This is the accepted manuscript made available via CHORUS. The article has been published as:

Contextuality and Wigner Negativity Are Equivalent for Continuous-Variable Quantum Measurements

Robert I. Booth, Ulysse Chabaud, and Pierre-Emmanuel Emeriau

Phys. Rev. Lett. **129**, 230401 — Published 29 November 2022

DOI: 10.1103/PhysRevLett.129.230401

Contextuality and Wigner negativity are equivalent for continuous-variable quantum measurements

Robert I. Booth,[1, 2, 3, *] Ulysse Chabaud,[4, †] and Pierre-Emmanuel Emeriau[5, ‡]

[1] *School of Informatics, University of Edinburgh, Edinburgh, United Kingdom*
[2] *LORIA CNRS, Inria-MOCQUA, Université de Lorraine, F-54000 Nancy, France*
[3] *Sorbonne Université, CNRS, LIP6, F-75005 Paris, France*
[4] *Institute for Quantum Information and Matter, Caltech, Pasadena, CA 91125 USA*
[5] *Quandela, 10 Boulevard Thomas Gobert, 91120, Palaiseau, France*

Quantum computers promise considerable speed-ups with respect to their classical counterparts. However, the identification of the innately quantum features that enable these speed-ups is challenging. In the continuous-variable setting—a promising paradigm for the realisation of universal, scalable, and fault-tolerant quantum computing—contextuality and Wigner negativity have been perceived as two such distinct resources. Here we show that they are in fact equivalent for the standard models of continuous-variable quantum computing. While our results provide a unifying picture of continuous-variable resources for quantum speed-up, they also pave the way towards practical demonstrations of continuous-variable contextuality, and shed light on the significance of negative probabilities in phase-space descriptions of quantum mechanics.

With the onset of quantum information theory, the weirdness of quantum mechanics has transitioned from being a bug to being a feature, and the first demonstrations of quantum speedup have recently been achieved [1, 2], building on inherently nonclassical properties of physical systems. While entanglement is used daily for the calibration of current quantum experiments, it was originally perceived as a 'spooky action at a distance' by Einstein. This led him, Podolsky and Rosen (EPR) to speculate about the incompleteness of quantum mechanics [3] and the existence of a deeper theory over 'hidden' variables reproducing the predictions of quantum mechanics without its puzzling nonlocal aspects.

During the same period, Wigner was also looking for a more intuitive description of quantum mechanics, and he obtained a phase-space description akin to that of classical theory [4]. However, a major difference with the classical case was that the Wigner function—the quantum equivalent of a classical probability distribution over phase space—could display negative values. These 'negative probabilities' seemingly prevented a classical phase-space interpretation of quantum mechanics.

More than thirty years later, the seminal results of Bell [5, 6] and Kochen and Specker [7] ruled out the possibility of finding the underlying hidden-variable model for quantum mechanics envisioned by EPR, thus establishing nonlocality, and its generalisation contextuality, as fundamental properties of quantum systems.

At an intuitive level, contextuality and negativity of the Wigner function are properties of quantum states that seek to capture similar characteristics of quantum theory: the non-existence of a classical probability distribution that describes the outcomes of the measurements of a quantum system.

In more operational terms, contextuality is present whenever any hidden-variable description of the behaviour of a system is inconsistent with the basic assumptions that (i) all of its observable properties may be assigned definite values at all times, and (ii) jointly measuring compatible observables does not disturb these global value assignments, or, in other words, these assignments are context-independent. Aside from its foundational importance, contextuality has been increasingly identified as an essential ingredient for enabling a range of quantum-over-classical advantages in information processing tasks, which include the onset of universal quantum computing in certain computational models [8–12].

Similarly, the negativity of the Wigner function, or Wigner negativity for short, is also crucial for quantum computational speedup as quantum computations described by nonnegative Wigner functions can be simulated efficiently classically [13].

Importantly, quantum information can be encoded with discrete but also continuous variables (CV) [14], using continuous quantum degrees of freedom of physical systems such as position or momentum. The study of contextuality has mostly focused on the simpler discrete-variable setting [15–21]. In [22], it was shown that generalised contextuality is equivalent to the non-existence of a nonnegative quasiprobability representation. The caveat is that to check if a system is indeed contextual, one would have to consider all possible quasi-probability distributions. Focusing on one particular quasiprobability distribution, Howard *et al.* [9] showed that, for discrete-variable systems of odd prime-power dimension, negativity of the (discrete) Wigner function [23] corresponds to contextuality with respect to Pauli measurements. The equivalence was later generalised to odd dimensions in [24] and to qubit systems in [25, 26]. Under the hypothesis of noncontextuality, it has also been shown that the discrete Wigner function is the only possible quasiprobability representation for odd prime dimensions [27].

However, the EPR paradox [3] and the phase-space description derived by Wigner [4] were originally formulated for CV systems. Moreover, from a practical point-of-view, CV quantum systems are emerging as very promising candidates for implementing quantum informational and computational tasks [28–33] as they offer unrivalled possibilities for quantum error-correction [34, 35], deterministic generation of large-scale entangled states over millions of subsystems [36, 37] and reliable and efficient detection methods, such as homodyne or heterodyne detection [38, 39].

Since contextuality and Wigner negativity both seem to play a fundamental role as nonclassical features enabling quantum-over-classical advantages, a natural question arises in the CV setting: What is the precise relationship between contextuality and Wigner negativity?

Here we prove that contextuality and Wigner negativity are equivalent with respect to CV Pauli measurements, thus unifying the quantum quirks that prevented Einstein and Wigner from obtaining a classically intuitive description of quantum mechanics. We build on the recent extension of the sheaf-theoretic framework of contextuality [15] to the CV setting [40]. Note that this treatment of contextuality is a strict generalisation of the standard notion of Kochen–Specker contextuality [7, 41], extended to CV systems. Using this framework, we prove the equivalence between contextuality and Wigner negativity with respect to generalised position and momentum quadrature measurements, i.e. CV Pauli measurements. These are amongst the most commonly used measurements in CV quantum information, in particular in quantum optics [32, 42], and for defining the standard models of CV quantum computing [14, 34].

Phase space and Wigner function

We fix $M \in \mathbb{N}^*$ to be the number of qumodes, that is, M CV quantum systems. For a single qumode, the corresponding state space is the Hilbert space of square-integrable functions $L^2(\mathbb{R})$ and the total Hilbert space for all M qumodes is then $L^2(\mathbb{R})^{\otimes M} \cong L^2(\mathbb{R}^M)$. To each qumode, we associate the usual position and momentum operators. We write $\hat{q}_k$ and $\hat{p}_k$ the position and momentum operators of the k^{th} qumode. In the context of quantum optics, any linear combination of such operators is called a quadrature of the electromagnetic field [38]. We use this terminology in the rest of the article: any $\mathbb{R}$-linear combination of position and momentum operators is called a quadrature.

The Wigner representation of a quantum state in the Hilbert space $L^2(\mathbb{R}^M)$ is a function defined on the phase space $\mathbb{R}^{2M}$, which can be intuitively understood as a quantum version of the position and momentum phase space of a classical particle. We equip this phase space with a symplectic form denoted Ω: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{2M}$, $\Omega(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} \cdot J\boldsymbol{y}$ where $J = \left(\begin{smallmatrix} 0 & \mathbb{1}_M \\ -\mathbb{1}_M & 0 \end{smallmatrix}\right)$, in a given basis $(\boldsymbol{e}_k, \boldsymbol{f}_k)_{k=1}^M$ of $\mathbb{R}^{2M}$, which is therefore a symplectic basis for the phase space. We also equip $\mathbb{R}^{2M}$ with its usual scalar product denoted by $- \cdot -$.

A Lagrangian vector subspace is defined as a maximal isotropic subspace, that is, a maximal subspace on which the symplectic form Ω vanishes. For a symplectic space of dimension $2M$, Lagrangian subspaces are of dimension M. See [43] for a concise introduction to the symplectic structure of phase space and [44] for a detailed review.

To any $\boldsymbol{x} \in \mathbb{R}^{2M}$ we associate a quadrature operator as follows. Assume w.l.o.g. that $\boldsymbol{x} = \sum_k q_k \boldsymbol{e}_k + \sum_k p_k \boldsymbol{f}_k$, and put $\hat{\boldsymbol{x}} = \sum_{k=1}^M q_k \hat{q}_k + \sum_{k=1}^M p_k \hat{p}_k$, where the indices indicate on which qumode each operator acts. Then, it is straightforward to verify, using the canonical commutation relations, that $[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}] = i\Omega(\boldsymbol{x}, \boldsymbol{y})\hat{\mathbb{1}}$, i.e. the symplectic structure encodes the commutation relations of quadrature operators.

The elements of $\mathbb{R}^{2M}$ can also be associated to translations in phase space. Firstly, for any $s \in \mathbb{R}^M$, define the Weyl operators, acting on $L^2(\mathbb{R}^M)$, by $\hat{X}(s)\psi(t) = \psi(t-s)$ and $\hat{Z}(s)\psi(t) = e^{ist}\psi(t)$, for all $t \in \mathbb{R}^M$. Then, define the displacement operator for any $\boldsymbol{x} = (q, p) \in \mathbb{R}^M \times \mathbb{R}^M$ in the symplectic basis $(\boldsymbol{e}_k, \boldsymbol{f}_k)_{k=1}^M$ by $\hat{D}(\boldsymbol{x}) = e^{-i\frac{q \cdot p}{2}} \hat{X}(q)\hat{Z}(p)$, so that $[\hat{D}(\boldsymbol{x}), \hat{D}(\boldsymbol{y})] = e^{i\Omega(\boldsymbol{x}, \boldsymbol{y})}\hat{\mathbb{1}}$.

There are several equivalent ways of defining the Wigner function of a quantum state [45–47]. We follow the conventions adopted in [48]. The characteristic function $\Phi_\rho : \mathbb{R}^{2M} \to \mathbb{C}$ of a density operator ρ on $L^2(\mathbb{R}^M)$ is defined as $\Phi_\rho(\boldsymbol{x}) := \mathrm{Tr}(\rho \hat{D}(-\boldsymbol{x}))$. The Wigner function W_ρ of ρ is then defined as the symplectic Fourier transform of the characteristic function of ρ: $W_\rho(\boldsymbol{x}) := \mathrm{FT}[\Phi_\rho](J\boldsymbol{x})$. The Wigner function is a real-valued square-integrable function on $\mathbb{R}^{2M}$, and one can recover the probabilities for quadrature measurements from its marginals: if W is the Wigner function of a pure state $\psi \in L^2(\mathbb{R}^M)$ such that W is integrable on $\mathbb{R}^{2M}$, then identifying $\boldsymbol{x}$ with $(q, p) \in \mathbb{R}^M \times \mathbb{R}^M$ in the same basis $(\boldsymbol{e}_k, \boldsymbol{f}_k)_{k=1}^M$ as before,

$$\frac{1}{(\sqrt{2\pi})^M} \int_{\mathbb{R}^M} W(q,p)\,\mathrm{d}p = |\psi(q)|^2, \tag{1}$$

$$\frac{1}{(\sqrt{2\pi})^M} \int_{\mathbb{R}^M} W(q,p)\,\mathrm{d}q = |\mathrm{FT}[\psi](p)|^2. \tag{2}$$

In general, if $\boldsymbol{x} \in \mathbb{R}^{2M}$ describes an arbitrary quadrature, the probability of obtaining an outcome x in $E \subseteq \mathbb{R}$ when measuring the quadrature $\hat{\boldsymbol{x}}$ is

$$\mathrm{Prob}[x \in E | \rho] = \frac{1}{(\sqrt{2\pi})^M} \int_A W_\rho(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{3}$$

where $A = \left\{ \boldsymbol{y} \in \mathbb{R}^{2M} \mid \boldsymbol{y} \cdot \boldsymbol{x} \in E \right\}$. This corresponds to marginalising the Wigner function over the axes orthogonal to $\boldsymbol{x}$. If the Wigner function only takes nonnegative values, it can therefore be interpreted as a simultaneous probability distribution for position and momentum measurements (and in general, any quadrature obtained as a linear combination of these).

Continuous-variable contextuality

In what follows, we use the contextuality formalism from [40], which is the extension of [15] to the CV setting. We refer to the Supplemental Material or Refs. [49, 50] for an introduction to this formalism and the associated tools of measure theory.

Measurement scenario.— In order to define 'contextuality' in a CV experiment, we need an abstract description of the experiment, called a measurement scenario, which is defined by a triple $\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle$ as follows: in a given setup, experimenters can choose different measurements to perform on a physical system. Each possible measurement is labelled and $\mathcal{X}$ is the corresponding (possibly infinite) set of measurement labels. Several compatible measurements can be implemented together (for instance, measurements on space-like separated systems). Maximal sets of compatible measurements define a context, and $\mathcal{M}$ is the set of all such contexts. For a measurement labelled by $\boldsymbol{x} \in \mathcal{X}$, the corresponding outcome space is $\boldsymbol{\mathcal{O}}_{\boldsymbol{x}} = \langle \mathcal{O}_{\boldsymbol{x}}, \mathcal{F}_{\boldsymbol{x}} \rangle$, which is a measurable space with an underlying set $\mathcal{O}_{\boldsymbol{x}}$ and its associated Lebesgue σ-algebra $\mathcal{F}_{\boldsymbol{x}}$. The collection of all outcome spaces is denoted $\boldsymbol{\mathcal{O}} = (\boldsymbol{\mathcal{O}}_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{X}}$. For various measurements labelled by elements of a set $U \subseteq X$, the corresponding joint outcome space is denoted $\boldsymbol{\mathcal{O}}_U := \prod_{\boldsymbol{x} \in U} \boldsymbol{\mathcal{O}}_{\boldsymbol{x}}$. In this article, we consider the following measurement scenario:

Definition 1.—The quadrature measurement scenario $\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle_{\text{quad}}$ is defined as follows: (i) the set of measurement labels is the symplectic phase space $\mathcal{X} := \mathbb{R}^{2M}$; (ii) the contexts are Lagrangian subspaces of $\mathbb{R}^{2M}$, so that the set of contexts $\mathcal{M}$ is the set of all Lagrangian subspaces of $\mathcal{X}$; (iii) for each $\boldsymbol{x} \in \mathcal{X}$, the corresponding outcome space is $\boldsymbol{\mathcal{O}}_{\boldsymbol{x}} := \langle \mathbb{R}, \sigma \rangle$ (σ being the Lebesgue sigma algebra of $\mathbb{R}$).

$\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle_{\text{quad}}$ is to be interpreted as follows: given a quantum state ρ, the measurement corresponding to the label $\boldsymbol{x} \in \mathcal{X}$ is given by the measurement of the corresponding quadrature $\hat{\boldsymbol{x}}$ of the state, while contexts correspond to maximal sets of commuting quadratures. This scenario consists in a continuum of possible measurements, each of which corresponds to a quadrature operator with continuous spectrum (see Fig. 1).

Empirical model.— While measurement scenarios describe experimental setups, empirical models capture in a precise way the probabilistic behaviours that may arise upon performing measurements on physical systems. In practice, these amount to tables of normalised frequencies of outcomes gathered among various runs of the experiment, or to tables of predicted outcome probabilities obtained by analytical calculation. Formally:

Definition 2.—An empirical model on a measurement scenario $\langle \mathcal{X}, \mathcal{M}, \mathcal{O} \rangle$ is a family $e = (e_C)_{C \in \mathcal{M}}$, where e_C is a probability measure on the space $\boldsymbol{\mathcal{O}}_C$ for each context $C \in \mathcal{M}$.

Informally, the empirical data is noncontextual whenever local descriptions (within a valid context) can be glued together consistently so that it can be described by a global probability measure (over all contexts).

Definition 3.—An empirical model $e = (e_C)_{C \in \mathcal{M}}$ on a $\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle$ is noncontextual if there exists a probability measure p on the space $\boldsymbol{\mathcal{O}}_{\mathcal{X}}$ such that marginalising p on a context gives back the empirical prediction i.e. $p|_C = e_C$ for every $C \in \mathcal{M}$.

Noncontextuality is equivalent to the existence of a deterministic hidden-variable model (HVM) [40].

Definition 4.—A HVM on a measurement scenario $\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle$ is a tuple $\langle \boldsymbol{\Lambda}, p, (k_C)_{C \in \mathcal{M}} \rangle$ where: (i) $\boldsymbol{\Lambda} = \langle \Lambda, \mathcal{F}_\Lambda \rangle$ is the measurable space of hidden variables; (ii) p is a probability distribution on $\boldsymbol{\Lambda}$; (iii) for each context $C \in \mathcal{M}$, k_C is a probability kernel between the measurable spaces $\boldsymbol{\Lambda}$ and $\boldsymbol{\mathcal{O}}_C$, i.e. k_C is a measurable function over $\boldsymbol{\Lambda}$ and a probability measure over $\boldsymbol{\mathcal{O}}_C$.

Determinism for the HVM is further ensured by requiring that each hidden variable gives a predetermined outcome. That is, for all contexts $C \in \mathcal{M}$ and for every $\lambda \in \Lambda$, $k_C(\lambda, -) = \delta_{\boldsymbol{x}}$ is a Dirac measure at some $\boldsymbol{x} \in \mathcal{O}_C$.

The space $\boldsymbol{\mathcal{O}}_{\mathcal{X}}$ can be thought of as a space of hidden variables, while the probability measure p provides probabilistic information about them. Hidden variables are supposed to provide an underlying description of the physical world at perhaps a more fundamental level than the empirical-level description via the quantum state. The motivation is that hidden variables could explain away some of the non-intuitive aspects of the empirical predictions of quantum mechanics, which would then arise from an incomplete knowledge of the true state of a system rather than being fundamental features.

Since we consider experiments arising from quadrature measurements of a quantum system ρ, we restrict our attention to empirical models $e = (e_C)_{C \in \mathcal{M}}$ reproducing the Born rule, which we refer to as quantum empirical models. We will use the notation $e^\rho = (e^\rho_C)_{C \in \mathcal{M}}$ to make explicit the dependence in the state ρ. If e^ρ is noncontextual in $\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle_{\text{quad}}$, we say that the density operator ρ is noncontextual for quadrature measurements.

In $\langle \mathcal{X}, \mathcal{M}, \boldsymbol{\mathcal{O}} \rangle_{\text{quad}}$, for each context $C \in \mathcal{M}$, the set $\mathcal{O}_C = \prod_{\boldsymbol{x} \in C} \mathbb{R}$ can be seen as the set of functions from C to $\mathbb{R}$ with the corresponding product σ-algebra. These functions are called local value assignments. In contrast, functions $\mathcal{X} \to \mathbb{R}$ which assign a tentative outcome to all quadratures simultaneously are called global value assignments. Contextuality then expresses the tension which arises when trying to explain different experimental predictions across distinct contexts (local value assignments) in terms of global value assignments.

Linearity of value assignments.— Before connecting the notions of contextuality and negativity of the Wigner function in $\langle \mathcal{X}, \mathcal{M}, \mathcal{O} \rangle_{\text{quad}}$, we must resolve the following mismatch: the Wigner function of a density operator ρ is a quasiprobability distribution over $\mathcal{X} = \mathbb{R}^{2M}$, while a density operator ρ which is noncontextual for quadrature measurements corresponds by Definition 3 to a global probability measure over the set $\mathcal{O}_{\mathcal{X}}$ of global assignments, which can be seen in this case as the set of functions $\mathcal{X} = \mathbb{R}^{2M} \to \mathbb{R}$. In general, the latter is much larger than the former.

To solve this issue, we show that we can restrict to linear global value assignments w.l.o.g. so that $\mathcal{O}_{\mathcal{X}}$ can be replaced by $\mathcal{X}^*$, the linear dual of $\mathcal{X}$. Since $\mathcal{X}^*$ is isomorphic to $\mathcal{X}$, this allows us to resolve the mismatch:

Proposition 1.—If $M \geqslant 2$, global value assignments can w.l.o.g. be taken to be linear functions $\mathcal{X} \to \mathbb{R}$, and the set of global value assignments then forms an $\mathbb{R}$-linear space of dimension $2M$, namely $\mathcal{X}^*$.

We refer to Appendix A for the proof. Therefore w.l.o.g. for any $U \subset \mathcal{X}$, we can restrict $\mathcal{O}_U$ to be the set of linear functions from $U \to \mathbb{R}$ that extend to $\mathbb{R}$-linear functions on the vector space generated by U. Thus, an empirical model will be a collection of probability measures on C^* for each $C \in \mathcal{M}$.

Equivalence

We may now prove our main result, i.e. the equivalence between contextuality and Wigner negativity in the quadrature measurement scenario $\langle \mathcal{X}, \mathcal{M}, \mathcal{O} \rangle_{\text{quad}}$:

Theorem 1.—For any $M \geqslant 2$, a density operator ρ on $L^2(\mathbb{R}^M)$ is noncontextual for quadrature measurements if and only if its Wigner function W_ρ is both integrable and nonnegative.

We refer to Appendix B for the full proof.

Sketch of proof.—Using Definition 3 and Proposition 1, a density operator ρ which is noncontextual for quadrature measurements corresponds to a probability measure p on the space $\mathcal{X}^*$ describing the outcomes of quadrature measurements on ρ. We show that the Fourier transform of p must be equal to the characteristic function Φ_ρ. Hence, p and W_ρ have the same Fourier transform, which gives $w_p = W_\rho$ and thus $W_\rho \geq 0$, since w_p is the density of a probability measure. Conversely, if the Wigner function is assumed to be integrable and nonnegative, we obtain the outcome probabilities for quadrature measurements by marginalising along the correct axes. $\square$

Experimental setup.—The quadrature measurement scenario requires measuring any linear combination of multimode position and momentum operators, e.g. $\hat{q}_1 + 2\hat{p}_1 + 5\hat{q}_2$. The measurement setup is represented in Fig. 1.

To do so experimentally, we first apply phase-shift operators $\hat{R}$ for each individual qumode to obtain the right quadrature for each mode. Then, we apply CZ gates of the form $e^{ig\hat{q}_k\hat{q}_l}$ for $g \in \mathbb{R}$ to pairs of qumodes k and l in order to sum them. This permits the construction of the desired linear combination, which is stored in one quadrature of a qumode. The measurement can then be implemented with standard homodyne detection, which consists in a Gaussian measurement of a quadrature of the field, by mixing the state on a balanced beam splitter (dashed line) with a strong coherent state (local osccilator LO). The intensities of both output arms are measured with photodiode detectors and their difference yields a value proportional to a quadrature $\hat{x}_\phi := (\cos\phi)\hat{q} + (\sin\phi)\hat{p}$ of the input qumode, depending on the phase ϕ of the local oscillator. All of these steps can be implemented experimentally with current optical technology [45, 51].

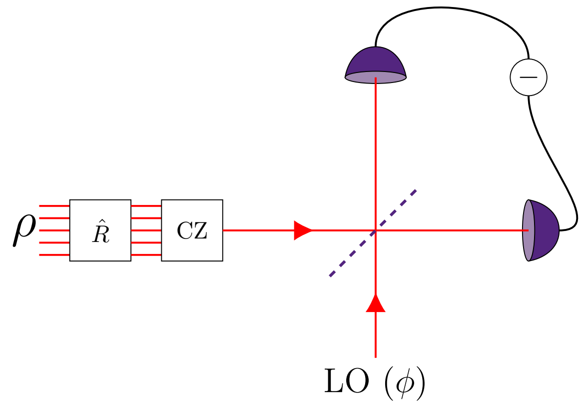

Figure 1. Experimental protocol corresponding to the quadrature measurement scenario $\langle \mathcal{X}, \mathcal{M}, \mathcal{O} \rangle_{\text{quad}}$.

Discussion.—We have shown that Wigner negativity is equivalent to contextuality with respect to CV quantum measurements which may be realised using homodyne detection, a standard detection method in CV [36], and the basis of several computational models in CV quantum information [52–55].

From a practical perspective, this implies that contextuality is a necessary resource for achieving a computational advantage within the standard model of CV quantum computation [14]. Like in the discrete-variable case [9], CV contextuality supplies the necessary ingredients for CV quantum computing.

From a foundational perspective, the failure of a local hidden-variable model describing quantum mechanical predictions, as enlightened by Bell regarding the EPR paradox, is very closely related to the impossibility of a nonnegative phase-space distribution, as described by Wigner. Hence, our result implies that the negativity of phase-space distributions can be cast as an obstruction to the existence of a noncontextual hidden-variable model.

The EPR state [3] describes a CV state that has a nonnegative Wigner function and still violates a Bell inequality [56]. This is possible since it necessitates parity operator measurements that do not have a nonnegative

Wigner representation [22], and thus is not in contradiction with our result. Indeed, our quadrature measurement scenario is nonnegatively represented in phase space: since homodyne detection (and quadrature measurements in general) is a Gaussian measurement, any possible quantum advantage is due to Wigner negativity being present before the detection setup.

Our results open up a number of future research directions. Firstly, our present argument requires considering a measurement scenario that comprises an uncountable family of measurement labels (the entire phase space $\mathcal{X} = \mathbb{R}^{2M}$). From an experimental perspective, it is crucial to wonder what happens if we restrict to a finite family of measurement labels and see whether we can derive a robust version of this theorem.

Another question concerns the link between quantifying contextuality and quantifying Wigner negativity. Quantifying contextuality for CV systems is possible via semidefinite relaxation [57]. Also, there exist various measures of Wigner negativity [58, 59]. In particular, witnesses for Wigner negativity have been introduced in [60], whose violation gives a lower bound on the distance to the set of states with nonnegative Wigner function. It would be highly desirable to establish a precise and quantified link between these different measures of nonclassicality.

This equivalence may also be useful in better understanding the problem of characterising those quantum-mechanical states whose Wigner function is nonnegative. This is a notoriously thorny issue when one considers mixed states, and, to the authors' knowledge, progress has mostly stalled since the 90s [48, 61–63]. Strong mathematical tools are being developed to detect contextuality from the theoretical description of a state [10, 64–69], although much work remains in applying them to CV and understanding their relation to machinery previously developed to tackle the positivity question.

On the practical side, our result paves the way for surprisingly simple demonstrations of nonclassicality. Contextual states are typically associated with violated Bell-like inequalities—although this result has only been formally proven in the case of a finite number of measurement settings [40], and needs to be generalised to CV. In principle, this means that one should be able to violate such an inequality with a setup as simple as a single photon and a heterodyne detection, necessitating only a single beamsplitter. The existence of such a genuinely continuous Bell inequality has been elusive, since previously-observed violations amount to encodings of discrete-variable inequalities in CV [70–76].

Acknowledgements. The authors would like to thank M. Howard, S. Mansfield and T. Douce for enlightening discussions. They are also grateful to D. Markham, E. Kashefi, E. Diamanti and F. Grosshans for their mentorship and the wonderful group they have created at LIP6. UC acknowledges interesting discussions with S. Mehraban and J. Preskill. UC acknowledges funding provided by the Institute for Quantum Information and Matter, a National Science Foundation Physics Frontiers Center (NSF Grant PHY-1733907). RIB was supported by the Agence Nationale de la Recherche VanQuTe project (ANR-17-CE24-0035).

Related work. An analogous, independent proof of our main theorem was subsequently derived in [77].

Appendix A: Proof of Proposition 1.—Let $L \subseteq \mathcal{X}$ be a Lagrangian subspace and let e^ρ be a quantum empirical model. For $U \in \mathcal{F}_L$ a Lebesgue measurable set of functions $L \to \mathbb{R}$ and for $\boldsymbol{x} \in L$, let $\pi_{\boldsymbol{x}}(U) := \{f(\boldsymbol{x}) \mid f \in U\} \subseteq \mathbb{R}$. We start by showing the following result:

Lemma 1.— Let $U \in \mathcal{F}_L$ be a Lebesgue measurable set of functions $L \to \mathbb{R}$ such that $\pi_{\boldsymbol{x}}(U)$ is distinct from $\mathbb{R}$ for a finite number of $\boldsymbol{x} \in L$. Then there exists a subset U_{lin} of linear functions $L \to \mathbb{R}$ such that for all $\boldsymbol{x} \in L$, $\pi_{\boldsymbol{x}}(U_{\text{lin}}) \subseteq \pi_{\boldsymbol{x}}(U)$ and $e^\rho_L(U_{\text{lin}}) = e^\rho_L(U)$.

Proof.—First, let $(\boldsymbol{e}_k)_{k=1,\ldots,M}$ be a basis of $L \cong \mathbb{R}^M$. Let P be the joint spectral measure of $\{\hat{\boldsymbol{e}}_{\boldsymbol{1}}, \ldots, \hat{\boldsymbol{e}}_{\boldsymbol{M}}\}$. For any $\boldsymbol{y} \in L$, define the function

$$f_{\boldsymbol{y}} : L \longrightarrow \mathbb{R} \tag{4}$$

$$\boldsymbol{x} \longmapsto \boldsymbol{x} \cdot \boldsymbol{y} \,, \tag{5}$$

where $- \cdot -$ is the usual Euclidean scalar product on $L \cong \mathbb{R}^M$. For any $\boldsymbol{x} \in L$, $P_{\hat{\boldsymbol{x}}}$ is the push-forward of P by the measurable function $f_{\boldsymbol{x}}$ by the functional calculus on M commuting observables (see Supplemental Material for a concise introduction to measure theory; this includes Ref. [78] for the product σ-algebra).

Then,

$$\operatorname{Tr}\left(\rho \prod_{\boldsymbol{x} \in L} P_{\hat{\boldsymbol{x}}} \circ \pi_{\boldsymbol{x}}(U)\right) = \operatorname{Tr}\left(\rho \prod_{\boldsymbol{x} \in L} P\left(f_{\boldsymbol{x}}^{-1}\left(\pi_{\boldsymbol{x}}(U)\right)\right)\right) \tag{6}$$

$$= \operatorname{Tr}\left(\rho P\left(\bigcap_{\boldsymbol{x} \in L} f_{\boldsymbol{x}}^{-1}(\pi_{\boldsymbol{x}}(U))\right)\right) \tag{7}$$

with

$$\bigcap_{\boldsymbol{x} \in L} f_{\boldsymbol{x}}^{-1}(\pi_{\boldsymbol{x}}(U)) = \{\boldsymbol{y} \in L \mid \forall \boldsymbol{x} \in L,\, \boldsymbol{x} \cdot \boldsymbol{y} \in \pi_{\boldsymbol{x}}(U)\}\,. \tag{8}$$

Now define

$$U_{\text{lin}} := \left\{ \begin{array}{l} L \longrightarrow \mathbb{R} \\ \boldsymbol{x} \longmapsto \boldsymbol{x} \cdot \boldsymbol{y} \end{array} \,\middle|\, \boldsymbol{y} \in \bigcap_{\boldsymbol{x} \in L} f_{\boldsymbol{x}}^{-1}(\pi_{\boldsymbol{x}}(U)) \right\}. \tag{9}$$

By construction,

$$\bigcap_{\boldsymbol{x}\in L} f_{\boldsymbol{x}}^{-1}(\pi_{\boldsymbol{x}}(U_{\mathrm{lin}})) = \bigcap_{\boldsymbol{x}\in L} f_{\boldsymbol{x}}^{-1}\left(\{\boldsymbol{x}\cdot\boldsymbol{y} \mid \boldsymbol{y}\in L \text{ s.t. } \forall \boldsymbol{z}\in L,\, \boldsymbol{y}\cdot\boldsymbol{z}\in\pi_{\boldsymbol{z}}(U)\}\right) \tag{10}$$

$$= \bigcap_{\boldsymbol{x}\in L} \{\boldsymbol{\alpha}\in L \mid \boldsymbol{x}\cdot\boldsymbol{\alpha} = \boldsymbol{x}\cdot\boldsymbol{y} \text{ with } \boldsymbol{y}\in L \text{ s.t. } \forall \boldsymbol{z}\in L,\, \boldsymbol{y}\cdot\boldsymbol{z}\in\pi_{\boldsymbol{z}}(U)\} \tag{11}$$

$$= \{\boldsymbol{\alpha}\in L \mid \forall \boldsymbol{x}\in L,\, \boldsymbol{x}\cdot\boldsymbol{\alpha} = \boldsymbol{x}\cdot\boldsymbol{y} \text{ with } \boldsymbol{y}\in L \text{ s.t. } \forall \boldsymbol{z}\in L,\, \boldsymbol{y}\cdot\boldsymbol{z}\in\pi_{\boldsymbol{z}}(U)\} \tag{12}$$

$$= \{\boldsymbol{\alpha}\in L \mid \forall \boldsymbol{z}\in L,\, \boldsymbol{\alpha}\cdot\boldsymbol{z}\in\pi_{\boldsymbol{z}}(U)\} \tag{13}$$

$$= \bigcap_{\boldsymbol{x}\in L} f_{\boldsymbol{x}}^{-1}(\pi_{\boldsymbol{x}}(U))\,, \tag{14}$$

where (13) follows from that fact that $(\forall \boldsymbol{x}\in L,\, \boldsymbol{x}\cdot\boldsymbol{\alpha} = \boldsymbol{x}\cdot\boldsymbol{y})$ implies $\boldsymbol{\alpha}=\boldsymbol{y}$. Also for all $\boldsymbol{x}\in L$, $\pi_{\boldsymbol{x}}(U_{\mathrm{lin}})\subseteq\pi_{\boldsymbol{x}}(U)$ so that we are indeed reproducing all value assignments from linear functions of U. Then, by the Born rule,

$$e_L^\rho(U_{\mathrm{lin}}) = \mathrm{Tr}\left(\rho\prod_{\boldsymbol{x}\in L} P_{\hat{\boldsymbol{x}}}\circ\pi_{\boldsymbol{x}}(U_{\mathrm{lin}})\right) \tag{15}$$

$$= \mathrm{Tr}\left(\rho\prod_{\boldsymbol{x}\in L} P\left(f_{\boldsymbol{x}}^{-1}\left(\pi_{\boldsymbol{x}}(U_{\mathrm{lin}})\right)\right)\right) \tag{16}$$

$$= \mathrm{Tr}\left(\rho P\left(\bigcap_{\boldsymbol{x}\in L} f_{\boldsymbol{x}}^{-1}\left(\pi_{\boldsymbol{x}}(U_{\mathrm{lin}})\right)\right)\right) \tag{17}$$

$$= \mathrm{Tr}\left(\rho P\left(\bigcap_{\boldsymbol{x}\in L} f_{\boldsymbol{x}}^{-1}\left(\pi_{\boldsymbol{x}}(U)\right)\right)\right) \tag{18}$$

$$= \mathrm{Tr}\left(\rho\prod_{\boldsymbol{x}\in L} P_{\hat{\boldsymbol{x}}}\left(\pi_{\boldsymbol{x}}(U)\right)\right) \tag{19}$$

$$= e_L^\rho(U). \tag{20}$$

□

We are now in position to prove Proposition 1.

The sheaf-theoretic framework for contextuality describes value assignments as a sheaf $\mathscr{E}$ where $\mathscr{E}(U)$ is the set of value assignments for the measurement labels in U, which can be viewed as a set of functions $U\to\mathbb{R}$. For any Lagrangian $L\in\mathcal{M}$, there is a restriction map $\mathscr{E}(\mathcal{X})\to\mathscr{E}(L): f\mapsto f|_L$ that simply restricts the domain of any function from $\mathcal{X}$ to L. Then $\mathscr{E}(L)$ must coincide with the set of possible value assignments $\mathcal{O}_L$.

By Lemma 1, $\mathscr{E}(L)$ consists in linear functions $L\to\mathbb{R}$ so that the set of global value assignments $\mathscr{E}(\mathcal{X})$ contains only functions $\mathcal{X}\to\mathbb{R}$ whose restriction to any Lagrangian subspace is $\mathbb{R}$-linear. Then, following [24, Lemma 1] (the Lemma is proven for the discrete phase-space $\mathbb{Z}_d^M\times\mathbb{Z}_d^M$ but its proof extends directly to $\mathbb{R}^M\times\mathbb{R}^M$), we conclude that if $M\geqslant 2$, $\mathscr{E}(\mathcal{X})$ contains only $\mathbb{R}$-linear functions $\mathcal{X}\to\mathbb{R}$, i.e. $\mathscr{E}(\mathcal{X})=\mathcal{X}^*$, where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$. □

Appendix B: Proof of Theorem 1.—The proof proceeds by showing both directions of the equivalence. We make use of standard properties of the Wigner function, which are reviewed in the Supplemental Material.

$(\implies)$ It follows from the definition of the Wigner function that, for $\boldsymbol{x}\in\mathcal{X}$, $\Phi_\rho(\boldsymbol{x}) = \mathrm{FT}^{-1}[W_\rho](-J\boldsymbol{x}) = \mathrm{FT}[W_\rho](J\boldsymbol{x})$. On the other hand, fix a noncontextual quantum empirical model e^ρ satisfying the Born rule associated to ρ and the quadrature measurement scenario $\langle\mathcal{X},\mathcal{M},\boldsymbol{\mathcal{O}}\rangle_{\mathrm{quad}}$. By Proposition 1, the set of global value assignments forms an $\mathbb{R}$-linear space of dimension $2M$, namely $\mathcal{X}^*$, which is isomorphic to its dual $\mathcal{X}$. Hence, by [40, Theorem 1] (restated in the Supplemental Material as Proposition 13) we have a deterministic HVM $\langle\boldsymbol{\mathcal{X}},p,(k_C)_{C\in\mathcal{M}}\rangle$ for e^ρ (see Definition 4), where $\boldsymbol{\mathcal{X}}=\langle\mathcal{X},\mathcal{F}_\mathcal{X}\rangle$ with $\mathcal{F}_\mathcal{X}$ the usual σ-algebra of $\mathcal{X}=\mathbb{R}^{2M}$. Moreover:

$$\Phi_\rho(\boldsymbol{x}) = \mathrm{Tr}\left(\hat{D}(-\boldsymbol{x})\rho\right) \tag{21}$$

$$= \mathrm{Tr}\left(\rho\int_{\lambda\in\mathbb{R}} e^{-i\lambda}\,\mathrm{d}P_{\hat{J}\boldsymbol{x}}(\lambda)\right) \tag{22}$$

$$= \int_{\mathbb{R}} e^{-i\lambda}\,\mathrm{d}p^\rho_{\hat{J}\boldsymbol{x}}(\lambda) \tag{23}$$

$$= \int_{\mathcal{X}} e^{-iJ\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{d}p(\boldsymbol{y}) \tag{24}$$

$$= \mathrm{FT}[p](J\boldsymbol{x}), \tag{25}$$

where the second line comes from the spectral theorem; the third line by letting $p^\rho_{\hat{J}\boldsymbol{x}}(E)=\mathrm{Tr}(P_{\hat{J}\boldsymbol{x}}(E)\rho)$ and the fact that the integral and the trace may be inverted by the definition of the integral with respect to the spectral measure [79]; the fourth line via the push-forward of measures; and the last line comes the definition of the Fourier transform of a measure.

Since the characteristic function Φ_ρ is square-integrable, we can apply [80, Lemma 1.1] (restated in the Supplemental Material as Lemma 7). Then, the measure p must have a density $w_p\in L^2(\mathbb{R}^M)$. As a result, for all $x\in\mathcal{X}$, $\mathrm{FT}[w_p](\boldsymbol{x}) = \mathrm{FT}[p](\boldsymbol{x}) = \Phi_\rho(-J\boldsymbol{x}) = \mathrm{FT}[W_\rho](\boldsymbol{x})$ and since w_p and W_ρ are both in $L^2(\mathcal{X})$ on which the Fourier transform is unitary, it must hold that $w_p=W_\rho$ d$\boldsymbol{x}$-almost everywhere. w_p is the density of a probability measure, so it follows that both functions must be almost everywhere nonnegative. Because the Wigner function is a continuous function from $\mathcal{X}$ to $\mathbb{R}$ [46], W_ρ must be nonnegative.

$(\impliedby)$ Conversely, the Wigner function provides the correct marginals for the quadratures and can be seen as a global probability density on phase space when it is nonnegative. Via the equivalence demonstrated in the first part of the proof (namely that the density of p is almost everywhere the Wigner function), the idea is to show that $\langle \boldsymbol{\mathcal{X}}, W_\rho \mathrm{d}\boldsymbol{x}, (\tilde{k}_L)_{L\in\mathcal{M}}\rangle$ is a valid deterministic HVM (see Definition 4) that reproduces the empirical predictions, where $\tilde{k}_L(\boldsymbol{x}, U) \coloneqq (\sqrt{2\pi})^{-M}\delta_{(\boldsymbol{x}\cdot -)|_L}(U)$.

For any $\boldsymbol{x} \in \mathcal{X}$, there is a special orthogonal and symplectic transformation S such that $\boldsymbol{x} = \|\boldsymbol{x}\| S\boldsymbol{e}_1$. For any $U \in \mathcal{F}_{\boldsymbol{x}}$,

$$
\begin{aligned}
e^\rho_{\boldsymbol{x}}(U) &= \mathrm{Tr}(P_{\hat{\boldsymbol{x}}} \circ \pi_{\boldsymbol{x}}(U)\rho) && (26)\\
&= \frac{1}{(\sqrt{2\pi})^M}\int_{\|\boldsymbol{x}\|^{-1}\pi_{\boldsymbol{x}}(U)\times\mathbb{R}^{2M-1}} W_\rho(S\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} && (27)\\
&= \frac{1}{(\sqrt{2\pi})^M}\int_{(\boldsymbol{e}_1\cdot -)^{-1}(\|\boldsymbol{x}\|^{-1}\cdot\pi_{\boldsymbol{x}}(U))} W_\rho(S\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} && (28)\\
&= \frac{1}{(\sqrt{2\pi})^M}\int_{(\|\boldsymbol{x}\|\boldsymbol{e}_1\cdot S^{-1} -)^{-1}(\pi_{\boldsymbol{x}}(U))} W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} && (29)\\
&= \frac{1}{(\sqrt{2\pi})^M}\int_{(\|\boldsymbol{x}\|S\boldsymbol{e}_1\cdot -)^{-1}(\pi_{\boldsymbol{x}}(U))} W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} && (30)\\
&= \frac{1}{(\sqrt{2\pi})^M}\int_{(\boldsymbol{x}\cdot -)^{-1}(\pi_{\boldsymbol{x}}(U))} W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, && (31)
\end{aligned}
$$

where we have used the symplectic covariance of the Wigner function (restated as Lemma 9 in the Supplemental Material) and the fact that the Jacobian change of variable in (29) is 1. By definition of $\tilde{k}$, for all $\boldsymbol{x} \in \mathcal{X}$ and all $U \in \mathcal{F}_{\boldsymbol{x}}$:

$$
\begin{aligned}
&\int_{\boldsymbol{z}\in\mathcal{X}} \tilde{k}_{\{\boldsymbol{x}\}}(\boldsymbol{z}, U) W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} && (32)\\
&\quad= \frac{1}{(\sqrt{2\pi})^M}\int_{\boldsymbol{z}\in\mathcal{X}} \delta_{(\boldsymbol{z}\cdot -)|_{\{\boldsymbol{x}\}}}(U) W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}. && (33)
\end{aligned}
$$

$U \in \mathcal{F}_{\boldsymbol{x}}$ consists of functions $\{\boldsymbol{x}\} \to \mathbb{R}$, which therefore extend to linear functions on the subspace generated by $\boldsymbol{x}$, so:

$$
\begin{aligned}
\{\boldsymbol{z}\in\mathcal{X} \mid (\boldsymbol{z}\cdot -)|_{\{\boldsymbol{x}\}} \in U\} &= \{\boldsymbol{z}\in\mathcal{X} \mid (\boldsymbol{z}\cdot\boldsymbol{x}) \in \pi_{\boldsymbol{x}}(U)\} && (34)\\
&= (-\cdot\boldsymbol{x})^{-1}(\pi_{\boldsymbol{x}}(U)). && (35)
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\int_{\boldsymbol{z}\in\mathcal{X}} \tilde{k}_{\{\boldsymbol{x}\}}(\boldsymbol{z}, U) W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} &= \frac{1}{(\sqrt{2\pi})^M}\int_{(-\cdot\boldsymbol{x})^{-1}(\pi_{\boldsymbol{x}}(U))} W_\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} && (36)\\
&= e^\rho_{\boldsymbol{x}}(U), && (37)
\end{aligned}
$$

While we have verified the calculation only for $e^\rho_{\boldsymbol{x}}(\mathrm{U})$ for $U \in \mathcal{F}_{\boldsymbol{x}}$, the same computation can be carried out for $e^\rho_L(U)$ for a Lagrangian subspace L and $U \in \mathcal{F}_L$ to retrieve the joint probability distributions from the HVM $\langle \boldsymbol{\mathcal{X}}, W_\rho \mathrm{d}\boldsymbol{x}, (\tilde{k}_L)_{L\in\mathcal{M}}\rangle$, using Proposition 1 to restrict the elements of U to linear functions. □

* robert.booth@ed.ac.uk

† uchabaud@caltech.edu

‡ pe.emeriau@quandela.com

[1] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. Brandao, D. A. Buell, *et al.*, Quantum supremacy using a programmable superconducting processor, Nature **574**, 505 (2019).

[2] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, *et al.*, Quantum computational advantage using photons, Science **370**, 1460 (2020).

[3] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Physical Review **47**, 777 (1935).

[4] E. Wigner, On the quantum correction for thermodynamic equilibrium, Physical Review **40**, 749 (1932).

[5] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics Physique Fizika **1**, 195 (1964), publisher: American Physical Society.

[6] J. S. Bell, On the problem of hidden variables in quantum mechanics, Reviews of Modern Physics **38**, 447 (1966).

[7] S. Kochen and E. P. Specker, The problem of hidden variables in quantum mechanics, in *The logico-algebraic approach to quantum mechanics* (Springer, 1975) pp. 293–328.

[8] R. Raussendorf, Contextuality in measurement-based quantum computation, Physical Review A **88**, 022322 (2013).

[9] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, Nature **510**, 351 (2014).

[10] S. Abramsky, R. S. Barbosa, and S. Mansfield, Contextual fraction as a measure of contextuality, Physical Review Letters **119**, 050504 (2017).

[11] J. Bermejo-Vega, N. Delfosse, D. E. Browne, C. Okay, and R. Raussendorf, Contextuality as a resource for models of quantum computation with qubits, Physical Review Letters **119**, 120505 (2017).

[12] S. Abramsky, R. S. Barbosa, N. de Silva, and O. Zapata, The quantum monad on relational structures, in *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017)*, Leibniz International Proceedings in Informatics (LIPIcs), Vol. 83, edited by K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin (Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017) pp. 35:1–35:19.

[13] A. Mari and J. Eisert, Positive Wigner functions render classical simulation of quantum computation efficient, Physical Review Letters **109**, 230503 (2012).

[14] S. Lloyd and S. L. Braunstein, Quantum computation over continuous variables, in *Quantum Information with Continuous Variables* (Springer, 1999) pp. 9–17.

[15] S. Abramsky and A. Brandenburger, The sheaf-theoretic structure of non-locality and contextuality, New Journal of Physics **13**, 113036 (2011).

[16] A. Cabello, S. Severini, and A. Winter, Graph-theoretic approach to quantum correlations, Physical Review Letters **112**, 040401 (2014).

[17] R. W. Spekkens, Contextuality for preparations, transformations, and unsharp measurements, Physical Review A **71**, 052108 (2005).

[18] X. Zhang, M. Um, J. Zhang, S. An, Y. Wang, D.-l. Deng, C. Shen, L.-M. Duan, and K. Kim, State-independent experimental test of quantum contextuality with a single trapped ion, Physical Review Letters **110**, 070401 (2013).

[19] Y. Hasegawa, R. Loidl, G. Badurek, M. Baron, and H. Rauch, Quantum contextuality in a single-neutron optical experiment, Physical Review Letters **97**, 230401 (2006).

[20] H. Bartosik, J. Klepp, C. Schmitzer, S. Sponar, A. Cabello, H. Rauch, and Y. Hasegawa, Experimental test of quantum contextuality in neutron interferometry, Physical Review Letters **103**, 040403 (2009).

[21] G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt, and C. F. Roos, State-independent experimental test of quantum contextuality, Nature **460**, 494 (2009).

[22] R. W. Spekkens, Negativity and contextuality are equivalent notions of nonclassicality, Physical Review Letters **101**, 020401 (2008).

[23] D. Gross, Hudson's theorem for finite-dimensional quantum systems, Journal of mathematical physics **47**, 122107 (2006).

[24] N. Delfosse, C. Okay, J. Bermejo-Vega, D. E. Browne, and R. Raussendorf, Equivalence between contextuality and negativity of the Wigner function for qudits, New Journal of Physics **19**, 123024 (2017).

[25] R. Raussendorf, D. E. Browne, N. Delfosse, C. Okay, and J. Bermejo-Vega, Contextuality and Wigner-function negativity in qubit quantum computation, Physical Review A **95**, 052334 (2017).

[26] N. Delfosse, P. Allard Guerin, J. Bian, and R. Raussendorf, Wigner function negativity and contextuality in quantum computation on rebits, Physical Review X **5**, 021003 (2015).

[27] D. Schmid, H. Du, J. H. Selby, and M. F. Pusey, The only noncontextual model of the stabilizer subtheory is gross's, Preprint arXiv:2101.06263 [quant-ph] (2021).

[28] S. L. Braunstein and P. Van Loock, Quantum information with continuous variables, Reviews of modern physics **77**, 513 (2005).

[29] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Reviews of Modern Physics **84**, 621 (2012).

[30] A. Crespi, R. Osellame, R. Ramponi, D. J. Brod, E. F. Galvao, N. Spagnolo, C. Vitelli, E. Maiorino, P. Mataloni, and F. Sciarrino, Integrated multimode interferometers with arbitrary designs for photonic boson sampling, Nature photonics **7**, 545 (2013).

[31] J. E. Bourassa, R. N. Alexander, M. Vasmer, A. Patil, I. Tzitrin, T. Matsuura, D. Su, B. Q. Baragiola, S. Guha, G. Dauphinais, K. K. Sabapathy, N. C. Menicucci, and I. Dhand, Blueprint for a Scalable Photonic Fault-Tolerant Quantum Computer, Quantum **5**, 392 (2021).

[32] M. Walschaers, Non-Gaussian quantum states and where to find them, Preprint arXiv:2104.12596 [quant-ph] (2021).

[33] U. Chabaud, Continuous variable quantum advantages and applications in quantum optics, PhD thesis arXiv:2102.05227 [quant-ph] (Sorbonne University, 2021).

[34] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A **64**, 012310 (2001).

[35] W. Cai, Y. Ma, W. Wang, C.-L. Zou, and L. Sun, Bosonic quantum error correction codes in superconducting quantum circuits, Fundamental Research **1**, 50 (2021).

[36] S. Yokoyama, R. Ukai, S. C. Armstrong, C. Sornphiphatphong, T. Kaji, S. Suzuki, J.-i. Yoshikawa, H. Yonezawa, N. C. Menicucci, and A. Furusawa, Ultra-large-scale continuous-variable cluster states multiplexed in the time domain, Nature Photonics **7**, 982 (2013).

[37] J.-i. Yoshikawa, S. Yokoyama, T. Kaji, C. Sornphiphatphong, Y. Shiozawa, K. Makino, and A. Furusawa, Invited article: Generation of one-million-mode continuous-variable cluster state by unlimited time-domain multiplexing, APL Photonics **1**, 060801 (2016).

[38] U. Leonhardt, *Essential Quantum Optics*, 1st ed. (Cambridge University Press, Cambridge, UK, 2010).

[39] M. Ohliger and J. Eisert, Efficient measurement-based quantum computing with continuous-variable systems, Physical Review A **85**, 062318 (2012).

[40] R. S. Barbosa, T. Douce, P.-E. Emeriau, E. Kashefi, and S. Mansfield, Continuous-variable nonlocality and contextuality, Communications in Mathematical Physics **391**, 1047 (2022).

[41] E. S. Simon Kochen, The problem of hidden variables in quantum mechanics, Indiana Univ. Math. J. **17**, 59 (1968).

[42] G. Adesso, S. Ragy, and A. R. Lee, Continuous variable quantum information: Gaussian states and beyond, Open Systems & Information Dynamics **21**, 1440001 (2014).

[43] R. Simon, E. C. G. Sudarshan, and N. Mukunda, Gaussian pure states in quantum mechanics and the symplectic group, Physical Review A **37**, 3028 (1988).

[44] M. A. De Gosson, *Symplectic geometry and quantum mechanics*, Vol. 166 (Springer Science & Business Media, 2006).

[45] A. Ferraro, S. Olivares, and M. G. Paris, Gaussian states in continuous variable quantum information, Preprint arXiv:0503237 [quant-ph] (2005).

[46] K. E. Cahill and R. J. Glauber, Density operators and quasiprobability distributions, Physical Review **177**, 1882 (1969).

[47] M. de Gosson, *The Wigner Transform* (World Scientific (Europe), 2017).

[48] M. A. de Gosson, *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, Pseudo-Differential Operators (Springer, Basel, 2011).

[49] P. Billingsley, *Probability and measure* (John Wiley & Sons, 2008).

[50] T. Tao, *An introduction to measure theory*, Vol. 126 (American Mathematical Society Providence, RI, 2011).

[51] X. Su, S. Hao, X. Deng, L. Ma, M. Wang, X. Jia, C. Xie, and K. Peng, Gate sequence for continuous variable one-way quantum computation, Nature communications **4**, 1 (2013).

[52] T. Douce, D. Markham, E. Kashefi, E. Diamanti, T. Coudreau, P. Milman, P. van Loock, and G. Ferrini, Continuous-variable instantaneous quantum computing is hard to sample, Physical Review Letters **118**, 070503 (2017).

[53] U. Chabaud, T. Douce, D. Markham, P. van Loock, E. Kashefi, and G. Ferrini, Continuous-variable sampling from photon-added or photon-subtracted squeezed states, Physical Review A **96**, 062307 (2017).

[54] L. Chakhmakhchyan and N. J. Cerf, Boson sampling with gaussian measurements, Physical Review A **96**, 032326

(2017).
[55] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Physical Review A **64**, 012310 (2001).
[56] K. Banaszek and K. Wódkiewicz, Nonlocality of the einstein-podolsky-rosen state in the wigner representation, Physical Review A **58**, 4345 (1998).
[57] R. S. Barbosa, *Contextuality in quantum mechanics and beyond*, Ph.D. thesis, University of Oxford (2015).
[58] A. Kenfack and K. Życzkowski, Negativity of the Wigner function as an indicator of non-classicality, Journal of Optics B: Quantum and Semiclassical Optics **6**, 396 (2004).
[59] A. Mari, K. Kieling, B. M. Nielsen, E. Polzik, and J. Eisert, Directly estimating nonclassicality, Physical Review Letters **106**, 010403 (2011).
[60] U. Chabaud, P.-E. Emeriau, and F. Grosshans, Witnessing Wigner negativity, Quantum **5**, 471 (2021).
[61] F. J. Narcowich and R. F. O'Connell, Necessary and sufficient conditions for a phase-space function to be a Wigner distribution, Physical Review A 3, **34** (1986).
[62] F. J. Narcowich and R. F. O'Connell, A unified approach to quantum dynamical maps and gaussian Wigner distributions, Physics Letters A **133**, 167 (1988).
[63] T. Bröcker and R. F. Werner, Mixed states with positive Wigner functions, Journal of Mathematical Physics **36**, 62 (1995).
[64] S. Abramsky, S. Mansfield, and R. S. Barbosa, The cohomology of non-locality and contextuality, in *Proceedings of 8th International Workshop on Quantum Physics and Logic (QPL 2011)*, Electronic Proceedings in Theoretical Computer Science, Vol. 95, edited by B. Jacobs, P. Selinger, and B. Spitters (Open Publishing Association, 2012) pp. 1–14.
[65] S. Abramsky, R. S. Barbosa, K. Kishida, R. Lal, and S. Mansfield, Contextuality, cohomology and paradox, in *24th EACSL Annual Conference on Computer Science Logic (CSL 2015)*, Leibniz International Proceedings in Informatics (LIPIcs), Vol. 41, edited by S. Kreutzer (Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2015) pp. 211–228.
[66] G. Carù, On the cohomology of contextuality, in *13th International Conference on Quantum Physics and Logic (QPL 2016)*, Electronic Proceedings in Theoretical Computer Science, Vol. 236, edited by R. Duncan and C. Heunen (Open Publishing Association, 2017) pp. 21–39.
[67] G. Carù, Towards a complete cohomology invariant for non-locality and contextuality, Preprint arXiv:1807.04203 [quant-ph] (2018).
[68] R. Raussendorf, Cohomological framework for contextual quantum computations, Quantum Information and Computation **19**, 1141 (2019).
[69] C. Okay and D. Sheinbaum, Classifying space for quantum contextuality, Annales Henri Poincaré **22**, 529 (2021), 1905.07723.
[70] Á. R. Plastino and A. Cabello, State-independent quantum contextuality for continuous variables, Physical Review A **82**, 022114 (2010).
[71] Q.-Y. He, E. G. Cavalcanti, M. D. Reid, and P. D. Drummond, Bell inequalities for continuous-variable measurements, Physical Review A **81**, 062106 (2010).
[72] G. McKeown, M. G. Paris, and M. Paternostro, Testing quantum contextuality of continuous-variable states, Physical Review A **83**, 062105 (2011).
[73] A. Asadian, C. Budroni, F. E. Steinhoff, P. Rabl, and O. Gühne, Contextuality in phase space, Physical Review Letters **114**, 250403 (2015).
[74] A. Laversanne-Finot, A. Ketterer, M. R. Barros, S. P. Walborn, T. Coudreau, A. Keller, and P. Milman, General conditions for maximal violation of non-contextuality in discrete and continuous variables, Journal of Physics A: Mathematical and Theoretical **50**, 155304 (2017).
[75] A. Ketterer, A. Laversanne-Finot, and L. Aolita, Continuous-variable supraquantum nonlocality, Physical Review A **97**, 012133 (2018).
[76] J. Wenger, M. Hafezi, F. Grosshans, R. Tualle-Brouri, and P. Grangier, Maximal violation of bell inequalities using continuous-variable measurements, Physical Review A **67**, 012105 (2003).
[77] J. Haferkamp and J. Bermejo-Vega, Equivalence of contextuality and wigner function negativity in continuous-variable quantum optics, Preprint arXiv:2112.14788 [quant-ph] (2021).
[78] A. Tychonoff, Über die topologische erweiterung von räumen, Mathematische Annalen **102**, 544 (1930).
[79] B. C. Hall, *Quantum theory for mathematicians*, Vol. 267 (Springer, 2013).
[80] N. Fournier and J. Printems, Absolute continuity for some one-dimensional processes, Bernoulli **16**, 10.3150/09-BEJ215 (2010), arXiv:0804.3037.