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Non-Abelian eigenstate thermalization hypothesis

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The eigenstate thermalization hypothesis (ETH) explains why nonintegrable quantum many-body systems thermalize internally if the Hamiltonian lacks symmetries. If the Hamiltonian conserves one quantity ("charge"), the ETH implies thermalization within a charge sector—in a microcanonical subspace. But quantum systems can have charges that fail to commute with each other and so share no eigenbasis; microcanonical subspaces may not exist. Furthermore, the Hamiltonian will have degeneracies, so the ETH need not imply thermalization. We adapt the ETH to noncommuting charges by positing a *non-Abelian ETH* and invoking the *approximate microcanonical subspace* introduced in quantum thermodynamics. Illustrating with SU(2) symmetry, we apply the non-Abelian ETH in calculating local operators' time-averaged and thermal expectation values. In many cases, we prove, the time average thermalizes. However, we find cases in which, under a physically reasonable assumption, the time average converges to the thermal average unusually slowly as a function of the global-system size. This work extends the ETH, a cornerstone of many-body physics, to noncommuting charges, recently a subject of intense activity in quantum thermodynamics.

Nonintegrable closed quantum many-body systems thermalize internally, in the absence of conserved observables, or *charges*. Few-body operators $\mathscr{O}$ equilibrate to the expectation values they would have in the canonical state $\rho_{\text{can}} \propto e^{-\beta H}$. H denotes the Hamiltonian, whose expectation value determines the inverse temperature β [1]. The eigenstate thermalization hypothesis (ETH) explains this thermalization [2–4]: Let $|\alpha\rangle$ denote the energy eigenstates; E_α, the eigenenergies; and $\mathscr{O}_{\alpha\alpha'} := \langle\alpha|\mathscr{O}|\alpha'\rangle$, matrix elements representing the operator. $\mathscr{O}$ and H satisfy the ETH if $\mathscr{O}_{\alpha\alpha'}$ has a certain structure, reviewed below. If $\mathscr{O}_{\alpha\alpha'}$ does and H is nondegenerate, $\mathscr{O}$ thermalizes: Its time-averaged expectation value approximately equals its thermal expectation value. The difference is of $O(N^{-1})$, if N denotes the global system size. (We use big-O notation as in many-body physics, meaning "scales as.") These results explain behaviors observed numerically and experimentally across condensed matter; atomic, molecular, and optical physics; and high-energy physics [1, 5–14].

The argument for thermalization relies on the Hamiltonian's nondegeneracy and on matrix-element structure. Both postulates are questionable if H conserves charges [15]. If H has an Abelian symmetry, the energy spectrum can lack degeneracies. Since the charges commute, they share eigenspaces—charge sectors. In each shared sector, the ETH applies. For example, consider N qubits (quantum two-level systems, or spins). H can conserve the total spin's z-component, S_z, by being U(1)-symmetric. The ETH is often applied in an S_z sector, wherein the ETH holds and implies thermalization.

A non-Abelian symmetry can eliminate our recourse to charge sectors: Such a symmetry is generated by charges that fail to commute with each other and so cannot necessarily have definite values simultaneously—cannot necessarily share sectors governable by the ETH. Moreover, non-Abelian symmetries force degeneracies on H, having multidimensional irreducible representations. Finally, how $\mathscr{O}$ transforms under the symmetry operations constrains the matrix elements $\mathscr{O}_{\alpha\alpha'}$ in opposition to the ETH.

For example, consider again an N-qubit system. H can conserve the total spin components $S_{a=x,y,z}$, by being SU(2)-symmetric. The energy spectrum splits into degenerate multiplets labeled by total spin quantum numbers s_α. Only the singlets, whose $s_\alpha = 0$, are simultaneous eigenspaces of $S_{x,y,z}$. Furthermore, the matrix elements $\mathscr{O}_{\alpha\alpha'}$ obey the Wigner–Eckart theorem [16], conflicting with the ETH.

Non-Abelian symmetries are ubiquitous in quantum many-body physics [17, 18]. They grace systems including complex nuclei and atoms [19], Heisenberg models in condensed matter [20, 21], gauge theories [22], and Wess-Zumino-Witten models [23–26]. Hence the apparent conflict between non-Abelian symmetries and the ETH impacts our basic understanding of diverse, prominent models.

To overcome the conflict, we propose a *non-Abelian ETH*. We apply it to SU(2) symmetry for simplicity, expecting results to generalize. Using the non-Abelian ETH, we compute two averages of few-body operators $\mathscr{O}$: time-averaged and thermal expectation values. For many operators and initial states, the time average agrees with the thermal average: Differences are $O(N^{-1})$, as without noncommuting charges [27, 28]. For certain operators and initial states, however, the time average may deviate from the thermal prediction by anomalously large corrections $\sim N^{-1/2}$. This result holds under a physically reasonable assumption about the non-Abelian analog of $\mathscr{O}_{\alpha\alpha'}$.

Below, we review the conventional ETH. We then introduce our setup, present the non-Abelian ETH [Eq. (14)], and apply it to calculate operators' thermal and time-averaged expectation values. Finally, we describe opportunities established by our results. This work extends the ETH, a mainstay of many-body physics, to the more fully quantum domain of noncommuting charges and so to a growing subfield of quantum-information thermodynamics [29–63].

Review of conventional ETH.—Let the Hamiltonian H, energy eigenstates $|\alpha\rangle$, eigenenergies E_α, operator $\mathscr{O}$, and matrix elements $\mathscr{O}_{\alpha\alpha'} := \langle\alpha|\mathscr{O}|\alpha'\rangle$ be defined as in the introduction. The operator and Hamiltonian satisfy the ETH if

$$\mathscr{O}_{\alpha\alpha'} = \mathcal{O}(\mathcal{E})\,\delta_{\alpha,\alpha'} + e^{-S_{\rm th}(\mathcal{E})/2}\,f(\mathcal{E},\omega)R_{\alpha\alpha'}\,. \tag{1}$$

The relevant energies average to $\mathcal{E} := (E_\alpha + E_{\alpha'})/2$, their difference is $\omega := E_\alpha - E_{\alpha'}$, $\mathcal{O}(\mathcal{E})$ and $f(\mathcal{E},\omega)$ are real functions that vary smoothly with the energy density $\mathcal{E}/N$, $S_{\rm th}(\mathcal{E})$ denotes the thermodynamic entropy (logarithm of the density of states) at energy $\mathcal{E}$, $\delta_{\alpha,\alpha'}$ denotes the Kronecker delta, and the $R_{\alpha\alpha'}$ are erratically varying $O(1)$ numbers [64–66]. The first, "diagonal" ($\alpha{=}\alpha'$) term in Eq. (1) contains the microcanonical expectation value $\mathcal{O}(\mathcal{E})$. The thermodynamic entropy $S_{\rm th}(\mathcal{E})$ exponentially suppresses the second, "off-diagonal" term.

If $\mathscr{O}$ and a nondegenerate H satisfy the ETH, $\mathscr{O}$ thermalizes [1, 28]: Let N denote the system's size. The system begins in a normalized state $|\psi(0)\rangle = \sum_\alpha C_\alpha|\alpha\rangle$ with an extensive energy $E := \langle H\rangle = O(N)$. We denote expectation values by $\langle\cdot\rangle := \langle\psi(0)|\cdot|\psi(0)\rangle$. Let the energy variance, $\text{var}(H) := \langle H^2\rangle - E^2$, be at most $O(N)$.

At time t, the operator's expectation value is

$$\langle\mathscr{O}\rangle_t = \sum_\alpha |C_\alpha|^2\mathscr{O}_{\alpha\alpha} + \sum_{\alpha\neq\alpha'} C_\alpha^* C_{\alpha'}\, e^{i(E_\alpha - E_{\alpha'})t/\hbar}\,\mathscr{O}_{\alpha\alpha'}\,. \tag{2}$$

Consider averaging this value over an infinite time: $\overline{\langle\mathscr{O}\rangle_t} := \lim_{t\to\infty}\frac{1}{t}\int_0^t dt'\,\langle\mathscr{O}\rangle_{t'}$. As H lacks degeneracies, phase cancellations make the second term average to zero: $\overline{\langle\mathscr{O}\rangle_t} = \sum_\alpha |C_\alpha|^2\mathscr{O}_{\alpha\alpha}$.

To the first term, we apply a strategy that will echo in our noncommuting-charge arguments. By the ETH (1), $\mathscr{O}_{\alpha\alpha} \approx \mathcal{O}(E_\alpha)$ can be Taylor-expanded about $E_\alpha = E$. The zeroth-order term yields $\overline{\langle\mathscr{O}\rangle_t} \approx \mathcal{O}(E)$, by the state's normalization. The first-order term vanishes, by the definition of E. All higher-order terms yield corrections $\leq O(N^{-1})$, by the energy-variance bound and the smoothness of $\mathcal{O}(\mathcal{E})$. Hence the time average $\overline{\langle\mathscr{O}\rangle_t} = \mathcal{O}(E) + O(N^{-1})$ approximately equals the microcanonical average. So does the canonical average, $\text{Tr}(\mathscr{O}\rho_{\rm can}) = \mathcal{O}(E) + O(N^{-1})$, by the ETH (1) and related arguments [28, 67–69]. Therefore, the time average $\overline{\langle\mathscr{O}\rangle_t}$ equals the thermal average plus $O(N^{-1})$ corrections.

Setup suited to noncommuting charges.—Consider a quantum system formed from $N \gg 1$ degrees of freedom. The Hamiltonian, H, is nonintegrable. It conserves a number $\ll N$ of charges Q_a that do not all commute: $[H, Q_a] = 0$, but $[Q_a, Q_{a'}] \neq 0$ for some $a' \neq a$. The charges generate a non-Abelian symmetry group.

We illustrate with an N-qubit system that has an SU(2) symmetry—whose total spin components $S_{a=x,y,z}$ are conserved. (Those components decompose as $S_a = \sum_{j=1}^N s_{j,a}$, if the $s_{j,a}$ denote qubit j's spin operators.) H, $\vec{S}^2$, and S_z share an eigenbasis $\{|\alpha, m\rangle\}$. If $\hbar = 1$,

$$H|\alpha,m\rangle = E_\alpha|\alpha,m\rangle, \tag{3}$$

$$\vec{S}^2|\alpha,m\rangle = s_\alpha(s_\alpha+1)|\alpha,m\rangle, \text{ and} \tag{4}$$

$$S_z|\alpha,m\rangle = m|\alpha,m\rangle, \text{ wherein} \tag{5}$$

$$m = -s_\alpha, -s_\alpha+1, \ldots, s_\alpha\,. \tag{6}$$

Ladder operators $S_\pm = S_x \pm iS_y$ raise and lower S_z

The normalized initial state decomposes as

$$|\psi(0)\rangle = \sum_{\alpha,m} C_{\alpha,m}|\alpha,m\rangle\,, \quad \text{wherein} \quad C_{\alpha,m}\in\mathbb{C}\,. \tag{7}$$

Operators $\mathscr{O}$ have time-t expectation values $\langle\mathscr{O}\rangle_t := \langle\psi(t)|\mathscr{O}|\psi(t)\rangle$. We drop the subscript from time constants:

$$\langle H\rangle =: E\,, \quad \text{and} \tag{8}$$

$$\langle S_z\rangle =: M\,. \tag{9}$$

Aligning the z-axis with $\langle\vec{S}\rangle$, we set $M \geq 0$ and $\langle S_x\rangle, \langle S_y\rangle = 0$, without sacrificing generality. The state has an extensive energy, $E = O(N)$, and is far from maximally spin-polarized: $N - M = O(N)$. (ETH-type statements tend to hold when the thermodynamic entropy is extensive [1]. $S_{\rm th}$ tends to be nonextensive when additive charges [e.g., E and $S_{x,y,z}$] lie near their extremes, which we therefore exclude.)

$|\psi(0)\rangle$ belongs to an *approximate microcanonical subspace*, which generalizes a microcanonical subspace for noncommuting charges [32, 45, 60]: Measuring any charge Q_a likely yields an outcome near $\langle Q_a\rangle$; the charges' variances are bounded as

$$\text{var}(H) \leq O(N), \tag{10}$$

$$\text{var}(S_z) \leq O(N), \quad \text{and} \tag{11}$$

$$\text{var}(S_{x,y}) \leq O(N)\,. \tag{12}$$

Conditions (10)–(12) govern typical many-body states prepared today, including all short-range-correlated states [70] [45, 60].

Having introduced the initial state, we profile operators expected to obey the non-Abelian ETH. Without sacrificing generality, we focus on symmetry-adapted operators: *Spherical tensor operators* consist of components $T_q^{(k)}$ that transform irreducibly under global SU(2) rotations [16]. For example, consider an atom absorbing a photon (of spin $k=1$), which imparts q=1 quantum of z-type angular momentum. $T_{q=1}^{(k=1)}$ represents the photon's effect. Generally, the index $q=-k,-k+1,\ldots,k$. Examples include single-spin operators: $s_{j,z}$ is a $T_0^{(1)}$, and the ladder operators $s_{j,\pm}=s_{j,x}\pm i s_{j,y}$ are proportional to $T_{\pm1}^{(1)}$ operators. Every operator equals a linear combination of $T_q^{(k)}$ operators [16].

We focus on few-body operators, commonly expected to satisfy ETH-type postulates [1, 71]. More precisely, we consider K-*local* operators $\mathscr{O}$, which have operator norms $\leq O(K)$. Examples include products of K single-spin operators, e.g., $s_{1,x}s_{2,y}\ldots s_{K,z}+\text{h.c.}$ Every K-local operator equals a linear combination of spherical-tensor components $T_q^{(\leq K)}$. We focus on $K=O(1)$ and hence on operators $T_q^{(k)}$ with $k,q=O(1)$.

Consider representing a $T_q^{(k)}$ operator as a matrix relative to the energy eigenbasis. The matrix elements obey the Wigner–Eckart theorem [16],

$$\langle\alpha,m|T_q^{(k)}|\alpha',m'\rangle=\langle s_\alpha,m|s_{\alpha'},m';k,q\rangle\,\langle\alpha||T^{(k)}||\alpha'\rangle. \tag{13}$$

The first factor, $\langle s_\alpha,m|s_{\alpha'},m';k,q\rangle$, is a *Clebsch–Gordan coefficient*, which encodes the rules of quantum angular-momentum addition: The coefficient is nonzero only if $m=m'+q$ and $s_\alpha=|s_{\alpha'}-k|,|s_{\alpha'}-k|+1,\ldots,s_{\alpha'}+k$ (only if, in the photon example, the atomic transition obeys selection rules). Whereas the Clebsch–Gordan coefficient is kinematic, the second factor in (13) is dynamical. This *reduced matrix element* $\langle\alpha||T^{(k)}||\alpha'\rangle$ depends on the operator $T_q^{(k)}$ and on H but not on the quantum numbers m, m', and q (e.g., not on how many quanta of z-type angular momentum the photon gives the atom).

Non-Abelian ETH.—We now posit that the reduced matrix element can obey the *non-Abelian ETH*. Define the average energy $\mathcal{E}:=\frac{1}{2}(E_\alpha+E_{\alpha'})$ and energy difference $\omega:=E_\alpha-E_{\alpha'}$. Analogously, define the average spin quantum number $\mathcal{S}:=\frac{1}{2}(s_\alpha+s_{\alpha'})$ and the difference $\nu:=s_\alpha-s_{\alpha'}$. Denote by $S_{\text{th}}(\mathcal{E},\mathcal{S})$ the thermodynamic entropy at energy $\mathcal{E}$ and spin quantum number $\mathcal{S}$. The operator $T_q^{(k)}$ and Hamiltonian H obey the non-Abelian ETH if

$$\begin{aligned}\langle\alpha||T^{(k)}||\alpha'\rangle&=\mathcal{T}^{(k)}(\mathcal{E},\mathcal{S})\,\delta_{\alpha,\alpha'}\\&\quad+e^{-S_{\text{th}}(\mathcal{E},\mathcal{S})/2}\,f_\nu^{(k)}(\mathcal{E},\mathcal{S},\omega)R_{\alpha\alpha'}\,.\end{aligned} \tag{14}$$

The real functions $\mathcal{T}^{(k)}$ and $f_\nu^{(k)}$ depend smoothly on the densities $\mathcal{E}/N$ and $\mathcal{S}/N$. The $R_{\alpha\alpha'}$ are erraticaly varying $O(1)$ numbers, as in the conventional ETH.

Unlike $\mathcal{E}$, $\mathcal{S}$ is nonextensive, so the $\mathcal{S}$ dependencies in (14) may be unexpected. Yet the Wigner–Eckart theorem (13) prevents $\langle\alpha||T^{(k)}||\alpha'\rangle$ from depending on m or m'. Hence only $\mathcal{S}$ can encode the non-Abelian-charge conservation here.

Thermal prediction.—Nonintegrable systems thermalize to the canonical state $\rho_{\text{can}}\propto e^{-\beta H}$ if just energy is conserved; to the grand canonical state $\rho_{\text{GC}}\propto e^{-\beta(H-\mu\mathcal{N})}$ if the energy and particle number $\mathcal{N}$ are conserved; etc. Which thermal state emerges depends on the charges [67, 72]. If they fail to commute, derivations of the thermal state's form break down [30, 32]. Certain derivations were generalized in quantum-information thermodynamics to accommodate noncommuting charges [31–33, 72, 73], leading to the *non-Abelian thermal state* (NATS),

$$\rho_{\text{NATS}}:=e^{-\beta(H-\sum_a\mu_aQ_a)}/Z. \tag{15}$$

β and the effective chemical potentials μ_a are defined by the charge expectation values, $\text{Tr}(H\rho_{\text{NATS}})=E$ and $\text{Tr}(Q_a\,\rho_{\text{NATS}})=\langle Q_a\rangle$ [45] [74]. The partition function is $Z:=\text{Tr}(e^{-\beta(H-\sum_a\mu_aQ_a)})$. The NATS shares its form with the generalized Gibbs ensemble [75–79], often defined for integrable Hamiltonians and usually used with commuting charges (see [80] for an exception). Since our charges fail to commute and our H is nonintegrable, we write "NATS" for clarity. Signatures of ρ_{NATS} have emerged dynamically in numerical simulations [45] and a trapped-ion experiment [60], yet full thermalization to ρ_{NATS} has not been observed in closed quantum systems. Furthermore, noncommuting charges were conjectured to alter thermalization [45].

Our z-axis choice simplifies ρ_{NATS} to $e^{-\beta(H-\mu S_z)}/Z$ (Suppl. Note 1) [81]. Although ρ_{NATS} now shares its mathematical form with ρ_{GC}, the physics differs significantly. Here, energy and three noncommuting charges are conserved globally and transported locally; during grand canonical thermalization, energy and particles—two commuting charges—are. In the grand canonical case, the global system begins in a microcanonical subspace. Here, no nontrivial microcanonical subspace (associated with $s_\alpha\neq0$) exists, and the global system begins in an approximate microcanonical subspace. These differences in setup, we show, permit differences in thermalization.

$T_q^{(k)}$ has a thermal expectation value $\langle T_q^{(k)}\rangle_{\text{th}}:=\text{Tr}(T_q^{(k)}\rho_{\text{NATS}})$, whose trace we calculate using the $|\alpha,m\rangle$ basis. We apply the Wigner–Eckart theorem (13), then the non-Abelian ETH (14). The Clebsch–Gordan coefficient vanishes if $q\neq0$, so

$$\begin{aligned}\big\langle T_q^{(k)}\big\rangle_{\text{th}}=\frac{\delta_{q,0}}{Z}\sum_{\alpha,m}e^{-\beta(E_\alpha-\mu m)}\,\langle s_\alpha,m|s_\alpha,m;k,0\rangle\\\times\,\mathcal{T}^{(k)}(E_\alpha,s_\alpha)\,.\end{aligned} \tag{16}$$

(We omit corrections exponentially small in N.)

Time-averaged expectation value.—After $|\psi(0)\rangle$ [Eq. (7)] evolves for a time t, the operator $T_q^{(k)}$ has an expectation value

$$\langle T_q^{(k)} \rangle_t = \sum_{\alpha,\alpha',m,m'} C^*_{\alpha,m} C_{\alpha',m'} \, e^{i(E_\alpha - E_{\alpha'})t} \times \langle \alpha, m | T_q^{(k)} | \alpha', m' \rangle. \tag{17}$$

We apply the Wigner–Eckart theorem (13), invoke the non-Abelian ETH (14), and average $\langle T_q^{(k)} \rangle_{t'}$ over an infinite time ($\lim_{t\to\infty} \frac{1}{t} \int_0^t dt'$). For all $\alpha' \neq \alpha$, the exponential in (17) dephases, so the "off-diagonal" terms vanish:

$$\overline{\langle T_q^{(k)} \rangle_t} = \sum_{\alpha,m} C^*_{\alpha,m+q} C_{\alpha,m} \langle s_\alpha, m+q | s_\alpha, m; k, q\rangle \times \mathcal{T}^{(k)}(E_\alpha, s_\alpha) \,. \tag{18}$$

Comparison.—We prove two results: (i) If $M = O(N)$, the time average (18) equals the thermal average (16), plus $O(N^{-1})$ corrections, as in the absence of noncommuting charges [27, 28]. (ii) If $M = 0$, the time average may deviate from the thermal average by anomalously large, $O(N^{-1/2})$ corrections. These corrections appear sourced by different physics: quantum uncertainty in noncommuting charges, rather than thermodynamic ensembles' distinguishability at finite N [27, 28]. Result (ii) holds under a physically reasonable assumption described and motivated below Eq. (21). Anomalous thermalization may occur also at intermediate scalings $M = O(N^\gamma)$, for exponents $0 < \gamma < 1$, but this regime lies outside this paper's scope.

Consider an extensive $M = O(N)$ and $s_{j,z}$-like operators $T_{q=0}^{(k)}$. We sketch the argument for thermalization here; details appear in Suppl. Note 2. The thermal average (16) and time average (18) share a crucial property: In each, $\mathcal{T}^{(k)}(E_\alpha, s_\alpha)$ is averaged over a sharply peaked probability distribution. The peaking follows primarily from the variance conditions (10)–(12). Near each peak, the smooth function $\mathcal{T}^{(k)}(E_\alpha, s_\alpha)$ can be Taylor-expanded, then averaged term by term. The leading term evaluates to $\mathcal{T}^{(k)}(E, M)$ in both averages, (16) and (18). All higher-order terms evaluate to $\leq O(N^{-1})$. Therefore, the averages equal each other to within the usual correction:

$$\overline{\langle T_0^{(k)} \rangle_t} - \langle T_0^{(k)} \rangle_{\text{th}} = O(N^{-1}). \tag{19}$$

Now, consider ladder-operator-like operators $T_{q\neq 0}^{(k)}$. The thermal average (16) vanishes, due to the Kronecker delta. The time average (18) is $\leq O(N^{-1})$, as shown in Suppl. Notes 3 and 4. Hence the time average equals the vanishing thermal average to within the ordinary $O(N^{-1})$ correction.

The correction can be anomalously large when $M = 0$. When $M = 0$, the thermal state is rotationally invariant. Only similarly invariant $T_0^{(0)}$ operators can have nonzero thermal averages [84]. Contrariwise, some states $|\psi(0)\rangle$ have $M = 0$ but are rotationally *noninvariant*. Intuitively, these states have vanishing magnetic dipole moments but nonzero magnetic quadrupole moments (or higher-order moments). Such states can endow operators $T_q^{(k>0)}$ with time averages of $O(N^{-1/2})$, in contrast with their vanishing thermal averages. Here is an example.

Consider an arbitrary Hamiltonian eigenspace labeled by $\alpha = A$, associated with an extensive energy $E_A = O(N)$ and a spin quantum number $s_A = O(N^{1/2})$ (chosen for reasons shown below). The following state has $M = 0$ but is rotationally noninvariant:

$$|\psi(0)\rangle := \sqrt{\frac{1}{3}}\, |A, m{=}s_A\rangle + \sqrt{\frac{2}{3}}\, \left| A, m{=} -\frac{s_A}{2} \right\rangle. \tag{20}$$

$|\psi(0)\rangle$ has the properties stipulated in the setup, one can check directly. Consider the local magnetic quadrupole moment $3 s_{i,z} s_{j,z} - \vec{s}_i \cdot \vec{s}_j$. The i and j label neighboring sites. This $T_0^{(2)}$ operator's time average (18) reduces to

$$\overline{\langle T_0^{(2)} \rangle_t} = O(1) \times \mathcal{T}^{(2)}(E_A, s_A). \tag{21}$$

Clebsch–Gordan coefficients determine the $O(1)$ factor.

The $\mathcal{T}^{(2)}(E_A, s_A)$ scales linearly with the spin density—as s_A/N—for some systems, we assume. Bound states motivate this assumption, as outlined here and detailed in Suppl. Note 5. $\mathcal{T}^{(2)}(E_A, s_A)$ approximately equals an eigenstate expectation value, by the Wigner–Eckart theorem and the non-Abelian ETH:

$$\mathcal{T}^{(2)}(E_A, s_A) \approx \langle A, s_A | 3 s_{i,z} s_{j,z} - \vec{s}_i \cdot \vec{s}_j | A, s_A \rangle. \tag{22}$$

The right-hand side is essentially the joint probability $P(i,j)$ of finding spin quanta at sites i and j. Semiclassically, $P(i,j) = P(i|j) \times P(j)$, if $P(j)$ denotes the probability of finding a quantum at j and $P(i|j)$ denotes the conditional probability of then finding a quantum at i. $P(i|j)$ can be $O(1)$ if the spin quanta form bound clusters: Just as attractive interactions can bind particles together, so may suitable (e.g., ferromagnetic) couplings bind spin quanta. In the high-energy eigenstate $|A, s_A\rangle$, clusters will be spread uniformly, with a density $\sim s_A/N \sim P(j)$. Combining these steps yields $\mathcal{T}^{(2)}(E_A, s_A) \sim P(i|j) \times P(j) = O(1) \times O(s_A/N) = O(N^{-1/2})$, by our choice $s_A = O(N^{1/2})$. Substituting into Eq. (21) yields the time average (21). It deviates from the vanishing thermal average by

$$\overline{\langle T_0^{(2)} \rangle_t} - \langle T_0^{(2)} \rangle_{\text{th}} = O(N^{-1/2}) > O(N^{-1}). \tag{23}$$

See Suppl. Note 6 for details and Suppl. Note 7 for another anomalous-thermalization example.

Anomalous $O(N^{-1/2})$ scaling characterizes also a kinematic bound in Ref. [32]. That work generalized a conventional derivation of the thermal state's form to accommodate noncommuting charges: The global system, formed from N identical subsystems, was assumed to be in a generalized microcanonical state. The average subsystem's reduced state was found to lie a distance $\leq$ (const.)$N^{-1/2}$ + (const.) from ρ_{NATS}. It is possible that our results, based on dynamics and the ETH, reflect the Hamiltonian-independent results in [32].

Outlook.—We have extended the eigenstate thermalization hypothesis, a cornerstone of many-body physics, to the more fully quantum scenario in which conserved charges fail to commute with each other. Noncommutation can prevent the charges from sharing an eigenspace (a sector) and invalidates the usual assumption of the Hamiltonian's nondegeneracy. We overcame these challenges by proposing a non-Abelian ETH and focusing on an approximate microcanonical subspace. Applying these tools to SU(2), we compared the long-time average of an operator's expectation value with the thermal expectation value. The averages agree in many cases, e.g., whenever $M = O(N)$. Yet the averages can disagree by anomalously large $O(N^{-1/2})$ corrections under a physically reasonable assumption.

This work establishes several research opportunities. First, our analytical results call for testing with numerics and quantum simulators. Trapped ions have been shown, and ultracold atoms and superconducting qudits have been argued, to be able to test noncommuting-charge thermodynamics [45, 59, 60]. Promising models include nonintegrable Heisenberg Hamiltonians [45, 59, 60] and many-electron atoms. One would verify the non-Abelian ETH (14); identify operators $T_q^{(k)}$ whose smooth functions $\mathcal{T}^{(k)}$ satisfy our assumptions, enabling anomalous thermalization; and observe deviations (23) from thermal predictions.

Second, those deviations may signal the retention, by local subsystems, of information about their initial conditions. Such retention might be leveraged. Noncommuting charges could enhance quantum memories, as many-body localization has been proposed to [85]. Localization resembles prethermalization [86], scars [87], and Hilbert-space fragmentation [88] in disrupting closed quantum many-body systems' thermalization. Noncommuting charges may belong on the list, our results indicate. Confirmation would hold fundamental interest, as disrupting thermalization effectively hinders time's arrow.

Third, our arguments merit generalization from SU(2). Fourth, the smooth function $f_\nu^{(k)}(\mathcal{E}, \mathcal{S}, \omega)$ [Eq. (14)] should reveal how non-Abelian symmetries influence thermalization *dynamics* and so merits investigation. This work extends the ETH to the more fully quantum regime of noncommuting charges, linking many-body physics to quantum-information thermodynamics [29–63].

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