac{U}{2}}\,. \tag{5}$$

The distinct feature of this expression is the presence W absent in the standard Kondo Hamiltonians. In addition, we perform a $\mathbf{k}$-expansion in the square bracket of Eq. 4, and keep only the zeroth and the linear order terms in $\mathbf{k}$; as was done in the expression for the hybridization matrix $\tilde{H}^{(fc,\eta)}(\mathbf{k})$ (Eq. 1). The zeroth order Kondo coupling has strength $\gamma^2/D_{\nu_c=0,\nu_f} = 42.3\text{meV}, 49.3\text{meV}, 98.6\text{meV}$ at $\nu = 0, -1, -2$ respectively [124, 125]. Since $\nu = -2$ is closer to the transition point (Fig. 1), we observe a larger Kondo coupling at $\nu = -2$.

The RKKY interactions and the Hund's rule— By integrating out the c-electrons in the Kondo Hamiltonian(Eq. 3), one induces an RKKY interaction [130–132] between the f-moments (details in SM [125]), Sec. V). In the chiral-flat limit ($v'_\star = 0, M = 0$), the RKKY interactions can be described by

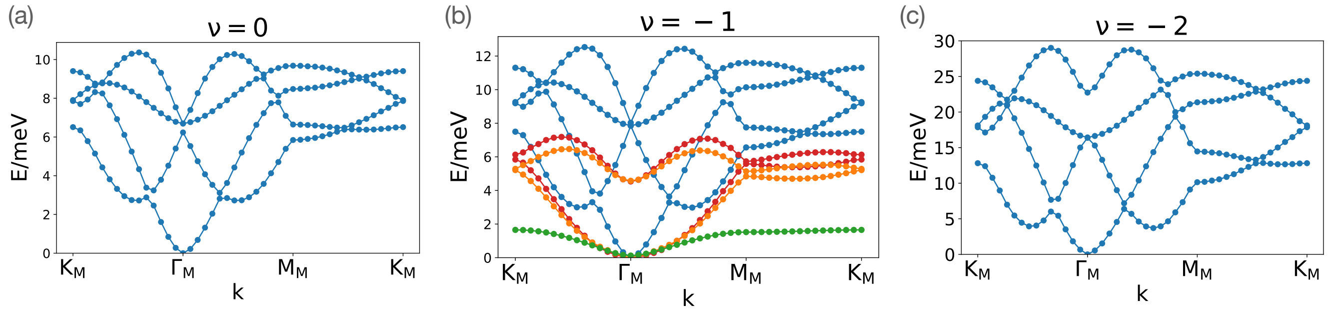

FIG. 2. Excitation spectrum at $\nu = 0, -1, -2$. Blue, orange, red and green denote the fluctuations in the full-empty sector, half-empty sector, full-half sector and half-half sector respectively.

the following $U(4) \times U(4)$ symmetric Hamiltonian

$$\hat{H}^{v'_\star=0,M=0}_{RKKY} = \sum_{\substack{\mathbf{R},\mathbf{R}'\\ \mu\nu,\xi\xi'}} [J_0^{RKKY}(\mathbf{R}') + J_1^{RKKY}(\mathbf{R}')\delta_{\xi,\xi'}] \\ :\Sigma^{(f,\xi\xi')}_{\mu\nu}(\mathbf{R}) :: \Sigma^{(f,\xi'\xi)}_{\mu\nu}(\mathbf{R}+\mathbf{R}'):, \quad (6)$$

where both $J_0^{RKKY}(\mathbf{R}')(\leq 0)$ and $J_1^{RKKY}(\mathbf{R}')(\leq 0)$ can be analytically obtained and are ferromagnetic (SM [125], Sec.V). Using bond operators $A^{\xi,\xi'}_{\mathbf{R},\mathbf{R}'} = \sum_i \psi^{f,\xi,\dagger}_{\mathbf{R},i}\psi^{f,\xi'}_{\mathbf{R}',i}$, we show that $\hat{H}^{v'_\star=0,M=0}_{RKKY}$ is a Positive Semidefinite Hamiltonian [4, 70], where the exact ground state can be obtained [4, 70, 87, 125]. The grounds states are ferromagnetic with the form of (SM [125], Sec. VI)

$$\prod_{\mathbf{R}} \left[\prod_{n=1}^{\nu_f^+ +2} \psi^{f,+,\dagger}_{\mathbf{R},i_n} \prod_{m=\nu_f^+ +3}^{\nu_f+4} \psi^{f,-,\dagger}_{\mathbf{R},i_m} \right] |0\rangle \,. \quad (7)$$

where ν_f^+ denotes the filling of the f-electrons with index $\xi = 1$, and $\nu_f^+ = 0$ at $\nu = 0$, $\nu_f^+ = -1, 0$ at $\nu = -1$, and $\nu_f^+ = -1$ at $\nu = -2$. $\{i_n\}$ are chosen arbitrarily and $|0\rangle$ is the vacuum with $\psi^{f,\xi}_{\mathbf{R},i}|0\rangle = 0$.

We then consider the nonchiral-flat limit ($v'_\star \neq 0$, $M = 0$), where the RKKY interactions are flat-$U(4)$ symmetric. They lift the ground-state degeneracy in Eq. 7. We obtain the ground states by treating $v'_\star$-induced RKKY interactions as perturbations (SM [125], Sec. VI) , and the true ground state is selected by the following RKKY interactions

$$\sum_{\substack{\mathbf{R},\mathbf{R}'\\ \mu\nu,\xi}} J_2^{RKKY}(\mathbf{R}') :\hat{\Sigma}^{(f,\xi\xi)}_{\mu\nu}(\mathbf{R}) :: \hat{\Sigma}^{(f,-\xi-\xi)}_{\mu\nu}(\mathbf{R}+\mathbf{R}'): \quad (8)$$

where $J_2^{RKKY}(\mathbf{R})(\leq 0)$ is analytically obtained and ferromagnetic (SM [125], Sec. V). $J_2^{RKKY}(\mathbf{R}_2 - \mathbf{R})$ tends to align two f-moments $:\hat{\Sigma}^{(f,++)}_{\mu\nu}(\mathbf{R}):$ and $:\hat{\Sigma}^{(f,--)}_{\mu\nu}(\mathbf{R}_2):$ and stabilize the ground states obeying the Hund's rule (see Fig. 1). The corresponding ground states are consistent with Ref. [70, 87] (nonchiral-flat limit).

Moreover, we also derive the RKKY interactions in the zero-hybridization limit ($\gamma = 0, v'_\star = 0$) with non-zero $J(\neq 0)$ (SM [125], Sec. III), where the corresponding ground states are consistent with Ref. [87] (chiral-nonflat limit).

In summary, the ground states, that we derived unbiasedly from the RKKY interactions, spontaneously break the flat-$U(4)$ symmetry. The long-range ordered ground state is an unavoidable consequence of the absence of Kondo screening at $\nu = 0, -1, -2$ (see also Ref. [133]). Finally, we mention that, to further distinguish the ground states within the same flat-$U(4)$ manifold, one needs to include the small flat-$U(4)$ breaking term ($M \neq 0$).

Fluctuations of the f-moments— We check the stability of the ferromagnetic ground state derived from RKKY Hamiltonian by studying small fluctuations (details in SM [125], Sec.VIII). We restrict ourselves to the flat-$U(4)$ nonchiral-flat limit $M = 0, v'_\star \neq 0$ at $\nu = 0, -1, -2$. To describe the fluctuations we introduce for each site a 8×8 traceless Hermitian matrix $u_{i\xi,j\xi'}(\mathbf{R})$, where $i, j \in \{1, 2, 3, 4\}$ are valley-spin flavors and $\xi, \xi' \in \{+, -\}$. The f-moments in Eq. 2 can then be written as (SM [125], Sec. VIII).

$$\hat{\Sigma}^{(f,\xi\xi')}_{\mu\nu}(\mathbf{R}) = \sum_{ij} T^{\mu\nu}_{ij} [e^{iu(\mathbf{R})}\Lambda e^{-iu(\mathbf{R})}]_{i\xi,j\xi'} \quad (9)$$

where Λ is an 8×8 matrix defined as $\Lambda_{i\xi,j\xi'} = \langle\psi_0| : \psi^{f,\xi,\dagger}_{\mathbf{R},i}\psi^{f,\xi'}_{\mathbf{R},j} : |\psi_0\rangle/2$ and $|\psi_0\rangle$ is the ground state. A non-zero $u_{i\xi,j\xi'}(\mathbf{R})$ will generate fluctuations by rotating the f-moments from their ground-state expectation values.

The ferromagnetic order in the ground state opens a gap in the single-particle spectrum of the c-electrons which allows us to safely integrate them out [125] and to develop an effective theory for small fluctuations by expanding the action to the second order in $u_{i\xi,j\xi'}(\mathbf{R}, \tau)$, where τ is the imaginary time (SM [125], Sec. VIII). The Lagrangian of the effective theory is provided in SM [125], Sec. VIII, where we find the diagonal components $u_{i\xi,i\xi}(\mathbf{R}, \tau)$ only contribute a total derivative and we will focus on the off-diagonal components: $u_{i\xi,j\xi'}(\mathbf{R}, \tau)$ with $i\xi \neq j\xi'$.

We then introduce two sets S_{fill} and S_{emp} to characterize the ground state, where S_{fill} and S_{emp} denote the sets of

$i\xi$ indices that are filled with one and zero number of the f-electrons at each site, respectively. A fluctuation generated by $u_{i\xi,j\xi'}(\mathbf{R})$ $(i\xi \neq j\xi')$ is described by the procedure of moving one f-electron from $j\xi'$ flavor at site $\mathbf{R}$ to $i\xi$ flavor at the same site (SM [125], Sec. VIII). This procedure can only be valid when $i\xi \in S_{emp}, j\xi' \in S_{fill}$. Consequently, only $u_{i\xi,j\xi'}(\mathbf{R},\tau)$ with $j\xi' \in S_{fill}, i\xi \in S_{emp}$ and also its complex conjugate $u_{j\xi',i\xi}(\mathbf{R},\tau)$ that describes the reverse procedure appear in our effective Lagrangian. We diagonalize the Lagrangian and plot the excitation spectrum in Fig. 2.

We first analyze the spectrum at $\nu = 0, -2$. The Lagrangian density in the long-wavelength limit and in momentum $(\mathbf{k})$ and frequency (ω) spaces has the form of $L = L_{Goldstone} + L_{gapped}$:

$$L_{Goldstone} = \sum_{\substack{j\xi\in S_{fill}\\ i\xi\in S_{emp}}} u^{\dagger}_{j,i}(\mathbf{k},\omega)\Big(\omega - \frac{k^2}{2m_0}\Big)u_{j,i}(\mathbf{k},\omega),$$

$$L_{gapped} = \sum_{\substack{j\xi\in S_{fill}\\ i\xi\in S_{emp}}} U^{\dagger}_{j,i}(\mathbf{k},\omega)[\omega\hat{I} - \hat{H}(\mathbf{k})]U_{j,i}(\mathbf{k},\omega),$$

$$H(\mathbf{k}) = \begin{pmatrix} \frac{k_+k_-}{2m_1}+\Delta_1 & Vk_+ & -Vk_- \\ Vk_- & \frac{k_+k_-}{2m_2}+\Delta_2 & \frac{k_-^2}{2m_3} \\ -Vk_+ & \frac{k_+^2}{2m_3} & \frac{k_+k_-}{2m_2}+\Delta_2 \end{pmatrix} \tag{10}$$

where $u_{j,i} = (u_{(j+,i+)} + u_{(j-,i-)})/\sqrt{2}$, and $U^T_{j,i} = ((u_{(j+,i+)} - u_{(j-,i-)})/\sqrt{2}, u_{(j+,i-)}, u_{(j-,i+)})$ and $k_\pm = k_x \pm ik_y$, $k = |\mathbf{k}|$. $m_0, m_1, m_2, V, \Delta_1, \Delta_2, V$ depend on the parameters of the THF Hamiltonian (SM [125], Sec. VIII). The Goldstone modes with quadratic dispersion decouple from the rest (gapped modes).

The gaps at $\mathbf{k} = 0$ are Δ_1 and Δ_2, with Δ_2 corresponding to two-fold degenerate modes. This feature is reproduced in Fig. 2 (c). The exceptional case when all three modes are degenerate $(\Delta_1 = \Delta_2)$ is depicted in Fig. 2 (a) and the condition for the degeneracy is $\alpha = [-0.37 + \sqrt{0.14 + 0.23(v'_\star)^2/(\gamma\lambda)^2}]\gamma^2/(JD_{\nu_c\nu_f}) = 1$. Using the realistic values of parameters, we find $\alpha = 1.07 \approx 1$. By directly evaluating the Lagrangian, we also observe "roton" minima in the gapped modes (most obviously at $\nu = -2$) at $|\mathbf{k}| \sim 0.3|\boldsymbol{b}_{M1}|$ with $\boldsymbol{b}_{M1}$ the moiré reciprocal lattice vector.

We next discuss the number of the Goldstone modes. Each combination of (i,j) that satisfies $i+, i- \in S_{emp}$, $j+, j- \in S_{fill}$ produce a Goldstone mode [125]. This leads to four Goldstone modes at $\nu = 0$ and three Goldstone modes at $\nu = -2$ [88]. Furthermore, all excitation modes depicted in Fig. 2 are four-fold degenerate at $\nu = 0$ and three-fold degenerate at $\nu = -2$, due to the remaining $U(2)\times U(2)$ symmetries at $\nu = 0$ and the remaining $U(1)\times U(3)$ symmetries at $\nu = -2$.

We now analyze the spectrum at $\nu = -1$. Unlike $\nu = 0, -2$ where each valley-spin flavor is filled with either two or zero f- electrons, at $\nu = -1$, there is one valley-spin flavor (denoted by $i = 1$) filled with two f-electrons and one valley-spin flavor (denoted by $i = 2$) filled with one f-electron and two empty valley-spin flavors (denoted by $i = 3, 4$) as shown in Fig. 1 (c). This allows us to classify $u_{i\xi,j\xi'}(\mathbf{R},\tau)$ at $\nu = -1$ into four sectors: (1) full-empty sector with $i = 3, 4, j = 1$ or $i = 1, j = 3, 4$; (2) half-empty sector with $i = 2, j = 3, 4$ or $i = 3, 4, j = 2$; (3) full-half sector with $i = 1, j = 2$ or $j = 2, i = 1$; (4) half-half sector with $i = 2, j = 2$. In Fig. 2, we label the excitation in different sectors with different colors. Due to the remaining $U(1)\times U(1)\times U(2)$ symmetry of the $\nu = -1$ ground state , each mode is two-fold degenerate in the full-empty and half-empty sectors and is non-degenerate in the full-half and half-half sectors. We find 2 degenerate Goldstone modes with stiffness $6.7\mathrm{meV}\cdot a_M^2$ in the full-empty sector, 2 degenerate Goldstone modes with stiffness $1.3\mathrm{meV}\cdot a_M^2$ in the half-empty sector, and 1 Goldstone mode with stiffness $1.8\mathrm{meV}\cdot a_M^2$ in the full-half sector [88].

Several remarks are in order. Firstly, some of the gapped modes at $\nu = 0, -1, -2$ are relatively flat as shown in Fig. 2. Secondly, at $\nu = -1$, one of the flat modes (green curve in Fig. 2 (b)) with eigenfunction $u_{2-,2+}(\mathbf{k})$ has a tiny gap $0.12\mathrm{meV}$ at Γ_M point and a very narrow bandwidth $1.5\mathrm{meV}$.

Summary and discussions— We have constructed and studied a Kondo lattice model for MATBLG. Its distinct feature is the Dirac character of the c-electron spectrum: at integer fillings, there is no Fermi surface. Hence the Kondo screening is irrelevant and the low energy physics is dominated by the RKKY interactions, which is also responsible for the Hund's rule and ferromagnetism of the ground states. Moreover, we have developed an effective theory describing small fluctuations of the local moments and found their excitation spectrum. We discuss the properties of the Goldstone and gapped modes. Unlike previous studies that mostly rely on numerical calculations, we are able to provide a more analytical understanding of the ground states and local-moment fluctuations here. We believe that our work provides insight into the correlated ground states and the local-moment fluctuations of the MATBLG.

We also comment on the connection with previous works [88, 134]. Some features, including the soft modes and accidental degeneracy of gapped modes at Γ_M, also appear in the projected Coulomb model [88], but the roton modes do not. We note that the projected Coulomb model [88] has ignored the effect of remote bands.

Roton modes have also been seen in Ref. [134], where the remote bands have been included. However, our model gives a larger bandwidth of the gapped modes than Ref. [134]. We point out that, we take a different set of parameter values (including dielectric constant, and gate distance). Besides, in our model, all Goldstone modes have quadratic dispersions, in contrast to Ref. [134] which found a linear one. The difference in the results has its origin in the different symmetry properties. We consider the flat limit of the model, and the quadratic dispersion comes from the broken flat $U(4)$ symmetry. Ref. [134] takes non-flat bands, and the linear Goldstone modes come from the broken $U(1)$ valley symmetry. However, the flat $U(4)$ symmetry is only weakly broken due to the small bandwidth of the flat bands, which makes it hard to detect this symmetry-breaking effect experimentally.

We conclude the paper with a discussion of the relevance of our results to the recent entropy experiments [119, 120]. Experimentally, the entropy in TBG near $\nu = -1$ has been found to be of the order of Boltzmann's constant and is suppressed by the applications of magnetic fields. This large entropy can be explained by the presence of soft mode at $\nu = -1$, which will be suppressed by the magnetic field. Finally, the fluctuations of the f-moments could potentially generate attractive interactions between c-electrons and drive the system to the superconducting phase. Thus, our work also establishes a platform for understanding the superconductivity.

Note added— After finishing this work, we have learned that related, but not identical, results had recently been obtained by the S. Das Sarma's [135], P. Coleman's [136] and Z. Song's groups [137].

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