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Generic nonadditivity of quantum capacity in simple channels

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Determining capacities of quantum channels is a fundamental question in quantum information theory. Despite having rigorous coding theorems quantifying the flow of information across quantum channels, their capacities are poorly understood due to super-additivity effects. Studying these phenomena is important for deepening our understanding of quantum information, yet simple and clean examples of super-additive channels are scarce. Here we study a family of channels called platypus channels. Its simplest member, a qutrit channel, is shown to display super-additivity of coherent information when used jointly with a variety of qubit channels. Higher-dimensional family members display super-additivity of quantum capacity together with an erasure channel. Subject to the "spin-alignment conjecture" introduced in the companion paper [1], our results on super-additivity of quantum capacity extend to lower-dimensional channels as well as larger parameter ranges. In particular, super-additivity occurs between two weakly additive channels each with large capacity on their own, in stark contrast to previous results. Remarkably, a single, novel transmission strategy achieves super-additivity in all examples. Our results show that super-additivity is much more prevalent than previously thought. It can occur across a wide variety of channels, even when both participating channels have large quantum capacity.

Introduction. A central aim of quantum information theory is to find out how much information a noisy quantum channel can transmit reliably—to find a quantum channel's capacity [2, 3]. In fact, a quantum channel has many capacities, depending on what sorts of information are to be transmitted and what additional resources are on hand. The primary capacities of a quantum channel are the classical [4–6], private [7–9], and quantum capacities [9–14]. This paper focuses on *unassisted* capacities, when no additional resources (such as free entanglement) are available.

The theory of quantum capacities is far richer and more complex than the corresponding classical theory [15, 16]. This richness includes many synergies and surprises: super-additivity of coherent information [17–31], private information [32–34], and Holevo information [35], super-activation of quantum capacity [36–40], and private communication at a rate above the quantum capacity [41, 42]. Over the past two decades, there have been numerous exciting discoveries about these phenomena, but they remain mysterious. As a result, we don't have a theory of how to best communicate with quantum channels, and can't answer many of the sorts of questions classical information theory does. For example, in quantum information theory random codes can be suboptimal, and we can only evaluate capacities in special cases [43–51]. Our understanding of error correction in the quantum setting is thus incomplete, whether the data is classical, private, or quantum.

Any quantum channel $\mathcal{B}$ can be expressed as an isometry $J\colon A \mapsto BE$ followed by a partial trace over the environment E: $\mathcal{B}(\rho) = \mathrm{Tr}_E(J\rho J^\dagger)$. Physically, it means that quantum noise arises from sharing the unclonable quantum data with the environment which is subsequently lost (i.e., traced out). Therefore, to understand quantum transmission we must also consider the environment's view of the channel, known as the complementary channel: $\mathcal{B}^c(\rho) = \mathrm{Tr}_B(J\rho J^\dagger)$. Together, the channel and its complement allow us to define the coherent information of a channel $\mathcal{B}$ on an input state ρ as $\Delta(\mathcal{B}, \rho) := S\big(\mathcal{B}(\rho)\big) - S\big(\mathcal{B}^c(\rho)\big)$, where $S(\sigma) = -\operatorname{tr}(\sigma \log \sigma)$ is the von Neumann entropy of σ. Mathematically, the coherent information signifies how much more information about the input is available in system B than in system E. Operationally, a random coding argument shows that indeed, for any input state ρ, the quantity $\Delta(\mathcal{B}, \rho)$ is an achievable rate for quantum transmission [9, 11–14]. Maximizing over all inputs ρ gives the channel coherent information $\mathcal{Q}^{(1)}(\mathcal{B})$.

If the channel coherent information is additive, that is, $\mathcal{Q}^{(1)}(\mathcal{B}_1 \otimes \mathcal{B}_2) = \mathcal{Q}^{(1)}(\mathcal{B}_1) + \mathcal{Q}^{(1)}(\mathcal{B}_2)$ for any two channels $\mathcal{B}_1$ and $\mathcal{B}_2$, then the theory of quantum capacity will resemble its classical analogue. However, a rich theory of quantum capacity originates from two distinct notions of nonadditivity: violations of *weak additivity* and violations of *strong additivity*.

We first discuss violations of *weak additivity*. The quantum capacity can be expressed as [9, 11–14, 52]

$$\mathcal{Q}(\mathcal{B}) = \lim_{n\to\infty} \frac{1}{n} \mathcal{Q}^{(1)}(\mathcal{B}^{\otimes n}), \tag{1}$$

where $\mathcal{B}^{\otimes n}$ is the n-fold tensor product of $\mathcal{B}$. If $\mathcal{Q}^{(1)}(\mathcal{B}^{\otimes n}) = n\mathcal{Q}^{(1)}(\mathcal{B})$ for all $n \in \mathbb{N}$, we say that $\mathcal{B}$ has weakly additive coherent information, in which case $\mathcal{Q}(\mathcal{B}) = \mathcal{Q}^{(1)}(\mathcal{B})$. However, there are channels $\mathcal{B}$ for which $\mathcal{Q}^{(1)}(\mathcal{B}^{\otimes n}) > n\mathcal{Q}^{(1)}(\mathcal{B})$ holds for some n [17–21, 23–26, 28–30]. Thus, the $n \to \infty$ limit is in general required in the above *regularized* expression for the quantum capacity. When a channel does *not* have weakly additive coherent information, special quantum codes can outperform the classical-inspired random coding strategy achieved by $\mathcal{Q}^{(1)}$. This unbounded optimization also means that we can rarely determine the quantum capacity of a quantum channel.

The second notion of nonadditivity, violations of strong additivity, can be phrased as follows. For two channels $\mathcal{B}_1$ and $\mathcal{B}_2$, we have the general inequality

$$\mathcal{Q}^{(1)}(\mathcal{B}_1 \otimes \mathcal{B}_2) \geq \mathcal{Q}^{(1)}(\mathcal{B}_1) + \mathcal{Q}^{(1)}(\mathcal{B}_2)\,. \tag{2}$$

Letting $\mathcal{B}_1$ be a fixed channel, if equality in (2) holds for all channels $\mathcal{B}_2$, we say that $\mathcal{B}_1$ has strongly additive coherent information. In this case, the quantum capacity satisfies $\mathcal{Q}(\mathcal{B}_1 \otimes \mathcal{B}_2) = \mathcal{Q}(\mathcal{B}_1) + \mathcal{Q}(\mathcal{B}_2)$. Note that strong additivity implies weak additivity. Violations of strong additivity imply that two different channels can have strictly superadditive coherent information, or even capacity. As a result, not only do we not know the capacity of most quantum channels, we also do not know when two channels used jointly can have capacity exceeding the sum of the individual channels. A more general notion of a channel's capability to transmit quantum data thus depends on the details of other resources available [36, 53, 54], and does not necessarily coincide with its capacity, a drastic deviation from the classical theory.

Similar to the quantum capacity, a channel's private and classical capacities can be defined as the highest rates of faithful transmission of private and classical information, respectively; expressions analogous to (1) are known [5, 6, 8, 9]. Both capacities require regularized expressions [35, 55], and the private capacity can be shown to be non-additive for some channels [32, 56].

For classical capacity, the underlying information quantity is the Holevo information, which was conjectured to be additive for a long time. In fact, strong additivity was proved for certain channels such as entanglement-breaking [44], depolarizing, [46], Hadamard [48, 49], and unital qubit channels [45]. As a result, for these channels the classical capacity completely characterizes their ability to faithfully send classical information. Furthermore, the only known proofs of violation of weak additivity of the Holevo information [35, 57, 58] are based on random channel constructions and no explicit example has been found yet [3, 35]. It is still open if the classical capacity can be non-additive. It is furthermore unclear if additivity is more prevalent for classical data transmission, or if proofs are simply harder to come by since the Holevo information involves a more complex optimization compared to coherent information.

The situation for quantum information transmission is quite different. There is a plethora of concrete channels with super-additive coherent information [17–20, 23–30]. The only known class of channels with strongly additive coherent information are the entanglement-breaking channels, but they are somewhat trivial – their quantum capacity is zero. Degradable channels [47, 59] have weakly additive coherent information, and two degradable channels have additive coherent information, yet surprisingly degradability does not imply strong additivity for a channel. Even weakly additive channels like some (anti-)degradable [47] and PPT channels [60] may have super-additive quantum capacity in combination with suitable channels [36, 39, 56]. A common feature in these violations of strong additivity is that one or both of the channels are manifestly noisy, that is, with vanishing or small quantum capacity. Most of these proofs come from a qualitative inability for the channels to transmit quantum data; in addition, nearly noiseless channels are indeed limited in their non-additivity [61].

In this paper, we provide qualitatively new examples of super-additivity of quantum capacity. The phenomenon seems prevalent, does not involve channels engineered to exhibit the effect, and can involve pairs of channels with large quantum capacity. Our findings show an even more complex landscape of non-additivity than hitherto appreciated. Yet, our channels and the proofs are simple, and thus we hope they improve our understanding of the subject.

Main results. Our first main result is that a simple qutrit 'platypus channel', defined via eq. (3) below, violates strong additivity of coherent information when used together with a variety of simple and well-known qubit channels such as the erasure, amplitude damping, depolarizing, and even randomly constructed qubit channels. Even more remarkably, the same simple code achieves non-additivity in all cases. Our findings strongly suggest that super-additivity is much more prevalent and generic than previously thought.

Second, as proved in our companion paper [1], platypus channels have weakly additive coherent information if the spin alignment conjecture introduced in [1] holds. As the erasure channel and the amplitude damping channel also have weakly additive coherent information, we have an example of non-additivity of quantum capacity between two weakly additive channels. The only known prior example revolves around superactivation [36], and requires substantial fine-tuning to demonstrate the effect. In contrast, our channel requires no such tuning, and both channels exhibit non-additivity over a wide range of parameters, including regimes where both channels have substantial capacity themselves.

Third, we show that higher-dimensional platypus channels have similar non-additive behavior. In particular, when used jointly with a higher-dimensional erasure channel, it exhibits super-additivity of quantum capacity unconditionally, i.e., without relying on the spin alignment conjecture. The underlying mechanism at work achieving all of these non-additivity results is qualitatively different from previous results in [36, 39, 56], as explained in the Discussion section.

In the following paragraphs we discuss our main results; see the Supplementary information [62] for additional details. MATLAB and Python code used to obtain the numerical results mentioned above will be made available at [63].

The qutrit platypus channel. The qutrit 'platypus channel' $\mathcal{N}_s$ is defined by the following isometry $F_s : \mathcal{H}_a \mapsto \mathcal{H}_b \otimes \mathcal{H}_c$:

$$\begin{aligned} F_s|0\rangle &= \sqrt{s}|0\rangle \otimes |0\rangle + \sqrt{1-s}|1\rangle \otimes |1\rangle \\ F_s|1\rangle &= |2\rangle \otimes |0\rangle \\ F_s|2\rangle &= |2\rangle \otimes |1\rangle, \end{aligned} \tag{3}$$

where $0 \leq s \leq 1/2$, and the input $\mathcal{H}_a$, output $\mathcal{H}_b$ and environment $\mathcal{H}_c$ have dimension $3, 3$, and 2, respectively. This channel [28, 64] is extensively studied in the companion paper [1]. From [1, 28], the channel coherent information is always positive and can be attained on inputs of the form $\sigma(u) := (1-u)|0\rangle\langle 0| + u|2\rangle\langle 2|$:

$$\mathcal{Q}^{(1)}(\mathcal{N}_s) = \max_{u\in[0,1]} \Delta(\mathcal{N}_s, \sigma(u)) > 0.$$

Conditioned on the spin-alignment conjecture (SAC) formulated in [1], the channel coherent information $\mathcal{Q}^{(1)}(\mathcal{N}_s)$ can be proved to be weakly additive, and thus $\mathcal{Q}(\mathcal{N}_s) = \mathcal{Q}^{(1)}(\mathcal{N}_s)$. Without the SAC, we have the upper bound $\mathcal{Q}(\mathcal{N}_s) \leq \log(1+\sqrt{1-s})$.

Violation of strong additivity. We find that $\mathcal{N}_s$ displays super-additivity in the strong sense,

$$\mathcal{Q}^{(1)}(\mathcal{N}_s \otimes \mathcal{K}) > \mathcal{Q}^{(1)}(\mathcal{N}_s) + \mathcal{Q}^{(1)}(\mathcal{K}), \tag{4}$$

when used with just about any small channel $\mathcal{K}$. Since $\mathcal{Q}^{(1)}(\mathcal{N}_s) > 0$, the additional channel $\mathcal{K}$ is said to amplify $\mathcal{Q}^{(1)}(\mathcal{N}_s)$. We consider various well-known and physically relevant channels $\mathcal{K}$, such as the qubit erasure channel, $\mathcal{E}_\lambda(\rho) = (1-\lambda)\rho + \lambda \mathrm{Tr}(\rho)|e\rangle\langle e|$ with erasure probability $\lambda \in [0,1]$, the qubit amplitude damping channel, $\mathcal{A}_\gamma(\rho) = N_0\rho N_0^\dagger + N_1\rho N_1^\dagger$ with damping probability $\gamma \in [0,1]$ and Kraus effects $N_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $N_1 = \sqrt{\gamma}|0\rangle\langle 1|$, and the qubit depolarizing channel, $\mathcal{D}_p(\rho) = (1-4p/3)\rho + 2p/3 I$ with depolarizing parameter $p \in [0,1]$. For erasure and amplitude damping channels the quantum capacity equals the channel coherent information [43, 47, 65]. The amplification in (4) not only occurs when each of the channels $\mathcal{E}_\lambda, \mathcal{A}_\gamma$, and $\mathcal{D}_p$ has zero coherent information (see Fig. 1), but it persists for a wide range of channel parameters $0 \leq s \leq 1/2$, $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$, $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$, and $p_{\min} \leq p \leq p_{\max}$ (see Supplementary material [62]).

Remarkably, the amplification of $\mathcal{Q}^{(1)}(\mathcal{N}_s)$ by all three channels $\mathcal{E}_\lambda$, $\mathcal{A}_\gamma$, and $\mathcal{D}_p$ can be achieved by a single input state ansatz for $\mathcal{N}_s \otimes \mathcal{K}$,

$$\begin{aligned} \rho(\epsilon, r_1, r_2) = r_1|00\rangle\langle 00| + r_2|01\rangle\langle 01| \\ + (1-r_1-r_2)|\chi_\epsilon\rangle\langle\chi_\epsilon|, \end{aligned} \tag{5}$$

where $|\chi_\epsilon\rangle = \sqrt{1-\epsilon}|20\rangle + \sqrt{\epsilon}|11\rangle$, and the parameters satisfy the constraints $\epsilon, r_1, r_2, r_1+r_2 \in [0,1]$. In more detail, we find that $\Delta^*(\mathcal{N}_s \otimes \mathcal{K}_x) := \max_{\epsilon,r_1,r_2} \Delta\big(\mathcal{N}_s \otimes \mathcal{K}_x, \rho(\epsilon, r_1, r_2)\big)$ exceeds $\mathcal{Q}^{(1)}(\mathcal{N}_s) + \mathcal{Q}^{(1)}(\mathcal{K}_x)$ where $\mathcal{K}_x$ is one of $\mathcal{E}_\lambda$, $\mathcal{A}_\gamma$, or $\mathcal{D}_p$. Since all three channels $\mathcal{E}_\lambda$, $\mathcal{A}_\gamma$, and $\mathcal{D}_p$ have well known symmetries, one may suspect that the amplification strategy (5) coincides because of these symmetries. We find this not to be the case. Numerics reveal that amplification of $\mathcal{Q}^{(1)}(\mathcal{N}_{1/2})$ using (5) occurs even when $\mathcal{K}$ is defined in terms of a random qubit channel. Super-additivity occurs both when $\mathcal{Q}^{(1)}(\mathcal{K}) > 0$ or when the coherent information of $\mathcal{K}$ itself vanishes.

Unconditional super-additivity of quantum capacity. In the previous section we showed super-additivity of the coherent information of $\mathcal{N}_s$ when used in parallel with other channels such as $\mathcal{E}_\lambda$ or $\mathcal{A}_\gamma$. The latter channels are known to satisfy $\mathcal{Q}(\mathcal{E}_\lambda) = \mathcal{Q}^{(1)}(\mathcal{E}_\lambda)$ and $\mathcal{Q}(\mathcal{A}_\gamma) = \mathcal{Q}^{(1)}(\mathcal{A}_\gamma)$. Moreover, conditioned on the spin alignment conjecture (SAC) [1], we also have $\mathcal{Q}^{(1)}(\mathcal{N}_s) = \mathcal{Q}(\mathcal{N}_s)$. Hence, the super-additivity of $\mathcal{Q}^{(1)}$ in (4) can be elevated to super-additivity of the quantum capacity $\mathcal{Q}$, provided the SAC is true.

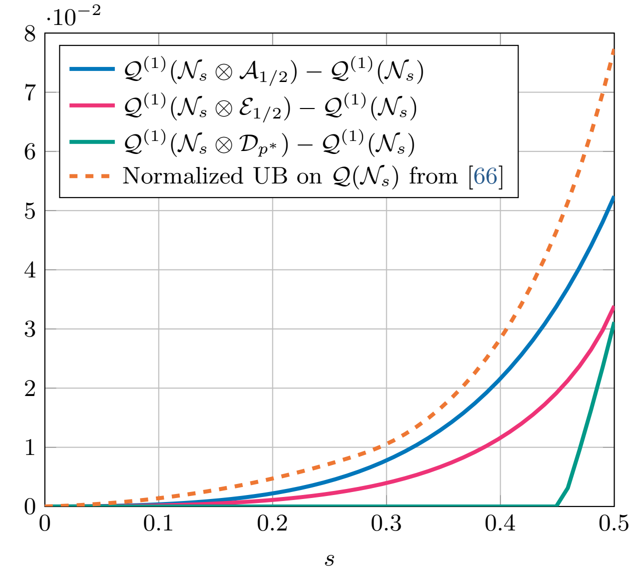

FIG. 1. Amplification of coherent information for the channel $\mathcal{N}_s$ and various additional channels. We plot $\mathcal{Q}^{(1)}(\mathcal{N}_s \otimes \mathcal{K}) - \mathcal{Q}^{(1)}(\mathcal{N}_s)$ for $\mathcal{K} = \mathcal{E}_{1/2}$ (solid magenta), $\mathcal{K} = \mathcal{A}_{1/2}$ (solid blue), and $\mathcal{K} = \mathcal{D}_{p^*}$ (solid green). Here, $\mathcal{E}_{1/2}$ and $\mathcal{A}_{1/2}$ are the symmetric erasure and amplitude damping channels respectively, $\mathcal{D}_{p^*}$ is the qubit depolarizing channel with $p^* \approx 0.1893$, so that all three channels have zero coherent information $\mathcal{Q}^{(1)}(\mathcal{K}) = 0$. We also plot $\hat{R}_\alpha(\mathcal{N}_s) - \mathcal{Q}^{(1)}(\mathcal{N}_s)$ (dashed orange), where $\hat{R}_\alpha(\cdot)$ with $\alpha = 1 + 2^{-5}$ is the upper bound (UB) on the quantum capacity $\mathcal{Q}(\cdot)$ derived in [66].

We now show that, remarkably, this result can be strengthened to an *unconditional* super-additivity of quantum capacity. To this end, we consider a channel $\mathcal{M}_d$ introduced in [1] that generalizes $\mathcal{N}_{1/2}$ to d input and output dimensions, and $d-1$ environment dimensions, with $d \geq 3$. The isometry $G\colon \mathcal{H}_a \to \mathcal{H}_b \otimes \mathcal{H}_c$ acts on an orthonormal input basis $\{|i\rangle\}_{i=0}^{d-1}$ as

$$\begin{aligned} G|0\rangle &= \frac{1}{\sqrt{d-1}} \sum_{j=0}^{d-2} |j\rangle \otimes |j\rangle, \\ G|i\rangle &= |d-1\rangle \otimes |i-1\rangle \quad \text{for } i = 1, \ldots, d-1, \end{aligned} \tag{6}$$

and defines the channel $\mathcal{M}_d(\cdot) := \mathrm{tr}_c(G \cdot G^\dagger)$.

Comparing (6) to the isometry (3) for $\mathcal{N}_{1/2}$, we see that $\mathcal{M}_3 = \mathcal{N}_{1/2}$, and hence $\mathcal{M}_d$ is indeed a d-dimensional generalization of $\mathcal{N}_{1/2}$. The coherent information $\mathcal{Q}^{(1)}(\mathcal{M}_d)$ is evaluated in [1], and similar to $\mathcal{N}_{1/2}$ we have $\mathcal{Q}(\mathcal{M}_d) = \mathcal{Q}^{(1)}(\mathcal{M}_d)$ modulo (a generalized version of) the spin alignment conjecture. However, we do not make use of this (conjectured) identity here and instead use the following upper bound on the quantum capacity of $\mathcal{M}_d$ derived in [1]:

$$\mathcal{Q}(\mathcal{M}_d) \leq \log\left(1 + \frac{1}{\sqrt{d-1}}\right) \leq \frac{1}{\ln 2} \frac{1}{\sqrt{d-1}}. \tag{7}$$

This upper bound follows from evaluating the "transposition bound" on the quantum capacity of a quantum channel [67]. It is phrased in terms of the diamond norm and can be evaluated using semidefinite programming techniques.

The quantum capacity of $\mathcal{M}_{d+1}$ is super-additive when used together with the d-dimensional erasure channel $\mathcal{E}_{\lambda,d}$ where $\lambda \in [0,1]$. More precisely, we show that

$$\mathcal{Q}(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) > \mathcal{Q}(\mathcal{M}_{d+1}) + \mathcal{Q}(\mathcal{E}_{\lambda,d}) \tag{8}$$

for suitable λ and d in two steps: First, using the upper bound (7) on $\mathcal{Q}(\mathcal{M}_d)$ and the fact that the quantum capacity of $\mathcal{E}_{\lambda,d}$ is given by $\mathcal{Q}(\mathcal{E}_{\lambda,d}) = \max\{(1-2\lambda)\log d, 0\}$ [43], we obtain an upper bound

$$u(\lambda, d) := \log\left(1 + 1/\sqrt{d}\right) + \max\{(1-2\lambda)\log d, 0\} \tag{9}$$

on the right-hand side of (8). Second, letting $\mathcal{H}_a$ and $\mathcal{H}_{a'}$ be the input Hilbert spaces for $\mathcal{M}_{d+1}$ and $\mathcal{E}_{\lambda,d}$, respectively, we find an input state $\rho_{aa'}$ with coherent information $\Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'})$ exceeding $u(\lambda, d)$,

$$\begin{aligned} \mathcal{Q}(\mathcal{M}_{d+1}) + \mathcal{Q}(\mathcal{E}_{\lambda,d}) &\leq u(\lambda, d) \\ &< \Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) \\ &\leq \mathcal{Q}(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}). \end{aligned} \tag{10}$$

This chain of inequalities proves (8).

The input state achieving (10) is $\rho_{aa'} = \mathrm{Tr}_{rr'}[\psi]_{ara'r'}$, where for $w \in [0,1]$ we define

$$\begin{aligned} |\psi\rangle_{ara'r'} = \sqrt{1-w}\; |0\rangle_r |0\rangle_a \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_{r'} |i\rangle_{a'} \\ + \sqrt{w}\; |1\rangle_r |0\rangle_{r'} \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_a |i-1\rangle_{a'}, \end{aligned} \tag{11}$$

and the reference spaces $\mathcal{H}_r$ and $\mathcal{H}_{r'}$ have dimensions two and d, respectively. The pure state $|\psi\rangle_{ara'r'}$ is a superposition of two orthogonal 'pieces' with amplitudes $\sqrt{1-w}$ and $\sqrt{w}$, respectively. By itself, the first piece only generates coherent information via $\mathcal{E}_{\lambda,d}$, as the input of $\mathcal{M}_{d+1}$ in $\mathcal{H}_a$ is in a product state with both the input to $\mathcal{E}_{\lambda,d}$ and the reference. The second piece by itself generates no coherent information, since the joint input system $\mathcal{H}_a \otimes \mathcal{H}_{a'}$ is unentangled with the reference $\mathcal{H}_r \otimes \mathcal{H}_{r'}$.

Optimizing over the parameter $w \in [0,1]$, this superposition of coding strategies results in a coherent information of the joint channel $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ that exceeds the upper bound $u(\lambda, d)$ on $\mathcal{Q}(\mathcal{M}_{d+1}) + \mathcal{Q}(\mathcal{E}_{\lambda,d})$. We first show this numerically for $\lambda \in [0.37, 0.57]$ and sufficiently large d. This is summarized in Fig. 2, where we plot the minimal values $\lambda^{\mathcal{Q}}_{\min}(d)$ (dashed blue) and $\lambda^{\mathcal{Q}}_{\max}(d)$ (dashed magenta) of λ as a function of d such that (8) holds numerically for all $\lambda \in [\lambda^{\mathcal{Q}}_{\min}(d), \lambda^{\mathcal{Q}}_{\max}(d)]$. Note that $\mathcal{E}_{\lambda,d}$ has positive quantum capacity when $\lambda < 1/2$, and hence for suitable d and λ we obtain super-additivity of quantum capacity (8) for two channels, $\mathcal{M}_d$ and $\mathcal{E}_{\lambda,d}$, each with strictly positive $\mathcal{Q}$.

In Fig. 2 we also plot the minimal values $\lambda_{\min}(d)$ (solid blue) and $\lambda_{\max}(d)$ (solid magenta) such that the coherent information of $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ is super-additive for all $\lambda \in [\lambda_{\min}(d), \lambda_{\max}(d)]$. While the interval $[\lambda_{\min}(d), \lambda_{\max}(d)]$ marks the 'true' extent of the super-additivity of quantum capacity (modulo the spin alignment conjecture), we stress once again that the super-additivity of quantum capacity within the interval $[\lambda^{\mathcal{Q}}_{\min}(d), \lambda^{\mathcal{Q}}_{\max}(d)]$ is unconditional.

We can further strengthen the numerical results of Fig. 2 by proving analytically that the super-additivity of quantum capacity in (8) indeed holds for all $\lambda \in (0, 1)$ and sufficiently large d. The proof is based on a log-singularity-like argument [28], and applied for any $\lambda \in (0, 1)$, by a suitable choice of the parameter w in the state (11). Details of this calculation can be found in the Supplementary material [62].

Discussion. Interestingly, a single ansatz (11) is responsible for super-additivity of $\mathcal{Q}^{(1)}$ when $\mathcal{N}_s$ is used with a variety of other channels $\mathcal{E}_\lambda, \mathcal{A}_\gamma, \mathcal{D}_p$, and randomly constructed qubit channels. A higher dimensional version of this ansatz gives rise to super-additivity of quantum capacity when $\mathcal{M}_d$ is used with $\mathcal{E}_{\lambda,d}$. The mechanism and extent of this super-additivity is distinct from super-activation, where the private capacity of a zero quantum capacity channel $\mathcal{N}$ is transformed into quantum capacity when used jointly with an anti-degradable channel $\mathcal{A}$. This transformation has efficiency at most $1/2$, and thus one obtains super-activation when $0 = \mathcal{Q}(\mathcal{N}) < \mathcal{P}(\mathcal{N})/2$. By contrast, $\mathcal{Q}(\mathcal{N}_s) > \mathcal{P}(\mathcal{N}_s)/2 > 0$, thus ruling out the super-activation mechanism as the cause for our super-additivity involving $\mathcal{N}_s$; our protocol (11) employs a different mechanism.

Like super-activation our protocol works robustly [39] when $\mathcal{A} = \mathcal{E}_{\lambda,d}$ and λ is varied, but unlike super-activation we find super-amplification, i.e., super-additivity even when both channels $\mathcal{M}_d$ and $\mathcal{E}_{\lambda,d}$ have non-zero quantum capacity. Similar super-additivity of quantum capacity arises in high-dimensional rocket and half-rocket channels when used with zero capacity channels [32, 42]. These noisy channels, carefully constructed to display super-additivity, have quantum capacity well below the dimensional bound $r = \mathcal{Q}/\log d \ll 1$. By contrast, $\mathcal{M}_d$ is simply constructed by hybridizing a degradable qubit channel with a useless channel, with the goal to support weak additivity of $\mathcal{Q}^{(1)}$. Yet, it exhibits super-additivity of $\mathcal{Q}$ even when it has modest input dimension and noise; for instance super-additivity occurs at $d = 5$ and $r > .2$. Our result on $\mathcal{M}_d$ also contrasts with those obtainable by extending super-activation via continuity arguments. The super-activating channels can be perturbed to have positive capacities, but these capacities are necessarily very small. Moreover, super-additivity involving $\mathcal{M}_d$ occurs over a wide range of erasure probabilities that is well beyond what one may expect from such perturbations. For instance, at $d = 10$, $r \simeq .075$, and super-additivity holds over erasure probabilities $.43 \leq \lambda \leq .53$, and the erasure channel can have substantial capacity. Using $\mathcal{M}_d$ with a symmetric channel, $\mathcal{S}$, of unbounded dimension leads to super-additivity, $\mathcal{Q}(\mathcal{M}_d \otimes \mathcal{S}) \geq \mathcal{Q}(\mathcal{M}_d) + \mathcal{Q}(\mathcal{S})$ for any $d \geq 7$ where $\mathcal{P}(\mathcal{M}_d)/2 > \mathcal{Q}(\mathcal{M}_d)$ [1], since $\mathcal{Q}(\mathcal{M}_d \otimes \mathcal{S}) > \mathcal{P}(\mathcal{M}_d)/2$ [36]. These super-additivity results can be strengthened and simplified further if the SAC is proven. The simplicity of the channels involved in super-additivity here raises the question of whether qualitatively similar constructions are possible for investigating super-additivity of private and classical capacities.

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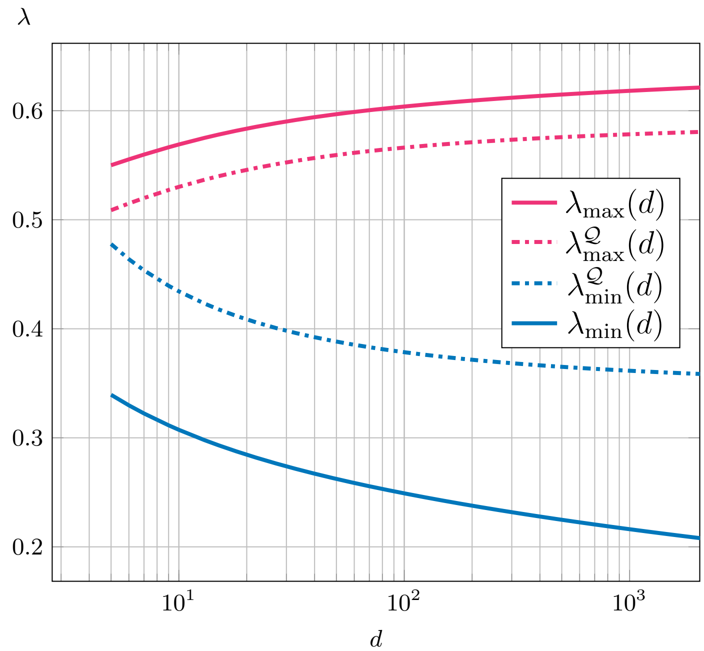

FIG. 2. Plot of the region of super-additivity of coherent information and quantum capacity of the quantum channel $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$. The solid lines are the minimal values $\lambda_{\min}(d)$ (blue) and maximal values $\lambda_{\max}(d)$ (magenta) between which $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ has super-additive coherent information, $\mathcal{Q}^{(1)}(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) > \mathcal{Q}^{(1)}(\mathcal{M}_{d+1}) + \mathcal{Q}^{(1)}(\mathcal{E}_{\lambda,d})$. The dashed lines are the minimal values $\lambda^{\mathcal{Q}}_{\min}(d)$ (blue) and maximal values $\lambda^{\mathcal{Q}}_{\max}(d)$ (magenta) between which $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ has super-additive quantum capacity, $\mathcal{Q}(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) > \mathcal{Q}(\mathcal{M}_{d+1}) + \mathcal{Q}(\mathcal{E}_{\lambda,d})$.

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