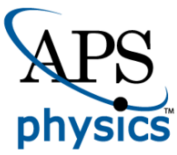

Static Effective Hamiltonian of a Rapidly Driven Nonlinear System

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On the static effective Hamiltonian of a rapidly driven nonlinear system

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We present a recursive formula for the computation of the static effective Hamiltonian of a system under a fast-oscillating drive. Our analytical result is well-suited to symbolic calculations performed by a computer and can be implemented to arbitrary order, thus overcoming limitations of existing time-dependent perturbation methods and allowing computations that were impossible before. We also provide a simple diagrammatic tool for calculation and treat illustrative examples. By construction, our method applies directly to both quantum and classical systems; the difference is left to a low-level subroutine. This sheds light on the relationship between seemingly disconnected independently developed methods in the literature and has direct applications in quantum engineering.

Driven nonlinear systems display a rich spectrum of phenomena which includes bifurcation, chaos, and topological order [1, 2]. Their behaviour is often counterintuitive and, beyond fundamental interest [3], yields important applications. A classic example is the Kapitza pendulum [4]. This system inverts its equilibrium position against gravity when driven by an appropriate fast-oscillating force and serves as a model for dynamical stabilization of mechanical systems [5]. The potency for novel applications transcends classical physics; recently the dynamical stabilization of a Schrödinger cat manifold [6–8], despite its famous fragility [9], has opened new perspectives for large-scale quantum computation [10]. In the promising field of quantum simulation, Floquet engineering of potentials [11] in ultracold atom experiments has permitted the realization of novel quantum systems with exotic properties unachievable otherwise [12]. Other important related phenomena are discrete time crystals [13] and many-body dynamical localization [14], just to name a few.

In general, driven nonlinear systems do not admit closed form solutions for their time evolution. But remarkably, under a rapid drive, their dynamics can be mapped to that generated by a time-independent effective Hamiltonian. This "Kamiltonian" [15] describes a *slow* dynamics of the system, corrected only perturbatively by a *fast* micromotion. Over the last century, different perturbation methods have been developed to construct such effective Hamiltonians and have succeeded in explaining several important nonlinear dynamical phenomena [1, 16–20]. However, these perturbation methods can hardly be carried out beyond the lowest orders in practice and a clear understanding of the connection between many of these methods is missing [21, 22]. The differences are exacerbated by the wide disparity in starting points of the classical [17, 23, 24] and quantum methods [18, 20, 25–30].

In this Letter, we construct a time-independent Kamiltonian perturbatively by seeking a pertinent *canonical* transformation. The small parameter of the expansion is the ratio of the typical rate of evolution of the driven system to the frequency of the driving force. We present a recursive formula for the Kamiltonian that allows its calculation to arbitrary order and is well-suited for symbolic manipulation. It can be applied indifferently to the classical and quantum cases, the change involving only a low-level subroutine of the symbolic algorithm. Our result unifies existing methods that have been developed solely in either the classical or quantum regimes.

We start with the equations governing time evolution of the classical or quantum state vector ρ under the action of a time-dependent Hamiltonian $H(t)$ that we write jointly as

$$\partial_t \rho = \{\!\{H, \rho\}\!\}, \tag{1}$$

where the double bracket can be understood as

$$\{\!\{H, \rho\}\!\} \to \begin{cases} \{\tilde{H}, \tilde{\rho}\} & \text{classical (Liouville)}, \\ \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] & \text{quantum (von Neumann)}. \end{cases} \tag{2}$$

Here, we have adopted the standard notation $\{\square, \square\}$ for the Poisson bracket over phase-space coordinates q and p and $[\square, \square]$ for the Hilbert space commutator. The state vector ρ can be taken to be either a phase-space distribution $\tilde{\rho}(q, p)$ or the density operator $\hat{\rho} = \sum_{x', x''} \rho_{x'x''} |x'\rangle\langle x''|$. Its time evolution is governed by the Hamiltonian H which is either the phase-space Hamiltonian $\tilde{H}(q, p, t)$ or the operator $\hat{H}(q, p, t)$. We note that one can also interpret $\{\!\{\square, \square\}\!\}$ as the Moyal bracket [31–33], in which case Eq. (1) describes the dynamics of the phase-space Wigner distribution.

In this formalism agnostic to the nature of the system, we seek a canonical transformation $\rho \to \varrho$ such that the time evolution of ϱ is governed, in the transformed frame, by the sought-after time-independent Kamiltonian. We thus consider the Lie transformation generated by a time-dependent generator S and parametrized by ϵ,

$$\begin{aligned} \varrho = e^{\epsilon L_S} \rho &= \sum_{k=0} \frac{\epsilon^k L_S^k}{k!} \rho; \\ &= \rho + \epsilon \{\!\{S, \rho\}\!\} + \frac{\epsilon^2}{2!} \{\!\{S, \{\!\{S, \rho\}\!\}\}\!\} + \cdots, \end{aligned} \tag{3}$$

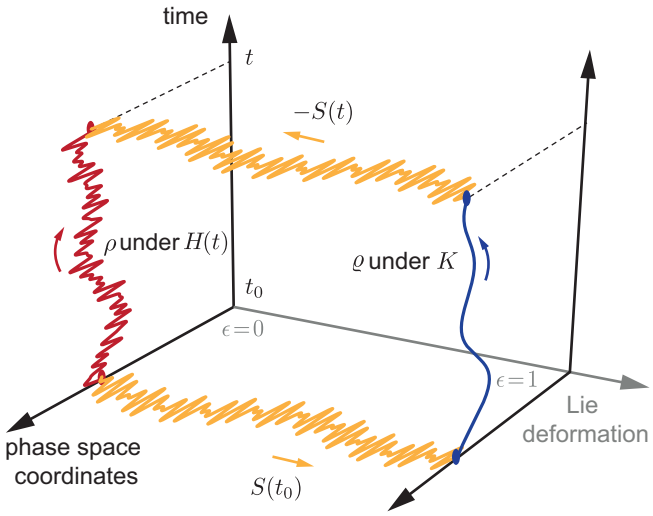

FIG. 1: Time evolution of the state vector in the transformed ($\epsilon = 1$) and un-transformed ($\epsilon = 0$) frames. The red curve represents the complicated time evolution of ρ under the time-dependent H. The blue curve represents the simpler time evolution of ϱ under the time-independent K. The transformation is exact. Under a sufficiently fast oscillating drive, the fast micromotion captured by S can be neglected and K can be taken to generate the time evolution of ρ in the un-transformed frame.

where $L_S\square = \{\!\{S, \square\}\!\}$ is the Lie derivative [34–38] generated by S. Here S is either a real phase-space function $\tilde{S}(q,p,t)$ or an Hermitian operator $\hat{S}(q,p,t)$. Equivalently, the transformed state ϱ is the solution to the differential equation $\partial_\epsilon \varrho = \{\!\{S, \varrho\}\!\}$, with initial condition $\varrho(\epsilon=0)=\rho$.

In the transformed representation, the dynamics obeys formally (1) as $\partial_t \varrho = \{\!\{K, \varrho\}\!\}$, with the Kamiltonian K given by

$$K = e^{L_S} H + \int_0^1 d\epsilon\, e^{\epsilon L_S} \dot{S}; \tag{4}$$

see Supplementary material Section A for the derivation. Note that in the quantum case, Eq. (4) reduces to the familiar expression $\hat{K} = \hat{U}^\dagger(\hat{H} - i\hbar\partial_t)\hat{U}$ with $\hat{U} = e^{-\hat{S}/i\hbar}$ [39].

We now carry out a perturbative expansion generated by S, while imposing that K is rendered time-independent. The transformation of the time evolution from $\rho \to \varrho$ is represented schematically in Fig. 1 and yields

$$\begin{aligned}\rho &= \mathcal{T} e^{\int_{t_0}^{t} dt'\, L_{H(t')}} \rho_0 \\ &= e^{L_{-S(t)}} e^{L_K (t-t_0)} e^{L_{S(t_0)}} \rho_0,\end{aligned} \tag{5}$$

where $\mathcal{T}$ is the time-ordering operator and ρ_0 is the initial state. The time evolution of ρ under H (a Lie transformation generated by H and parametrized by t) can be understood as being decomposed into three successive Lie transformations generated by $S(t_0)$, K, and $-S(t)$. Under this decomposition, the time-ordering operator drops out in the time evolution under K, providing an important simplification.

To carry out the perturbative expansion, we consider the Hamiltonian

$$H(t) = \sum_{m\in\mathbb{Z}} H_m e^{im\omega t} \tag{6}$$

with period $T = 2\pi/\omega$. For the perturbative treatment to be valid, the rate of evolution under any one H_m needs to be much smaller than ω. In the case of an unbounded Hamiltonian, either quantum or classical, the corresponding space will require truncation. We focus on the case of a periodic drive for simplicity but we note that our treatment can be generalized to include quasiperiodic or non-monochromatic drives; see Supplement section B III for a concrete example. We take the following ansatz for S and K:

$$S = \sum_{n\in\mathbb{N}} S^{(n)}, \qquad K = \sum_{n\in\mathbb{N}} K^{(n)}, \tag{7}$$

where we take $S^{(0)} = 0$ and the n^{th} terms to be of order n in the perturbation parameter, here taken to be $1/\omega$ [11, 27]. Substituting Eqs. (7) into Eq. (4) separates the problem into orders of $1/\omega$. At each order, $K^{(n)}$ can further be expressed as a sum of terms generated by a Lie series as in Eq. (3), which we write as $K^{(n)} = \sum_k K^{(n)}_{[k]}$. Demanding K to be time-independent to all orders, we find, after a few lines of algebra, the following coupled recursive formulas:

$$K^{(n)}_{[k]} = \begin{cases} H & n = k = 0 \\ \dot{S}^{(n+1)} + L_{S^{(n)}} H & k = 1 \\ \sum_{m=0}^{n-1} \frac{1}{k} L_{S^{(n-m)}} K^{(m)}_{[k-1]} & 1 < k \le n+1 \\ 0 & \text{otherwise,} \end{cases} \tag{8a}$$

$$S^{(n+1)} = \begin{cases} -\int dt\, \mathbf{osc}(H) & n = 0 \\ -\int dt\, \mathbf{osc}\Big(L_{S^{(n)}} H & \\ \quad + \sum_{k>1}^{n+1} \sum_{m=0}^{n-1} \frac{1}{k} L_{S^{(n-m)}} K^{(m)}_{[k-1]} \Big) & n > 0, \end{cases} \tag{8b}$$

where $\mathbf{osc}(f) \coloneqq f - \overline{f}$, and $\overline{f} = \frac{1}{T}\int_0^T dt\, f$. Note that H is taken to be of order zero in the perturbation parameter, but this hypothesis can be relaxed in a more elaborate treatment; see Supplement section B III.

By construction, taking the time-derivative of Eq. (8b), substituting the result into Eq. (8a), and summing over k yields a time-independent $K^{(n)}$. All in all, the computations of K and S are interleaved so that the computation of $K^{(n)}$ requires as an input the value of $S^{(m\le n)}$. Demanding the time-independence of $K^{(n)}$ fixes $\dot{S}^{(n+1)}$, allowing the recursion to be carried out to the next order.

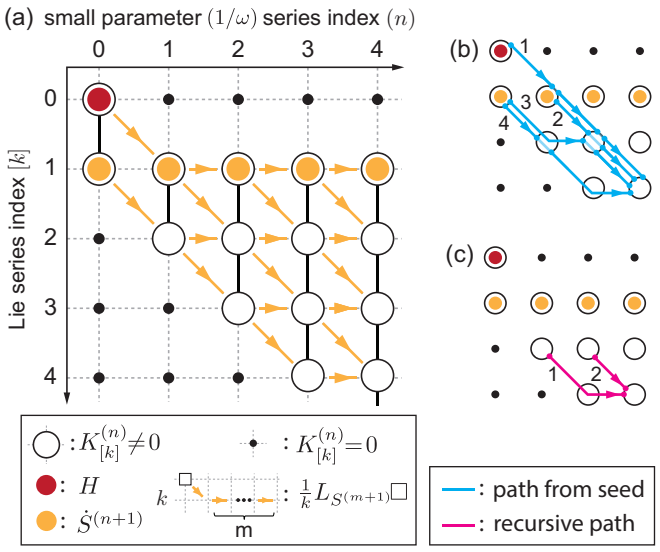

FIG. 2: (a) Grid for the diagrammatic construction of K and S. Colored circles represent the seeds generating the series to all orders. (b) As an example, all the paths contributing to the calculation of $K^{(3)}_{[3]}$ are highlighted. (c) Here, only the subpaths contributing to the recursive expression of the aforementioned term are highlighted.

The coupled recursive formula in Eq. (8) constructs, as announced, S and K order-by-order.

The mathematical structure of the recursive formula Eq. (8) is shown diagrammatically in Fig. 2, as we now explain. The figure consists of a grid indexed by the integers n and k. The grid supports a graph. Each node (n,k) of the graph corresponds to a summand $K^{(n)}_{[k]}$, and the colored ones represent the "seeds" of the calculation. The summand $K^{(n)}_{[k]}$ is itself a sum of terms, each corresponding to a path connecting the node (n,k) to a seed. Evaluating a path corresponds to taking Lie derivatives over H or $\dot{S}^{(n+1)}$ as dictated by the seed color. The rule is that each Lie derivative is specified by a valid subpath, which must start "downwards" and, when followed by m horizontal edges at row k, contributes with $L_{S^{(m+1)}}/k$. Finally, if the considered node is itself colored, either H or $\dot{S}^{(n+1)}$ must be added to the sum. We note that our grid construction is inspired by [40], where the construction is limited to completely classical and time-independent systems.

Let us discuss, as an example, how $K^{(3)}_{[3]}$ is evaluated from the figure. As indicated by panel Fig. 2(b), $K^{(3)}_{[3]}$ contains only four terms corresponding to the concatenations of the valid subpaths (in blue). The sum reads

$$K^{(3)}_{[3]} = \frac{L_{S^{(1)}}}{1}\frac{L_{S^{(1)}}}{2}\frac{L_{S^{(1)}}}{3}H + \frac{L_{S^{(1)}}}{2}\frac{L_{S^{(1)}}}{3}\dot{S}^{(2)} + \frac{L_{S^{(1)}}}{2}\frac{L_{S^{(2)}}}{3}\dot{S}^{(1)} + \frac{L_{S^{(2)}}}{2}\frac{L_{S^{(1)}}}{3}\dot{S}^{(1)},$$

where the terms are ordered as enumerated in the figure.

Alternatively, one could have expressed $K^{(3)}_{[3]}$ recursively by directly applying Eq. (8a). The computation of $K^{(3)}_{[3]}$ then involves only the two pink subpaths shown in Fig. 2(c) and yields

$$K^{(3)}_{[3]} = \frac{L_{S^{(2)}}}{3}K^{(1)}_{[2]} + \frac{L_{S^{(1)}}}{3}K^{(2)}_{[2]}.$$

At this stage, once all entries of the n^{th} column are computed, the calculation proceeds by demanding the time-independence of $K^{(n)}$ computed as their row-sum over column n and represented by the vertical bold lines in Fig. 2(a). This is required by Eq. (8b). For the column $n=3$ the algorithm yields

$$S^{(4)} = -\int dt\, \mathbf{osc}\left(L_{S^{(3)}}H + K^{(3)}_{[2]} + K^{(3)}_{[3]} + K^{(3)}_{[4]}\right),$$

which is a necessary ingredient to compute $K^{(5)}$ and so, the calculation proceeds.

We further illustrate our formulation by treating three concrete examples in the Supplement.

First, in Section B I, we treat a standard dynamical system: the Kapitza pendulum. Previous works [4, 21] find the effective Hamiltonian in the classical case by averaging its equation of motion. This method is known to become unwieldy even at the lowest orders, and moreover lacks an equivalent quantum counterpart, which veils any comparison to a quantum effective Hamiltonian. We apply Eq. (8) to the Kapitza pendulum to find both the classical and quantum effective Hamiltonians, which to our knowledge is two orders beyond what is available in previous literature. Only by going to such a high order, we are able locate terms in the quantum static effective Hamiltonian that cannot be obtained by any quantization prescription [41–43] applied to its classical counterpart: a consequence of Groenewold's theorem [31]. This result explains the difficulty encountered in stating the quantum-classical correspondence in the context of effective Hamiltonians [21, 22], underlining the necessity of exploiting the shared underlying Lie algebra.

Second, in Section B II, we treat a driven Duffing oscillator, an archetype of several driven superconducting circuits, and find the effective Hamiltonian to order 5 in the perturbation parameter. In particular, we focus on the renormalization of the frequency and Kerr coefficients of the oscillator in the presence of the drive. We demonstrate with this toy example the convergence of our series to numerical simulations obtained via exact Floquet numerical diagonalization.

Finally, in Section B III, we show the remarkable agreement with state-of-the-art experiments [44] that independently measured the effective Hamiltonian of a driven transmon-cavity superconducting circuit [45] and could only be analyzed numerically until our work. With this final example, we demonstrate the generality of the recursive formula: by applying it to a multi-mode, non-

monochromatic Hamiltonian, and developing the expansion for a perturbation parameter other than $1/\omega$, we are able to predict and explain to a high accuracy the measured effective Hamiltonian even at large intra-cavity photon numbers.

We stress again that Eq. (8) is agnostic to the choice of Lie bracket in Eq. (2). A Lie-based formulation is thus well-suited to unify seemingly disconnected perturbation methods, in particular those that, to be linked, require the quantum-classical correspondence to be made explicit.

Exploiting this property, we now turn to discuss the connection between several time-dependent perturbation methods developed independently. We consider their common starting point to be the additive ansatz

$$Z = \mathcal{Z} + \zeta(\mathcal{Z}, {}^{c}\mathcal{Z}), \tag{9}$$

where Z is the state variable, ${}^{c}\mathcal{Z}$ is the conjugate variable to $\mathcal{Z}$ and ζ is a correction. For time-varying problems, it is customary to take $\mathcal{Z}$ to describe the slow dynamics and the correction to describe the fast dynamics with vanishing time-average ($\bar{\zeta}=0$). To compute ζ, different methods proceed in vastly different ways: the classical ones rely on partial derivatives of phase-space functions [4, 21, 23, 24]; this draws a line separating them from the quantum methods which rely on matrix products [21, 25–29]. These procedural differences hide their shared structure.

We uncover the connection between these methods by realizing that the procedural differences stem from *premature* specifications of a particular Lie bracket. Identifying this feature allows us to relate the correction ζ, specified by each method, to the generator S of the Lie transformation as $\zeta=(e^{L_S}-I)\mathcal{Z}$. In other words, the ansatz in Eq. (9) corresponds to the additive representation of the exponential map in Eq. (3). It follows that, if carried out to all orders, these methods correspond to invertible canonical transformations. See Section C of the Supplementary material for a detailed discussion on the relationship between the different ansätzs.

In Fig. 3, we show how we can collect seemingly disconnected perturbation methods and unify them under the umbrella of Lie series. The two main branches, colored in red (right) and blue (left), group the quantum and classical methods. The ones based on equations of motion (EOM) correspond to different choices of Z in Eq. (9). Among the quantum methods, secular averaging theory (SAT) [46] corresponds to $Z=\hat{\rho}$. The perturbative expansion can also be developed at the level of the wave function $Z=|\phi\rangle$, allowing for the derivation of higher-order rotating wave approximations (RWA) [26]. In this case, the ansatz is $\mathcal{Z}=|\varphi\rangle$ and $\zeta=\hat{\delta}|\varphi\rangle$ and it can be mapped to our approach by taking $\zeta=(e^{-\hat{S}/i\hbar}-1)|\varphi\rangle$. This last relation can be understood by noting that in the quantum case Eq. (3) reduces to $e^{L_S}\hat{\rho}=e^{-\hat{S}/i\hbar}\hat{\rho}e^{\hat{S}/i\hbar}$. Classically,

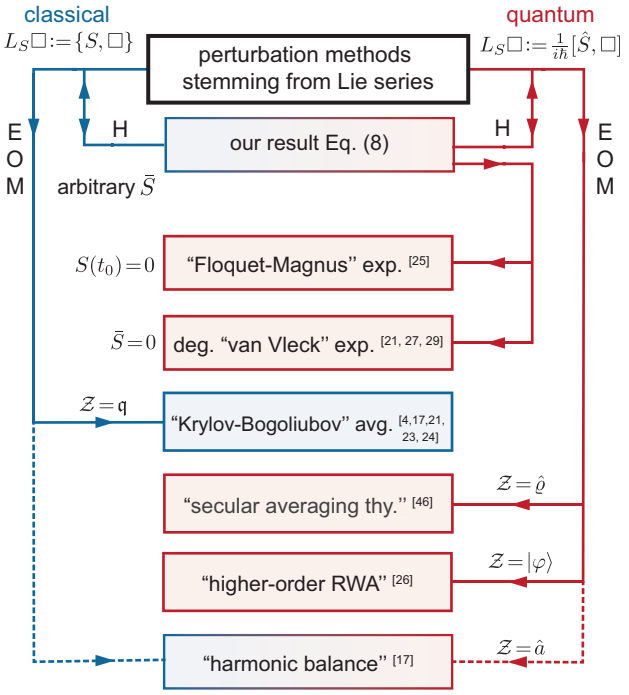

FIG. 3: Relationship between various perturbation methods for systems submitted to a time-dependent oscillatory drive. All the methods in the tree can be seen as falling under the umbrella of Lie series. The methods can be divided between classical (blue lines/left) and quantum (red lines/right) ones. Other forks separate the methods based on the equation of motion (EOM) from the methods based on the Hamiltonian (H). The dashed lines refer to a class of methods whose effective Hamiltonians have not yet been identified and they are left for future work. The methods labeled by a $\mathcal{Z}$ symbol are additive ansatz-based unlike the methods containing S which can be thought as multiplicative. For the Hamiltonian methods, the associated integration constant for S in Eq. (8b) is specified as a condition. The double arrow refers to a bidirectional relationship and the acronym RWA stands for rotating wave approximation.

the Krylov-Bogoliubov (KB) method [4, 17, 21, 23, 24] averages the equation of motion of the position coordinate $\mathcal{Z}=\mathfrak{q}$ and $\zeta=\zeta(\mathfrak{q},\mathfrak{p},t)$, where $\mathfrak{p}$ is the conjugate momentum, and it maps to our approach with $\zeta=(e^{L_S}-I)\mathfrak{q}$ (the change of sign in S is simply a change from the active representation used so far to the passive representation used in KB).

Besides the EOM methods, we also include the most common quantum Hamiltonian methods in the genealogy of Fig. 3. They are characterized by the utilization of Floquet theorem [47], which guarantees the existence of a unitary transformation rendering the Kamiltonian time-independent. They map naturally to the exponential representation discussed in this work. We find that using the freedom in the integration constant of Eq. (8b), as specified in Fig. 3, we recover the so-called Floquet-Magnus expansion ($S(t_0)=0$) [25] or the van Vleck ex-

pansion ($\bar{S} = 0$) [21, 27, 29]. This gives formal ground to the observations made in [22, 28] on the connection between these two methods.

Note that, with the exception of the Floquet-Magnus expansion, all the aforementioned methods were limited to the lowest orders. Instead, our symbolic formula can be readily used to carry out the calculation to arbitrary order with computer algebra software [48, 49]. We illustrate this by taking as an example the widely employed van Vleck expansion. We have explicitly written a symbolic algebra algorithm, made available in [50], and used it to explicitly display the expansion up to order five, automatedly (see Section D of the Supplementary material). To the best of our knowledge, the expansion could only be found to order three in the literature [28] until this paper.

In summary, we developed a perturbation method to efficiently treat rapidly driven nonlinear systems. It yields a double coupled recursive formula well-suited for automated symbolic computation to arbitrary order. We achieve this result by constructing a canonical transformation that explicitly decouples the relevant dynamics, governed by a time-independent effective Hamiltonian, from the complicated micromotion. Our treatment is completely agnostic to the classical or quantum nature of the problem and sheds light on the longstanding discussion of the relationship between well-known perturbation methods developed independently of each other. We note that an application of the structural correspondence to out-of-equilibrium driven systems had been suggested by [16] but was not carried out until this paper. We further remark that our formula can be generalized to treat Hamiltonians of arbitrary order in the perturbation parameter. This is particularly powerful when treating parametric processes in driven nonlinear bosonic oscillators [51], and is thus relevant to Hamiltonian engineering in superconducting quantum circuits [19, 52–54]. Finally, we note that there is a strong relationship between time-periodic (Floquet) and space-periodic (Bloch) systems, and thus, our recursive formula can be adapted to compute in this context the Schrieffer-Wolff transformation [55] to arbitrary order and potentially get new results in quantum many-body problems.

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