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Discrete time crystals enforced by Floquet-Bloch scars

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We analytically identify a new class of quantum scars protected by spatiotemporal translation symmetries, dubbed Floquet-Bloch scars. They distinguish from previous (quasi-)static scars by a rigid spectral pairing only possible in Floquet systems, where strong interaction and drivings equalize the quasienergy corrections to all scars and maintain their spectral spacings against generic bilinear perturbations. Scars then enforce the spatial localization and rigid discrete time crystal (DTC) oscillations as verified numerically in a trimerized kagome lattice model relevant to recent cold atom experiments. Our analytical solutions offer a potential scheme to understand the mechanisms for more generic translation-invariant DTCs.

Introduction — Systems far from equilibrium have become a fertile ground cultivating unexpected phenomena recently. Among them, discrete time crystals (DTC) [1–7] constitute an intriguing example. As foundational concepts of ground state and temperature fall apart in the absence of thermal equilibrium, Landau's theory of symmetry breaking [8] is replaced by new principles like spectral pairing and eigenstate orders [1, 2] in handling time translation symmetries. That results in the DTC phenomena where Hamiltonians $\hat{H}(t+T) = \hat{H}(t)$ give rise to observables $O(t+NT) = O(t)$ $(1 < N \in \mathbb{Z})$ oscillating like a temporal charge/spin density wave. Crucially, the periodicity NT demands no fine-tuning and withstands generic perturbations.

The concept of DTCs has been considered in several physical realizations [9–15]. While the strongly disordered cases are relatively well understood [1–3, 16], the possibility of DTCs in translation-invariant ordered systems is less clear. Empirical evidence for DTCs is accumulating in both physical and numerical experiments [9, 10, 17–23]. However, analytical explanations based on many-body localization (MBL) [24] or prethermalization [25] do not seem to apply to these cases. Recently, it was indicated that quasi-conservation laws [26, 27], which can be enhanced by single-particle terms, may help protect phenomena pertinent to DTCs. Meanwhile, the initial state dependence of clean DTCs [26, 28] has been reexamined in terms of scar physics [29] in recent numerics [30, 31]. Altogether, continued investigation on DTCs in non-disordered systems, with the objectives of uncovering the underlying mechanism that supports the DTC and the specific role of many-body (vs. single-particle) effects, is warranted.

In this Letter, we gain insights on these two research objectives by studying a small cluster of soft-core bosons on driven trimerized kagome lattices, relevant to recent experiments [32] and feasible for numerical verifications. We find analytically that it is a class of quantum scars *protected by* spatiotemporal translation invariance, dubbed Floquet-Bloch scars (FBS), that gives rise to DTC behaviors for sublattice density oscillations. FBS's identified here *neither* exploit a static scar (i.e. "PXP" model [29, 33, 34]) *nor* end up with engineered static Hamiltonians. Instead, these FBS's exhibit a unique DTC feature. Specifically, each scar quasienergy may be shifted considerably under perturbation. However, the interplay of strong interactions and drivings equalizes the scar level shifts, which is proved to all orders in our perturbative treatment. Then, the quasienergy difference ω_0 between FBSs remains invariant and enforces the persisting $2\pi/\omega_0$-periodic DTC. Rigid scar level spacing here resembles the "spectral pairing rigidity" for all Floquet eigenstates in MBL DTCs [1, 16]. Also, such a mechanism allows for rather generic perturbations compared with preexisting scar models typically relying on microscopic details to achieve configuration separations [29, 33–44]. Thus, our analytical solutions not only offer a more definitive understanding of clean DTC mechanisms, but also point out a new way of constructing scars showing peculiar spectral orders characteristic of Floquet systems.

Model and phenomena — We consider bosons evolving under a Hamiltonian $\hat{H}(t+T) = \hat{H}(t)$ that is toggled between two settings repetitively within each period T:

$$\hat{H}_1 T/2\hbar = \phi_1 \sum_{\boldsymbol{r},\mu\neq\nu} i f_{\mu\nu} \left[\hat{\psi}^\dagger_{\boldsymbol{r}\mu}\hat{\psi}_{\boldsymbol{r}\nu} + \lambda \hat{\psi}^\dagger_{\boldsymbol{r}+\boldsymbol{e}_\mu,\mu}\hat{\psi}_{\boldsymbol{r}+\boldsymbol{e}_\nu,\nu}\right],\ t \in [0, T/2)$$

$$\hat{H}_2 T/2\hbar = \sum_{\boldsymbol{r},\mu} \left[\phi_2 \hat{n}_{\boldsymbol{r}\mu}(\hat{n}_{\boldsymbol{r}\mu} - 1) + \theta_\mu \hat{n}_{\boldsymbol{r}\mu}\right],\ t \in [T/2, T). \qquad (1)$$

Here, $\hat{H}_1$ describes the hopping of non-interacting bosons in a trimerized kagome lattice with complex hopping amplitudes, as shown in Fig. 1(a), while $\hat{H}_2$ describes the combination of on-site single-particle and interaction energy shifts. Dimensionless parameters $(\phi_1, \lambda, \phi_2, \theta_\mu)$ characterize the Floquet operator $\hat{U}_F = P_t e^{-(i/\hbar)\int_0^T dt \hat{H}(t)} = e^{-i\hat{H}_2 T/2\hbar} e^{-i\hat{H}_1 T/2\hbar}$. $\hat{\psi}_{\boldsymbol{r}\mu}$ and $\hat{n}_{\boldsymbol{r}\mu} = \hat{\psi}^\dagger_{\boldsymbol{r}\mu}\hat{\psi}_{\boldsymbol{r}\mu}$ are annihilation and particle number operators respectively, for L^2 unit cells $\boldsymbol{r} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2$ $(m_{1,2} = 0, 1, \ldots, L-1)$ and three sublattices $\mu, \nu = 0, 1, 2$. Here $\boldsymbol{e}_{1,2}$ are Bravais vectors for kagome lattices and $\boldsymbol{e}_0 \equiv \boldsymbol{0}$. $i f_{\mu\nu} = (1 + 2e^{2\pi i(\mu-\nu)/3})/\sqrt{3} = \pm i$ specifies the $+i$ hopping directions in Fig. 1 (a). Note that $\sum_\mu \theta_\mu = 0$ can always be achieved by subtracting $(N_b/3)\sum_\mu \theta_\mu$ from $\hat{H}_2 T/2\hbar$, where total bosons $N_b = \sum_{\boldsymbol{r}\mu} \hat{n}_{\boldsymbol{r}\mu}$.

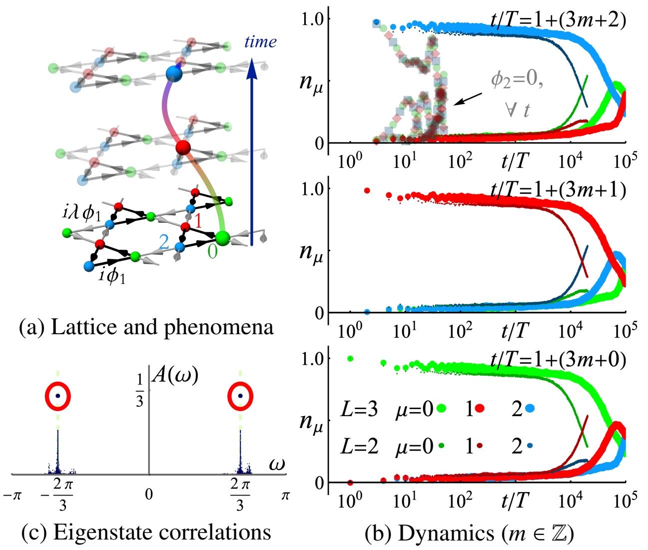

FIG. 1. (a) Trimerized kagome lattice and the schematic illustration of DTC dynamics. Triangles with strong/weak bonds are denoted by black/gray colors, with $+i$ hopping directions indicated by arrows. (b) Particle dynamics $n_\mu(NT) = \sum_{\boldsymbol{r}} \langle\psi_{\rm ini}|(\hat{U}_F^\dagger)^N n_{\boldsymbol{r}\mu} \hat{U}_F^N|\psi_{\rm ini}\rangle$, with the initial state $|\psi_{\rm ini}\rangle$ at $t/T = 1$ that all particles locate at a single site $\boldsymbol{r} = \boldsymbol{0}, \mu = 0$. To facilitate reading, data is grouped into 3 sets at t mod $3T = 0, 1, 2$ respectively. For comparison, the non-interacting $\phi_2 = 0, L = 3$ case is plotted at all t in the upper panel as translucent dots. (c) Temporal correlation functions indicating infinite-time response frequencies ($L = 3$). Unless denoted otherwise, $N_b = 5$, $\phi_1 = 2\pi/3\sqrt{3}$, $\lambda = 0.1$, $\phi_2 = 1.1$, $\theta_{1,2,3} = (0.1, 0.2, -0.3)$.

DTC dynamics obtained by exact diagonalization is briefly shown in Fig. 1. When $\lambda \to 0$, $\hat{H}_1$ enters the strongly trimerized regime composed of disconnected triangles, where $\pi/2$-fluxes equalize the spacing between single-particle flat bands $\omega_n = 0, \pm\sqrt{3}\phi_1$ ($\hat{U}_F|\omega_n\rangle = e^{i\omega_n}|\omega_n\rangle$). Then, $\phi_1 = 2\pi/3\sqrt{3}$ leads to $3T$ ballistic oscillations for particles $\hat{U}_F^\dagger \hat{\psi}^\dagger_{\boldsymbol{r},\mu=0,1,2}\hat{U}_F = \hat{\psi}_{\boldsymbol{r},\mu=1,2,0}$ breaking the Hamiltonian time translation symmetry of T, as in Fig. 1 (a). Frequencies given by single-particle physics are, of course, unstable against perturbations. It is then the hallmark for DTC where strong interactions ϕ_2 stabilize the $3T$ periodicity without fine-tuning, see Fig. 1 (b). Late time dynamics can be further confirmed by the temporal correlation functions $C(\omega) = \sum_{N=-\infty}^{\infty} \frac{e^{i\omega N}}{2\pi} \sum_n \langle\omega_n|\hat{P}(N)\hat{P}(0)|\omega_n\rangle = \sum_{mn} \delta(\omega - \omega_{mn}) A(\omega_{mn})$ for the sublattice density bias, i.e. $\hat{P}(N) = \left(\hat{U}_F^\dagger\right)^N N_b^{-1} \sum_{\boldsymbol{r}} (\hat{n}_{\boldsymbol{r}0} - \hat{n}_{\boldsymbol{r}1}) \hat{U}_F^N$. Note that the summation N is over *infinite* time without truncation. The spectral weight $A(\omega_{mn}) = |\langle\omega_m|\hat{P}|\omega_n\rangle|^2$, $\omega_{mn} = \omega_m - \omega_n$ in Fig. 1 (c) showing strong peaks at frequencies $\omega_0 \to \pm 2\pi/3 + O(1/D)$ verifies long-time oscillation periods $2\pi T/|\omega_0| \to 3T$. The small deviation $O(1/D)$ suppressed by Hilbert space dimension D gives an envelop modulation in Fig. 1 (b) as noticed previously for both MBL [2, 16] and clean [17] DTCs.

The above phenomena may be viewed from several angles. Particularly, in the case of complete trimerization, $\lambda = 0$, the two-dimensional lattice breaks up into isolated trimers. DTCs observed in this case is then explained simply as that of a microscopic three-site chiral system similar to Ref. [45]. If we were to regard intertrimer coupling as simply opening up each one-trimer DTC to an external bath composed of other trimers, we might expect the overall DTC dynamics to be destroyed over short time at $\lambda \neq 0$ [46]. Yet, such expectations contradict results in Fig. 1 (b) (c). Below, we offer an explanation that DTCs in the coupled-trimer regime is stabilized by a special class of scar Floquet eigenstates each spanning over the entire two-dimensional lattice.

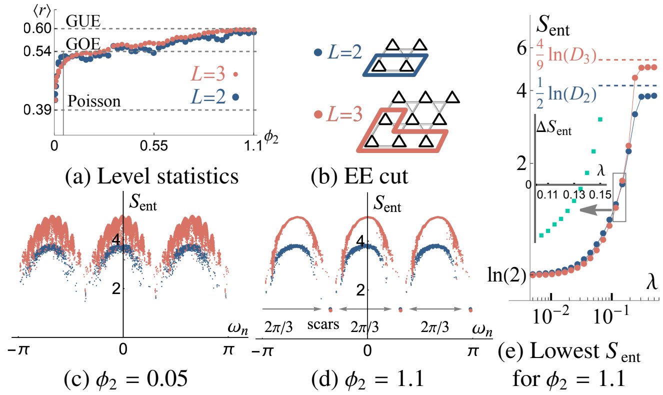

FIG. 2. (a) $\langle r\rangle$ shows generic ergodicity. (b) Subsystem for computing EE. (c) – (d) Eigenstate EE in (c) proximate-integrable and (d) DTC regimes, where low-entropy scars in (d) are highlighted by larger dots. (e) Lowest EE approaching size-insensitive values at small λ and volume law $S_{\rm ent} \sim (N_{\rm sub}/3L^2)\ln(D_L)$ at large λ. Here D_L is the total Hilbert space dimension and $N_{\rm sub}$ the subsystem site number enclosed in (b). Inset shows $\Delta S_{\rm ent} = S_{\rm ent}^{(L=3)} - S_{\rm ent}^{(L=2)}$ near the crossing $\lambda_0 \approx 0.135$. Unless specified otherwise, in all plots parameters are the same as in Fig. 1. Blue (or red) colors denote $L = 2$ (or $L = 3$) respectively.

Identifying scars — Quantum scars are rare non-ergodic eigenstates within an eigenstructure that is otherwise thermalizing [47]. Numerical calculations confirm the overall thermalizing, non-integrable nature of our model system. Specifically, we point to two signatures of non-integrability: level-spacing statistics and entanglement entropy.

Consider first the level spacing. Ordering quasi-energies as $\omega_n < \omega_{n+1}$, following Ref. [48], we test for ergodicity by calculating the level spacing ratios $r_n = \min(\delta_n, \delta_{n+1})/\max(\delta_n, \delta_{n+1})$ for consecutive gaps $\delta_n = \omega_{n+1} - \omega_n$. Clearly from Fig. 2 (a), except for a vanishingly small region in proximity to single particle limit $\phi_2 \to 0$, our model is generically far from the integrable Poissonian case $\langle r\rangle \to 0.39$. We also note a crossover between two ergodic Gaussian orthogonal/unitary ensembles (GOE/GUE) purely by different drivings, an interesting feature previous seen in spin models [49].

We next exploit the entanglement entropy (EE) to examine each Floquet eigenstate $|\omega_n\rangle$. Reduced density matrices $\rho_A = {\rm Tr}_B(|\omega_n\rangle\langle\omega_n|)$ for subsystem A (region enclosed by highlighted paths in Fig. 2 (b)) can be formed by tracing out the remaining part B in real space. The EE $S_{\text{ent}} = -\text{Tr}(\rho_A \ln \rho_A)$ then shows that in both proximate-integrable (Fig. 2 (c)) and DTC (Fig. 2 (d)) regimes, majority eigenstates do exhibit the typical arch shape for S_{ent} whose values increase with Hilbert space dimensions [50]. The narrow distribution of EE for eigenstates of similar quasi-energy in the DTC regime confirms that majority arch eigenstates are ergodic [50], in consistent with $\langle r\rangle$ results previously.

However, in the DTC regime, additional non-ergodic states are observed. As exhibited in Fig. 2(d), we identify precisely $3L^2$ low S_{ent} scar states (each scar dot in the figure is L^2-fold degenerate). Each set of scars separates from the others by quasienergy $|\Delta E| \to 2\pi/3$, corresponding to exactly the DTC frequency in Fig. 1 (c). The scaling of lowest EE in Fig. 2 (e) shows a system size L insensitive scar EE for $\lambda \to 0$. With increasing λ, a possible transition is observed around $\lambda_0 \approx 0.135$ [51], after which all eigenstates approach the volume law ergodic limit.

We have confirmed numerically that parameters in Fig. 2 (c) give rather short DTC lifetime, unlike the lifetime shown in Fig. 1 (b) for parameters in Fig. 2 (d). It strongly indicates that the DTC behaviors here are intimately associated with scars rather than (approximate) overall integrability.

Analytical results for FBS — To characterize these quantum scars further, we work in the many-body momentum basis [52] $|\boldsymbol{k}, \{n_{\boldsymbol{r},\mu}\}\rangle = (1/L)\sum_{m_1,m_2=0}^{L-1} e^{(2\pi i/L)(k_1 m_1 + k_2 m_2)} \left|\{n_{\boldsymbol{r}+m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2,\mu}\}\right\rangle$ constructed from Fock basis $|\{n_{\boldsymbol{r}\mu}\}\rangle = \prod_{\boldsymbol{r}\mu}\left[\left(\hat{\psi}^\dagger_{\boldsymbol{r}\mu}\right)^{n_{\boldsymbol{r}\mu}} / \sqrt{n_{\boldsymbol{r}\mu}!}\right]|0\rangle$. Here $\{n_{\boldsymbol{r}\mu}\}$ specifies occupation numbers at different sites, and $\boldsymbol{k} \sim k_{1,2} = 0, 1, \ldots, L-1$. Then, translation-invariant $\hat{U}_F$ are block-diagonalized $\langle \boldsymbol{k}, \{n_{\boldsymbol{r}\mu}\}|U_F|\boldsymbol{k}', \{n'_{\boldsymbol{r}\mu}\}\rangle \sim \delta_{\boldsymbol{k},\boldsymbol{k}'}$. Each $\boldsymbol{k}$ sector would be shown later to host 3 scar states, leading to the $3 \times L^2$-fold scars in Fig. 2 (d).

It is helpful to write down the solution $U_F|\boldsymbol{k}, \ell, \{n_{\boldsymbol{r}\mu}\}\rangle = e^{iE(\ell,\{n_{\boldsymbol{r}\mu}\})}|\boldsymbol{k}, \ell, \{n_{\boldsymbol{r}}\}\rangle$ to Eqs. (1) at the anchor point $\lambda = 0$,

$$\left|\boldsymbol{k}, \ell, \{n_{\boldsymbol{r}\mu}\}\right\rangle = \frac{1}{\sqrt{3}} \sum_{m=0,1,2} e^{-i(\frac{2\pi m}{3}\ell - \alpha_m)}|\boldsymbol{k}, \{n_{\boldsymbol{r},\mu+m \bmod 3}\}\rangle, \quad (2)$$

$$E\left(\ell, \{n_{\boldsymbol{r}\mu}\}\right) = \frac{2\pi}{3}\ell + \phi_2 \sum_{\boldsymbol{r}\mu} n_{\boldsymbol{r}\mu}(n_{\boldsymbol{r}\mu} - 1), \qquad \ell = 0, \pm 1, \quad (3)$$

where $\alpha_0 = 0$, $\alpha_1 = \sum_{\boldsymbol{r}\mu} \theta_\mu n_{\boldsymbol{r}\mu}$, and $\alpha_2 = -\sum_{\boldsymbol{r}\mu} \theta_\mu n_{\boldsymbol{r},\mu+2 \bmod 3}$. Each eigenstate populates 3 sublattices $\mu = 0, 1, 2$ coherently, and therefore an arbitrary Fock state $|\{n_{\boldsymbol{r}\mu}\}\rangle$, usually taken as initial states, will simultaneously overlap with all three branches $\ell = 0, \pm 1$ separating from each other by quasi-energy $|\Delta E| = 2\pi/3$. Then, observables diagonal in the Fock basis, such as $\hat{O} = \hat{n}_{\boldsymbol{r}\mu}$ or $\hat{O} = \hat{P} = N_b^{-1}\sum_{\boldsymbol{r}}(\hat{n}_{\boldsymbol{r}0} - \hat{n}_{\boldsymbol{r}1})$, will demonstrate an oscillation $\langle \hat{O}\rangle(t) \sim c_1^* c_2 \langle \boldsymbol{k}_1, \ell_1, \{n_{\boldsymbol{r}\mu}\}|\hat{O}|\boldsymbol{k}_2, \ell_2, \{n_{\boldsymbol{r}\mu}\}\rangle e^{-i\Delta E t/T} + c.c.$ with periodicity $2\pi T/\Delta E = 3T$.

Spectral pairing ΔE for majority eigenstates in Eqs. (2) (3) is, as expected, unstable against perturbations. The crucial difference here from the disordered case [1–3, 16] is the uniform interaction strength ϕ_2 in Eq. (3), which results in an enormous Floquet emergent degeneracy. Specifically, consider the combination $Q_a = \{(q_j^{(a)}, N_j^{(a)})|j = 1, 2, \ldots, M\}$ for, i.e. $q_j^{(a)}$ copies of sites each hosting $N_j^{(a)} \geq 0$ particles. In terms of the Hubbard interaction $\phi_2 \sum_{\boldsymbol{r}\mu} n_{\boldsymbol{r}\mu}(n_{\boldsymbol{r}\mu} - 1) = \phi_2 \sum_j q_j^{(a)} N_j^{(a)}(N_j^{(a)} - 1)$, each Q_a manifold contains degenerate levels of different $\{n_{\boldsymbol{r}\mu}\}$ as $\deg(Q_a) = (3L^2)!/\prod_{j=1}^M q_j^{(a)}!$. The degeneracy, though partially lifted by $2\pi\ell/3$ in Eq. (3), leads to the instability that a small perturbation could generally trigger a reconstruction for extensive numbers of eigenstates in Eq. (2) with different configurations $\{n_{\boldsymbol{r}\mu}\}$, leading to the ergodicity as indicated by Fig. 2. Correspondingly, a Fock initial state would overlap with large numbers of eigenstates with different quasienergies without rigid spectral pairings.

To identify FBS, we then seek for manifolds with low degeneracy. Except for a homogeneous distribution $n_{\boldsymbol{r}\mu} = N_b/3L^2$ (deg=1) without dynamical signatures, the lowest degenerate $\{n_{\boldsymbol{r}\mu}\}$ deposit all N_b bosons into a single site $n_{\boldsymbol{r}\mu} = \delta_{\boldsymbol{r},\boldsymbol{r}_0}\delta_{\mu,\mu_0}N_b$. There are apparently $3L^2$ such $\{n_{\boldsymbol{r}\mu}\}$ with N_b bosons allocated into different sites $(\boldsymbol{r}_0, \mu_0)$. They compose the FBS eigenstates

$$|\boldsymbol{k}, \ell, N_b\rangle = \frac{1}{\sqrt{3}} \sum_{m=0,1,2} e^{-i(\frac{2\pi m}{3}\ell - \alpha_m)}|\boldsymbol{k}, \{n_{\boldsymbol{r}\mu} = \delta_{\boldsymbol{r},\boldsymbol{0}}\delta_{\mu,m}N_b\}\rangle, \quad (4)$$

with quasienergy $E_{\text{scar}}(\ell) = 2\pi\ell/3 + \phi_2 N_b(N_b - 1)$. The $3L^2$ FBSs equally partition into L^2 conserved many-body momentum $\boldsymbol{k}$ sectors, each hosting 3 scars with $\ell = 0, \pm 1$. Spatial translation symmetry then forbids hybridizing eigenstates of different $\boldsymbol{k}$, and temporal translation symmetry protects the conserved quasienergy separating different ℓ by $|\Delta E| = |E_{\text{scar}}(\ell + 1) - E_{\text{scar}}(\ell)| = 2\pi/3$. Therefore, FBS's experience no degenerate-level perturbations.

It still remains to consider non-degenerate perturbations. In particular, the periodicity 2π of Floquet quasienergy constrains Hubbard-interaction gap for different Q_a to be of the order unity. Then, one may expect each scar level to receive an energy correction $\sim \lambda^2$ (of Fermi golden rule type), resulting in fast detuning within $t \sim T/\lambda^2 \sim 100T$ for $\lambda = 0.1$. However, such estimations directly contradict Fig. 1 (b).

The resolution turns out to be that all three $\ell = 0, \pm 1$ scars are shifted identically, such that their quasienergy difference, dubbed *spectral pairing* gap $|\Delta E| = 2\pi/3$ [53], is unchanged. In Supplemental Materials (SM) [54], we construct the strong-drive perturbation theory. For conciseness, we illuminate the essential physics below by elaborating results up to the second order in the perturbation series, while higher orders cases are left to SM [54].

Arrange a Floquet operator in the form $U_F = U_0 U'$, where U_0 corresponds to Eq. (1) at $\lambda = 0$, and perturbations are factored into $U' \equiv e^{i\lambda H'}$. For our purposes, it is more than enough to take H' as a generic hopping Hamiltonian $H' = \sum J_{\boldsymbol{r}\mu \neq \boldsymbol{r}'\mu'} \hat{\psi}^\dagger_{\boldsymbol{r}\mu}\hat{\psi}_{\boldsymbol{r}'\mu'}$. (See SM [54] for factorization process). Scar quasienergy corrections $e^{i\tilde{E}_{\text{scar}}(\ell)} = e^{i(E_{\text{scar}}(\ell) + \sum_{\alpha=1}^{\infty}\lambda^\alpha E_\ell^{(\alpha)})}$ up to the second order read $E_\ell^{(1)} = \langle \boldsymbol{k}, \ell, N_b|H'|\boldsymbol{k}, \ell, N_b\rangle$, $E_\ell^{(2)} =$

$-\frac{1}{2}\sum'_{(\ell',\{n_{r\mu}\})}|\langle \boldsymbol{k},\ell',\{n_{r\mu}\}|H'|\boldsymbol{k},\ell,N_b\rangle|^2\cot\frac{E_{\rm scar}(\ell)-E(\ell',\{n_{r\mu}\})}{2}$, where summation $\sum'$ excludes the scar eigenstate in consideration. Here, $E^{(1)}_\ell = 0$ is trivially identical for all ℓ. Importantly, Eqs. (2)–(4) show that each term for $E^{(2)}_\ell$ depends only on the difference $(\ell-\ell')$. Due to 2π quasienergy periodicity, quantum numbers ℓ in Eqs. (2) and (3) are only defined modulo 3. That allows for shifting dummy indices ℓ' in the summation $E^{(2)}_{\ell_1} \equiv \sum'_{\ell',\{n_{r\mu}\}}\varepsilon(\ell_1-\ell') = \sum'_{\ell',\{n_{r\mu}\}}\varepsilon(\ell_2-(\ell'-\ell_1+\ell_2)) = \sum'_{\ell'',\{n_{r\mu}\}}\varepsilon(\ell_2-\ell'') = E^{(2)}_{\ell_2}$, proving the equality of energy corrections for all scars. SM [54] also numerically verifies spectral pairing rigidity for Eq. (1) and against more generic bilinear perturbations.

Importantly, it is exactly the Floquet spectrum periodicity that allows for shifting all three ℓ's in Eq. (3) by the same integer and end up with an identical set of levels, which is crucial for the above proof. In SM [54], we prove that the spectral pairing rigidity persists to *all perturbation orders* for FBS's. Therefore, $O(L^2)$ initial states overlapping with multiple FBS's separating by a rigid $\omega_0 = 2\pi/3T$ will exhibit persisting $2\pi/\omega_0 = 3T$ DTC oscillations.

Analytical identification of FBS's and proof for their spectral pairing rigidity are the main results of our work. They rely on three pivotal factors. First, *strong interactions* validate the starting point from Eqs. (2) and (3) for kicked Fock states. Second, *strong Floquet drivings* produce three identical $\ell = 0, \pm 1$ spectral plethora at $\lambda = 0$, and the 2π quasienergy periodicity intrinsic of Floquet nature enables the rigid spectral pairing for FBS against perturbations. Third, *spatiotemporal translation symmetry* prevents FBS from mutual hybridization. Therefore, FBS's describe genuine strongly interacting Floquet matters in clean systems.

Numerical verification — Revisiting previous numerics can now be illuminating. Spectral function peaks in Fig. 1 (c) derive from pairs of FBS's in Eq. (4), $A(\omega_0)|_{\lambda\to 0} = |\langle \boldsymbol{k},\ell_1,N_b|\hat{P}|\boldsymbol{k},\ell_1\pm 1,N_b\rangle|^2 = 1/3, |\omega_0| = 2\pi/3$. The spectral pairing rigidity then stabilizes $|\omega_0|$ against perturbation up to finite size effects, resulting in DTC oscillations in Fig. 1 (b). Also, Eq. (4) prescribes an L-independent EE for FBS at $\lambda \to 0$ (see SM [54] for analytical calculation) as in Fig. 2 (e).

Finally, we offer an efficient way to benchmark FBS by exploiting their peculiar $\boldsymbol{k}$ space localization. A natural measure is then the momentum space inverse participation ratio IPR $= \sum_{\{n_r\}}|\langle \boldsymbol{k},\{n_{r\mu}\}|\omega_n\rangle|^4$, where scars would show exceptionally large IPR as in Fig. 3 (a). Due to the absence of degenerate level hybridization, the original scar components in Eq. (4) still dominate upon perturbation as in Fig. 3 (b) and (c). The scaling of largest IPRs in Fig. 3 (d) reproduces the reference transition $\lambda_0 \approx 0.135$ as in Fig. 2 (e).

Experimental relevance — Small clusters studied above can be readily realized using the latest technology of quantum gas microscopes [55–57], which allows for manipulation and detection with single-site resolutions. We now further discuss cases with finite filling fractions relevant to wider ranges of experiments.

In principle, previous analytical results show that initial states populating more than one unit cell will chiefly overlap with non-scar ergodic eigenstates. Therefore, a finite filling fraction among all unit cells will eventually lead to a thermalizing behavior without dynamical signatures. However, there could exist a finite and predictable time window before decay to observe the scar DTCs due to scar localization.

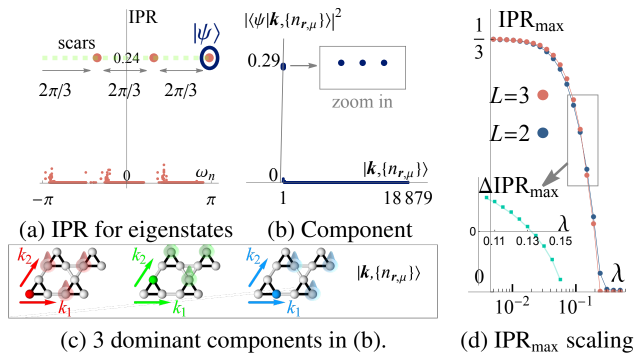

FIG. 3. Structure of Floquet eigenstates in $\boldsymbol{k} = \boldsymbol{0}$ sector. Other $\boldsymbol{k}$ sectors show essentially the same results. **(a)** Most eigenstates involve extensive number of basis $|\boldsymbol{k},\{n_{r\mu}\}\rangle$ leading to vanishing IPR, except for the 3 FBSs. **(b)** Expand for instance one FBS in the basis $|\boldsymbol{k},\{n_{r\mu}\}\rangle$, we see it is dominated by 3 components depicted in **(c)**, exactly as given by Eq. (4). **(d)** Scaling of the maximal IPR, where $\Delta\mathrm{IPR}_{\max} = \mathrm{IPR}^{(L=3)}_{\max} - \mathrm{IPR}^{(L=2)}_{\max}$ in the inset. Parameters are the same as in Fig. 1, and $L = 3$ for (a) (b).

To show it, we first take a closer look at Fig. 1 (b). The initial state of putting N_b bosons on one site overlaps with all FBS's (perturbed Eq. (4)) in different $(\boldsymbol{k},\ell)$ sectors; they interfere destructively everywhere except for the unit cell $\boldsymbol{r} = \boldsymbol{0}$, resulting in a real-space localization. As such, two scar DTCs localized in different regions will take time to sense the presence of and affect each other $\phi_2 n_{\rm ol}(n_{\rm ol}-1)t_0 \sim 1$ by interactions, giving rise to the characteristic time scale t_0 to observe DTC's before decays. Here $n_{\rm ol}$ is the density overlap for two scar DTCs hypothetically left alone in a lattice. Then, one can predict that larger distance gives a smaller density overlap $n_{\rm ol}$, which prolongs the scar DTC lifetime. Such expectations are verified numerically in SM [54] for two lattice settings relevant to the Berkeley platform. It confirms the possibility of observing DTC signatures with finite filling fractions over the experimentally accessible time, and further point out theoretically the controlling parameter for DTC lifetime therein: the distance of initially populated cells.

Conclusion — We show a distinct DTC phenomenon enforced by the analytically discovered FBS's. Its intrinsic Floquet and many-body nature stabilizes spectral pairings against translation-invariant bilinear perturbations. Moreover, the new scheme of checking Floquet emergent degeneracy and scar spectral pairing indicates a possible procedure to unveil the long-sought universal mechanism behind clean DTCs in arbitrary dimensions. It is also tantalizing to incorporate more intricate crystalline spacegroup symmetries aside translations into designing DTCs with unique structures and phenomena in clean systems.

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