Interaction-Induced Non-Hermitian Topological Phases from a Dynamical Gauge Field

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Interaction-induced non-Hermitian topological phases from a dynamical gauge field

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We present a minimal non-Hermitian model where a topologically nontrivial complex energy spectrum is induced by inter-particle interactions. Our model consists of a one-dimensional chain with a dynamical non-Hermitian gauge field with density dependence. The model is topologically trivial for a single particle system, but exhibits nontrivial non-Hermitian topology with a point gap when two or more particles are present in the system. We construct an effective doublon model to describe the nontrivial topology in the presence of two particles, which quantitatively agrees with the full interacting model. Our model can be realized by modulating hoppings of the Hatano-Nelson model; we provide a concrete Floquet protocol to realize the model in atomic and optical settings.

Non-Hermitian Hamiltonians have been found to host a rich variety of topological phases [1–14]. While some non-Hermitian phases are direct analogues of Hermitian phases, there are many uniquely non-Hermitian phases. These result when the system has a point gap in the complex energy plane, which allows for non-trivial winding of the energy spectrum. Non-Hermitian topology manifests in an analogous bulk-boundary correspondence known as the skin effect wherein a macroscopic number of states localize at the boundary [15–19]. These phenomena have been primarily investigated as single particle effects.

Comparatively little work has been done to understand the role of correlations and interactions in non-Hermitian topological phases [20–24]. Interactions have played an important role in Hermitian topological physics, giving rise to many paradigmatic phases including the fractional quantum Hall effect and quantum spin liquids. Such strongly interacting systems have led to many advancements in physics, including developments in gauge theories. Given the richness of Hermitian interacting systems, it remains to be seen how analogous non-Hermitian interactions can enrich the topology of open systems. Experimentally, two body loss terms are ubiquitous in optical lattices and photonics with an increasing degree of control, further motivating investigations of the topology of many body open systems.

In this letter, we report on a minimal 1D non-Hermitian model exhibiting interaction induced topology. We demonstrate that our model is topologically trivial for a single particle, but gains a non-trivial winding number in the complex energy plane for two or more particles. We characterize the spectrum by the clustering properties of the eigenstates. This leads us to derive an effective SSH model of Doublons with an emergent sublattice symmetry, which quantitatively captures the complex energy ring of the full spectrum. The winding number of the interacting model corresponds with the winding of this effective model analogous to interaction induced topology in Hermitian systems [25–27]. We conclude by proposing a two-frequency Floquet protocol that realizes our model as an effective Hamiltonian. As an intermediate step, this Floquet protocol realizes a Hatano-Nelson model.

Model.— Our model consists of bosons populating a 1D chain with hoppings dependent on the gradient of the density, which can be interpreted as a density-dependent synthetic dynamical gauge field. The Hamiltonian of our system is

$$H = \sum_j a_{j+1}^\dagger \Big[-t + i\gamma_R(n_{j+1} - n_j) \Big] a_j + a_j^\dagger \Big[-t + i\gamma_L(n_j - n_{j+1}) \Big] a_{j+1} \tag{1}$$

where a_j and $a_j^\dagger$ are bosonic annihilation and creation operators, respectively, t is the single particle hopping parameter, n_j is the density operator $a_j^\dagger a_j$ on the jth site, and $\gamma_{R/L}$ are the couplings to the gauge field for right and left hoppings. To realize a non-Hermitian model, we take $\gamma_R \neq \gamma_L^*$ in similar fashion to the Hatano-Nelson model [28, 29]. Description in terms of non-Hermitian Hamiltonians can be obtained through a post-selection procedure on quantum trajectories [21, 30]. Later we present a concrete experimental protocol combining quantum trajectory and Floquet theory to realize this Hamiltonian. In the present model, when there is only one particle present in the system, the density terms are identically zero. Therefore, for a single particle the Hamiltonian is Hermitian and corresponds to a free boson. The single particle spectrum is shown in Fig. 1a which reproduces the free boson result for a periodic chain of length $L = 20$.

Exact Diagonalization.— Let us contrast this result with the two particle spectrum shown in Fig 1b. We consider fixed particle number as the Hamiltonian has U(1) symmetry. Physically, particle number is conserved between quantum jumps during which the non-Hermitian Hamiltonian description is valid. Here we find that the energy spectrum consists of a sector where energies are nearly real and a sector consisting of a ring of complex energies. We project each eigenstate into the subset of basis states where particles lie on the same site or adjacent sites, and find that the states with complex energies largely lie in this subspace while those with nearly real energies have almost zero weight in this subspace. To

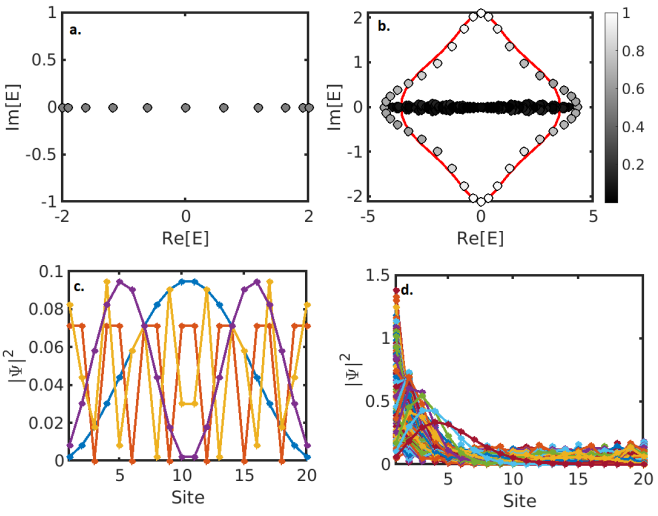

FIG. 1. Summary of key results. Panels a and b are energy spectra for periodic boundary conditions plotted in the complex plane for 1 and 2 particles, respectively. For one particle, the spectrum is real while for two particles, the spectrum is complex with a point gap. The coloring in panel b is obtained projecting each eigenstate onto the subspace of basis states where particles lie on the same site or adjacent sites. The red line is the spectrum obtained from the effective doublon model described below. Panel c and d are the real space profiles of the eigenstates in the open boundary geometry for 1 and 2 particles respectively. For one particle, eigenstates are typical standing waves while for two particles we observe a skin effect. The chosen parameters are $t = 1$, $\gamma_L = 1.5$ and $\gamma_R = 0$ for a lattice with 20 sites.

further characterize this separation we calculate the correlator $a_j^\dagger a_k^\dagger a_j a_k$ for each eigenstate. Representative correlators for complex energy states and nearly real energy states are shown as a function of j for a fixed k in Fig 2. This correlator confirms that the states with complex energy occur when the particles cluster while those with nearly real energies occur when the particles separate.

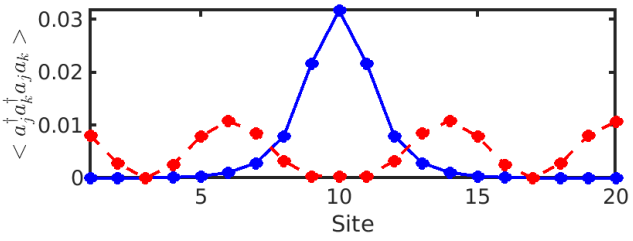

FIG. 2. Four point correlator $\langle a_j^\dagger a_k^\dagger a_j a_k \rangle$ for $k = 10$ with periodic boundary conditions. The solid line is representative of states with corresponding energies on the ring while the dashed line is representative of states with nearly real eigenenergies.

The energy spectrum has a point gap indicating the presence of nontrivial topology. To verify the nontrivial topology, we calculate the winding number following the flux insertion procedure outlined in Ref [31]. We define $H(\phi)$ by multiplying $e^{-i\phi}$ ($e^{i\phi}$) to the boundary hopping term in the first (second) term of Eq.(1), where ϕ is the strength of the inserted magnetic flux. We then calculate

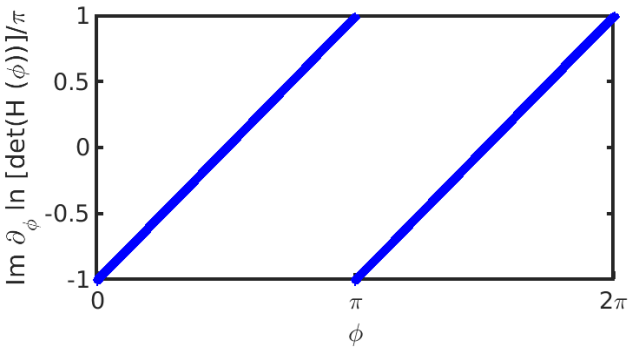

FIG. 3. Plot demonstrating the nontrivial winding number. A jump from 1 to -1 increases the winding by 1 while a jump from -1 to 1 descreases the winding by 1.

$$\frac{1}{\pi}\Im\Big[\partial_\phi \ln\det[H(\phi) - \delta I]\Big] \tag{2}$$

as a function ϕ, where $\Im[\cdot]$ stands for the imaginary part, and I is the identity matrix. The signed number of jumps in this quantity give the winding of the phase about the point δ, chosen to lie within the point gap, in the complex plane. In Fig 3, we plot this quantity versus the flux, ϕ, and clearly see that it jumps twice as the flux is tuned from 0 to 2π, giving a winding number of 2 and confirming that the system is topologically non-trivial when there are two particles.

A non-trivial winding number implies the existence of the skin effect in the open boundary geometry. The open-boundary eigenstates are plotted in Figs 1c and 1d for the one and two particle systems respectively. The one particle eigenstates are exactly those obtained for a free Boson model while the two particle eigenstates demonstrate a clear skin effect. In fact, all states will localize on the edge for strong enough gauge coupling. For completeness, we present the energy spectrum in the open boundary geometry in Fig 4, which shows that the spectrum does not cleanly separate along particle clustering properties as both particles localize on the edge. Unlike the single-particle Hatano-Nelson model, the spectrum in the open boundary condition does not lie on the real axis only; we discuss below the origin of this complex spectrum.

Effective Doublon Model.— To understand the topology of the system, we derive an effective doublon model that captures the physics of the complex energy ring. We obtain this by restricting our basis to states where the particles lie on the same site or on adjacent sites. The effective doublon model consists of two sublattices; we

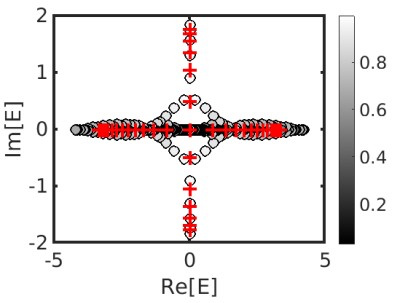

FIG. 4. Energy spectrum for open boundary geometry. The parameters are the same as in Fig. 1 and the coloring of points is obtained through the same projection method. Red crosses are obtained from the effective Doublon SSH model.

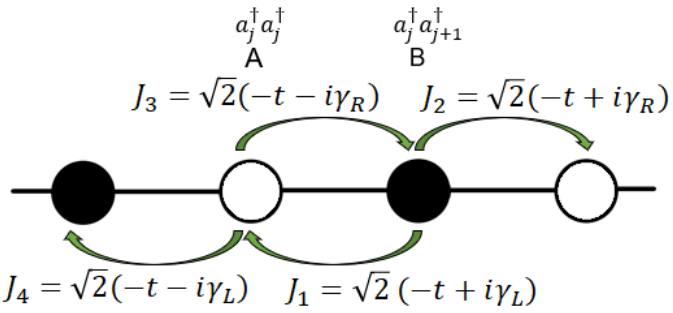

FIG. 5. Diagram of the effective doublon SSH model. Sublattice A maps to basis states with particles on the same site while sublattice B maps to basis states with particles on adjacent sites.

define sublattice A to be the set of states with particles on the same site, and sublattice B to be the set of states with two particles occupying adjacent sites, as shown in the diagram in Fig 5. We define the creation operator of a particle in j-th site of sublattice A as $a^\dagger_{A,j} \equiv a^\dagger_j a^\dagger_j$ and that of sublattice B as $a^\dagger_{B,j} \equiv a^\dagger_j a^\dagger_{j+1}$. The resulting effective model is given by the Hamiltonian

$$H_{Doublon} = \sum_j \Psi^\dagger_{j,j} \begin{pmatrix} 0 & J_1 \\ J_3 & 0 \end{pmatrix} \Psi_{j,j} + \Psi^\dagger_{j+1,j} \begin{pmatrix} 0 & J_2 \\ J_4 & 0 \end{pmatrix} \Psi_{j+1,j}, \tag{3}$$

where $\Psi^\dagger_{j,l} \equiv \left(a^\dagger_{A,j}, a^\dagger_{B,l}\right)$ and $J_1 = \sqrt{2}(-t + i\gamma_L)$, $J_2 = \sqrt{2}(-t+i\gamma_R)$, $J_3 = \sqrt{2}(-t-i\gamma_R)$, and $J_4 = \sqrt{2}(-t-i\gamma_L)$, which is an SSH model with asymmetric hoppings on a chain of length $2L$. We note that the effective doublon model has emergent sublattice symmetry in which hopping within the same sublattice is absent. The energy spectrum of this model is plotted in red on Figs. 1b and 4 for periodic and open boundaries, respectively, demonstrating that key features of the many body spectrum are captured by this single particle doublon model. The accuracy of the Doublon model increases for stronger density-dependent hopping.

To characterize the topology of this effective theory, we calculate the winding number of the energy spectrum in momentum space. The Fourier transform of the effective Hamiltonian is

$$\sum_k \tilde{\Psi}^\dagger_{k,k} \begin{pmatrix} 0 & J_1 + J_2 e^{-ik} \\ J_3 + J_4 e^{ik} & 0 \end{pmatrix} \tilde{\Psi}_{k,k} \tag{4}$$

with $\tilde{\Psi}_{k,k} \equiv (\tilde{a}_{A,k}, \tilde{a}_{B,k})$, where $\tilde{a}_{A,k}$ and $\tilde{a}_{B,k}$ are the Fourier transforms of $a_{A,j}$ and $a_{B,j}$. The details for determining the topology of such a system with sublattice symmetry are outlined in Ref [1] where we have the upper diagonal matrix $H^+ = J_1 + J_2 e^{-ik}$ and the lower diagonal matrix $H^- = J_3 + J_4 e^{ik}$. If the point gap in the many body system is well-defined, the winding of the full system without considering the sublattice symmetry is twice the winding of the doublon Hamiltonian. The doublon model tells us approximately which points in the complex energy plane we should consider to find non-trivial winding in the full many-body system. Additionally, the point gaps in the many body system are often larger than in the doublon effective theory, providing additional points about which there is non-trivial winding.

In the Hatano-Nelson model, the spectrum under open boundary conditions is real implying there exists an imaginary gauge transformation between the Hatano-Nelson Hamiltonian and a Hermitian Hamiltonian [28, 29, 32]. In general the energy spectrum of $H_{Doublon}$ has both imaginary and real energies, implying that the Hamiltonian cannot be made Hermitian by an imaginary gauge transformation, i.e. our effective doublon model is not related to any Hermitian matrix by a similarity transformation. With a similarity transformation similar to the one used for the ordinary Hatano-Nelson model [32], one can transform our effective doublon Hamiltonian under an open boundary condition to a tri-diagonal form:

$$H_{Doublon} \sim \begin{pmatrix} 0 & \sqrt{J_1 J_3} & 0 & \cdots \\ \sqrt{J_1 J_3} & 0 & \sqrt{J_2 J_4} & \cdots \\ 0 & \sqrt{J_2 J_4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

This matrix is real and symmetric when both $\sqrt{J_1 J_3}$ and $\sqrt{J_2 J_4}$ are real, which gives a sufficient condition for the reality of the spectrum. When γ_L and γ_R are real, which is the situation relevant in this paper, $J_2 J_4 = (J_1 J_3)^*$, in which case we numerically confirm that $J_1 J_3 > 0$ gives the necessary and sufficient condition for the spectrum being entirely real.

Floquet Protocols.— To realize our model, we have identified a Floquet protocol whose effective Hamiltonian can be identified with Eq. 1. This Floquet protocol is inspired by previous efforts towards realizing Hermitian density dependent gauge fields in the setting of optical lattices [33–39]. The system consists of a static

Bose-Hubbard Hamiltonian and a time periodic Hatano-Nelson model.

We first describe how to realize Hatano-Nelson model from a Floquet protocol, which consists of a repeated three step process as presented in Ref. [40] with total frequency $\Omega = 2\pi/T_\Omega$.[41] The time-dependent Hamiltonian is given as

$$H(t) = \Delta \sum_j (a_{j+1}^\dagger a_j + a_{j+1} a_j^\dagger) + U \sum_j n_j(n_j - 1) + V(t) \tag{6}$$

where Δ is the hopping parameter, U is the strength of the interaction and $V(t)$ is the modulation given by

$$V(t) = \begin{cases} \Delta_1 \sum_j a_{j+1}^\dagger a_j + a_{j+1} a_j^\dagger, & 0 \le t < T_\Omega/3 \\ \sum_j i\mu_j a_j^\dagger a_j, & T_\Omega/3 \le t < 2T_\Omega/3 \\ 0, & 2T_\Omega/3 \le t < T_\Omega \end{cases} \tag{7}$$

corresponding to a free gas of bosons with periodically modulated hopping and site dependent loss μ_j. Note t is defined mod T_Ω. We obtain an effective Hamiltonian to order $1/\Omega$ as

$$\begin{aligned} H_{eff} &= \sum_j \left[\Delta - \frac{\pi\Delta_1}{27\Omega}(\mu_j - \mu_{j+1})\right] a_{j+1}^\dagger a_j \\ &+ \left[\Delta - \frac{\pi\Delta_1}{27\Omega}(\mu_{j+1} - \mu_j)\right] a_j^\dagger a_{j+1} + U \sum_j n_j(n_j - 1) \end{aligned} \tag{8}$$

Choosing the site-dependent loss strength to be $\mu_j = j\mu_0$, we obtain a Hatano-Nelson model. A non-Hermitian Hamiltonian with site dependent loss can be implemented in ultracold gases by applying near-resonant light with position-dependent intensity [42]. Modulating this electric field in time as proposed above should give rise to the effective time-dependent Hamiltonian in Eqs. 6 and 7.

By further modulating the hopping of the effective Hatano-Nelson Hamiltonian, we obtain the desired hopping as we now show. The full time dependent Hamiltonian is given by

$$\begin{aligned} H(t) =& \Delta \sum_j \left[a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right] + U \sum_j a_j^\dagger a_j (a_j^\dagger a_j - 1) \\ &+ \sin(\omega t) \sum_j \left[\Delta_R a_{j+1}^\dagger a_j + \Delta_L a_j^\dagger a_{j+1}\right] \end{aligned} \tag{9}$$

where Δ is the hopping parameter, U is the interaction strength, ω is the frequency of the drive, and $\Delta_R \neq \Delta_L^*$ describes a non-Hermitian drive analogous to a Hatano-Nelson model. We assume $\omega \ll \Omega$ so that in the time scale of $1/\omega$ the system is effectively described by a static Hatano-Nelson model. From the Magnus expansion[43, 44], we can obtain the effective Hamiltonian to order $1/\omega$

$$\begin{aligned} H_{\text{eff}} =& \sum_j a_{j+1}^\dagger \left[\Delta - \frac{2}{i\hbar\omega} U \Delta_R (n_j - n_{j+1})\right] a_j \\ &+ a_j^\dagger \left[\Delta - \frac{2}{i\hbar\omega} U \Delta_L (n_{j+1} - n_j)\right] a_{j+1} \\ &+ U \sum_j a_j^\dagger a_j (a_j^\dagger a_j - 1) \end{aligned} \tag{10}$$

which maps to our original system with $\Delta = -t$, $\gamma_L = \frac{2}{\hbar\omega} U \Delta_L$ and $\gamma_R = \frac{2}{\hbar\omega} U \Delta_R$. We note here if one considers this Floquet protocol for fermions instead of bosons, the Hatano-Nelson model can be realized from the fast frequency oscillation, but the density-dependent hopping is absent as particles cannot lie on the same site. This Floquet protocol introduces an additional interaction term in the effective Hamiltonian. The physics of our proposed model remains unchanged for sufficiently small interaction strength U.

Besides one dimensional quantum systems with controlled modulation, such as ultracold atomic gases, the above protocols can also be realized by mapping the models to two dimensional classical systems, where density-induced Hermitian models have been previously investigated [25, 26].

Conclusion.— We have presented a minimal model where the interaction induces non-trivial non-Hermitian topology. The density-dependent non-Hermitian gauge field played a crucial role in the emergence of two-body topological phases. Our work paves a way toward understanding the role of gauge fields in non-Hermitian many-body systems. In Hermitian systems, effects of gauge fields are pronounced in two or higher dimensions, giving rise to exotic many-body phases such as fractional quantum Hall phases and topological order. It is therefore of great interest to extend the study of many-body physics in non-Hermitian systems under gauge fields to two and higher dimensions, in which we expect interplay of topological orders and non-Hermiticity. The Floquet protocol we provide also elucidated that such non-Hermitian many-body physics can be studied in experimentally viable setups under suitable time modulations. Furthermore, density-dependent gauge fields are simplest examples of dynamical gauge fields where gauge fields do not take externally fixed values. In Hermitian systems, dynamical gauge fields play a fundamental role in understanding a wide variety of systems, from high-energy to condensed matter physics. Their relations to non-Hermitian physics has been little explored; our work opens an avenue toward exploring this uncharted field.

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