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Lindblad master equations for quantum systems coupled to dissipative bosonic modes

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We present a general approach to derive Lindblad master equations for a subsystem whose dynamics is coupled to dissipative bosonic modes. The derivation relies on a Schrieffer-Wolff transformation which allows to eliminate the bosonic degrees of freedom after self-consistently determining their state as a function of the coupled quantum system. We apply this formalism to the dissipative Dicke model and derive a Lindblad master equation for the atomic spins, which includes the coherent and dissipative interactions mediated by the bosonic mode. This master equation accurately predicts the Dicke phase transition and gives the correct steady state. In addition, we compare the dynamics using exact diagonalization and numerical integration of the master equation with the predictions of semiclassical trajectories. We finally test the performance of our formalism by studying the relaxation of a NOON state and show that the dynamics captures quantum metastability.

Introduction.— The description of open many-body quantum systems dynamics is a formidable challenge for modern physics. Typical out-of-equilibrium scenarios are a quantum system (QS) interacting with an environment of bosonic modes (BM) [1] [see Fig. 1(a)]. This is the common setup of quantum electrodynamics, where the BM are the electromagnetic field [2, 3]. Furthermore, it is the basis of prominent implementations of quantum simulators because it allows to tailor the interactions between the constituents of the QS [4–7]. Examples include quantum gases in optical cavities [7–11], optomechanical arrays [12], phonon-mediated interactions of trapped ions [4–6, 13], polaritons or nitrogen-vancancy centers in diamond coupled to microcavities or mechanical elements [14–16], and photonic crystals [17].

A powerful tool to analyze open many-body QS is the Keldysh approach [18, 19], which employs methods of quantum field theory and is very successful in predicting their asymptotic behavior. The dynamics and metastability are instead accessed by full simulations or so-called effective master equations. The latter dispose of a large part of the Hilbert space by eliminating the BM [20–23] and include interactions, noise, and dissipation they mediate. The derivation of effective master equations is an active field [24, 25] with various emphases, such as high-precision metrology [26, 27], exact solutions and validity [28–31], multi-mode configurations [32–35], and coherent many-body QS [21–23].

Recently, in cavity quantum electrodynamics, effective Redfield master equations were derived [21, 22]. While describing the correct low-frequency behavior, they are not necessarily positive. Attempts to make them positive, e.g., by bringing them into Lindblad form, resulted in incorrect predictions of the asymptotics. Other effective descriptions add fluctuations around a mean-field treatment of the BM [23]. Here, the problem of positiveness was resolved by assuming a thermalization of the QS, which is questionable regarding the existence of non-thermal metastable states [36]. This highlights the need to identify general effective descriptions that preserve positivity. With such, one could determine the spectrum of the open system or simulate the master equation using quantum state diffusion models [37]. This can then be used to analyze critical properties of driven-dissipative QS [38–40], study prethermalization and metastability [36, 41, 42], and shed light on aspects that cannot be accessed easily otherwise, including measurement-induced phase transitions [43–45].

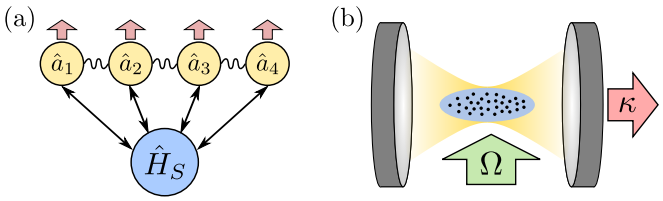

FIG. 1. (a) The general model includes coupled dissipative bosonic modes $\hat{a}_k$ interacting with a quantum system described by $\hat{H}_S$. (b) Example: a dissipative optical cavity mode couples to a cloud of driven atoms.

In this Letter, we identify a general procedure which allows derivation of effective master equations for an arbitrary QS that is coupled to dissipative BM. We use a specific type of Schrieffer-Wolff transformation [46] to reduce the coupling between the QS and the BM such that we can eliminate the latter. This transformation is a displacement that depends, in general, on the eigenstates and eigenenergies of the decoupled QS. The resulting master equation has the Lindblad form and the specific procedure allows us to systematically include retardation effects between the QS and BM. As an example, we derive an effective master equation for the dissipative Dicke model and benchmark our results by comparing the spectrum and dynamics with the one of the composite system.

Derivation of the effective master equation.— We start by considering a set of BM, described by the annihilation (creation) operators $\hat{a}_k$ ($\hat{a}_k^\dagger$), with eigenenergies ω_k, that exchange energy at the finite rate κ_k with an external thermal bath at temperature $1/\beta$. The dynamics for the density matrix $\hat{\rho}$ is described by $\mathcal{L}_\mathrm{d}\hat{\rho} = \sum_k \{\kappa_k(n_k+1)\mathcal{D}[\hat{a}_k]\hat{\rho} + \kappa_k n_k \mathcal{D}[\hat{a}_k^\dagger]\hat{\rho}\}$, where we introduced $\mathcal{D}[\hat{O}]\hat{\rho} = 2\hat{O}\hat{\rho}\hat{O}^\dagger - \hat{O}^\dagger\hat{O}\hat{\rho} - \hat{\rho}\hat{O}^\dagger\hat{O}$.

In this Letter, we consider the case where $n_k = [\exp(\beta\omega_k) - 1]^{-1} \approx 0$, which is valid if the ω_k are optical frequencies. On a timescale that is longer than the typical relaxation time $1/\kappa_k$, the BM couple coherently to a QS described by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \hat{H}_S + \sum_k \left(\sum_{k'} \hat{a}^\dagger_{k'} \hat{\Omega}_S^{k',k} \hat{a}_k + \hat{a}^\dagger_k \hat{S}_k + \hat{S}^\dagger_k \hat{a}_k \right). \quad (1)$$

The Hamiltonian in absence of the BM is denoted by $\hat{H}_S$. The term proportional to $\hat{\Omega}_S^{k',k} = (\hat{\Omega}_S^{k,k'})^\dagger$ denotes the frequencies and mode-mode coupling that may depend on the QS's degrees of freedom. The last term in Eq. (1) represents the coupling of the BM to QS operators $\hat{S}_k$. The dynamics of the density matrix $\hat{\rho}$ is then described by the master equation

$$\frac{\partial \hat{\rho}}{\partial t} = \mathcal{L}\hat{\rho} := -i[\hat{H}, \hat{\rho}] + \mathcal{L}_{\rm d}\hat{\rho}. \quad (2)$$

We want to eliminate the BM degrees of freedom and derive an effective master equation describing the dynamics of the QS. The steps for the derivation are as follows: (i) We derive the master equation for $\tilde{\rho} = \hat{D}^\dagger \hat{\rho} \hat{D}$ where $\hat{D} = \exp[\sum_k (\hat{a}^\dagger_k \hat{\alpha}_k - \hat{\alpha}^\dagger_k \hat{a}_k)]$ is a displacement operator that correlates the BM to the QS by establishing an effective-field operator $\hat{\alpha}_k$. We assume $\|\hat{\alpha}_k\| \sim \epsilon \ll 1$ and apply a perturbation theory where we discard all terms that are of third order in ϵ or higher. (ii) In the displaced picture, we project the BM onto the thermal state. Here, we assume that the displaced BM are to good approximation in a thermal state, whereas they are not necessarily thermal in the original picture due to the interaction with the QS. For the parameter regime considered here the thermal state is essentially the vacuum state $|\text{vac}\rangle$ and we can define $\hat{\rho}_{\rm sys} = \langle \text{vac}|\tilde{\rho}|\text{vac}\rangle$. We systematically include the coupling of $\hat{\rho}_{\rm sys}$ to higher Fock states in the displaced BM and optimize the operators $\hat{\alpha}_k$, such that $\hat{\rho}_{\rm sys}$ is decoupled up to third order in ϵ. This decoupling procedure is reminiscent to a Schrieffer-Wolf transformation. In the Supplemental Material (SM) [47] we show that these steps result in solving

$$\frac{\partial \hat{\alpha}_k}{\partial t} = -i[\hat{H}_S, \hat{\alpha}_k] - i \sum_{k'} \hat{\Omega}_S^{k,k'} \hat{\alpha}_{k'} - i\hat{S}_k - \kappa_k \hat{\alpha}_k. \quad (3)$$

With the solution $\hat{\alpha}_k$ of the above equation, we obtain a master equation for the density matrix $\hat{\rho}_{\rm sys}$ that reads

$$\frac{\partial \hat{\rho}_{\rm sys}}{\partial t} = \mathcal{L}_{\rm eff}\hat{\rho}_{\rm sys} := -i[\hat{H}_{\rm eff}, \hat{\rho}_{\rm sys}] + \sum_k \kappa_k \mathcal{D}[\hat{\alpha}_k]\hat{\rho}_{\rm sys} \quad (4)$$

and the effective Hamiltonian

$$\hat{H}_{\rm eff} = \hat{H}_S + \frac{1}{2}\sum_k (\hat{\alpha}^\dagger_k \hat{S}_k + \hat{S}^\dagger_k \hat{\alpha}_k). \quad (5)$$

This master equation is the main result of this Letter that we now discuss in greater depth. We first observe that Eq. (4) is of the Lindblad form, thereby it describes a completely positive divisible quantum process if the $\hat{\alpha}_k$ are bounded operators. This is usually fulfilled since we require $\|\hat{\alpha}_k\| \ll 1$, which is physically the case if κ_k exceeds the coupling $\|\hat{S}_k\|$. In the discussion below, we will focus on the single-mode case [48] but we provide a multi-mode example in the SM [47]. The terms proportional to κ and $\hat{\Omega}_S$ in Eq. (3) describe the relaxation of the BM to the thermal state in absence of $\hat{S}$. During this relaxation, the QS evolves according to $\hat{H}_S$ such that the BM sees a retardation effect determined by $[\hat{H}_S, \hat{\alpha}]$. This term is a principal finding because it shows that the BM carries information about the evolution of the QS. In fact, solving Eq. (3) for the steady state, assuming that $[\hat{H}_S, \hat{\alpha}]$ can be ignored, results in the adiabatic elimination [49–51] given by $\hat{\alpha} = -i\hat{S}/(i\hat{\Omega}_S + \kappa)$ and includes quantum noise due to κ, visible by the proportional incoherent part in Eq. (4). For $\|\hat{\Omega}_S\| \gg \kappa$, it also recovers the dispersive limit, where the QS evolves coherently with $\hat{H}_{\rm eff}$. Using Eqs. (3) and (4), we can systematically take retardation and noise effects into account by treating $[\hat{H}_S, \hat{\alpha}]$ and κ either in arbitrary order, or as a perturbation. We remark, that first-order perturbation in retardation effects has been studied in semiclassical descriptions, giving rise to collective cavity cooling and dissipation-assisted prethermalization [20, 34–36, 52, 53]. However, the effective master equation (4) is a full quantum description and therefore complementary to the results of Refs. [21–23] that derive effective quantum descriptions. Similar to Ref. [23], we use a displacement operation to eliminate the BM, however, our "displacement" is not based on a mean-field assumption. Instead, "$\hat{\alpha}$" is an operator that intrinsically includes fluctuations. Our approach requires thermalization of the displaced BM, but no thermalization of the QS, allowing Eq. (4) to describe metastable dynamics. To show the potential of Eq. (4) we will analyze an example, namely the dissipative Dicke model.

Application to the dissipative Dicke model.— The dissipative Dicke model describes a single mode coupled to N two-level atoms. It can be realized with driven atoms interacting with an optical cavity [8, 54] [see Fig. 1(b)]. We therefore denote the QS by atoms and the BM by cavity mode. With our definitions in Eq. (1) we use $\hat{H}_S = \omega_0 \hat{S}^z$, the cavity frequency $\hat{\Omega}_S = \omega_c$, and coupling $\hat{S} = 2g\hat{S}^x/\sqrt{N}$. We have introduced the spin operators $\hat{S}^a = \sum_{j=1}^N \hat{\sigma}^a_j/2$ with $a \in \{x, y, z\}$, where $\hat{\sigma}^a_j$ denote the Pauli matrices of the jth atom. The dissipative Dicke model exhibits a phase transition in the thermodynamic limit $N \to \infty$ from a normal ($g < g_c$) to a superradiant phase ($g > g_c$) [18, 54–56], with a critical value $g_c^2 = \omega_0(\omega_c^2 + \kappa^2)/(4\omega_c)$. In contrast to the quantum phase transition of the Dicke model [55, 56], the dissipative Dicke model exhibits different critical exponents and a damping rate at steady state [18, 57, 58].

In Ref. [21], it was shown that an atom-only Redfield master equation for this model gives the correct low-frequency behavior. On the other hand, this cannot be achieved by a Lindblad master equation derived after making a large-detuning or a secularization approximation, which are obtained assuming $\omega_0/\omega_c = 0$ or dropping the co-rotating and off-resonant $\hat{a}^\dagger \hat{S}^+$

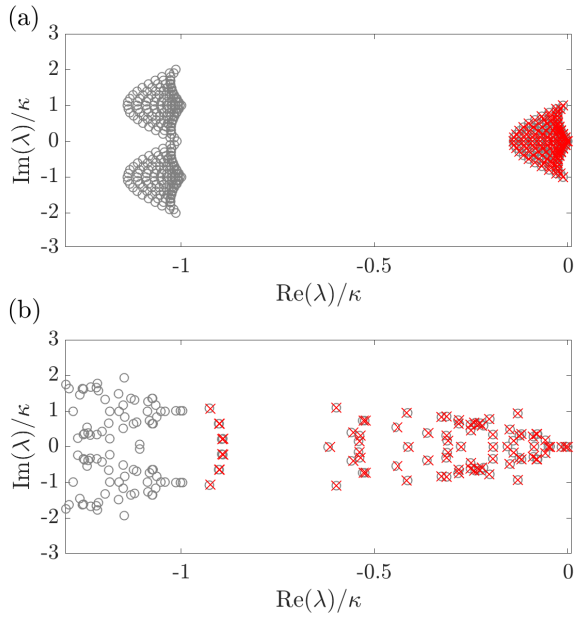

FIG. 2. Eigenvalues λ in units of κ of Eq. (2) (gray "o") and Eq. (4) (red "x") for the dissipative Dicke model. The parameters are $N = 10$, $\omega_c = \kappa$, $\omega_0 = 0.1\kappa$, and (a) $g = 0.5g_c$, (b) $g = 2g_c$.

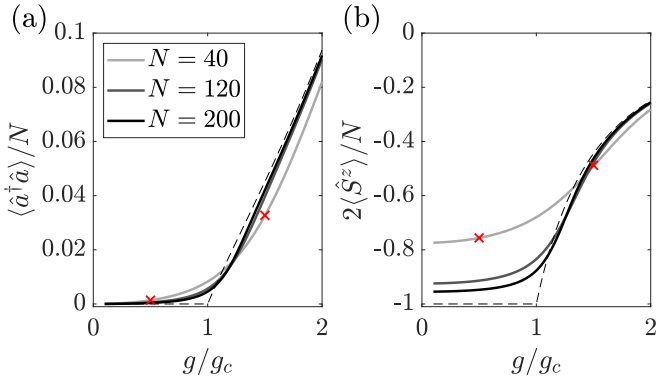

FIG. 3. (a) Photon number $\langle \hat{a}^\dagger \hat{a} \rangle$ and (b) inversion $\langle \hat{S}^z \rangle$ as a function of g in units of the critical coupling strength g_c. Dashed lines are the mean-field results for $N \to \infty$ and solid lines are obtained by finding the steady-state of Eq. (4) for the dissipative Dicke model with various atom numbers N (see inset). Red crosses are obtained by finding the steady state of the full master equation (2) for $N = 40$. The remaining parameters are $\omega_c = \kappa$, $\omega_0 = 0.1\kappa$.

and $\hat{a}\hat{S}^-$ terms ($\hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y$), respectively. Based on this, it was conjectured that correct, atom-only master equations for the dissipative Dicke model require a non-Lindblad form. We will show that the Lindblad master equation (4) goes beyond the large-detuning and secularization approximation and is a counter example for this conjecture.

We determine $\hat{\alpha}$ using Eq. (3) whose steady state is

$$\hat{\alpha} = \alpha_+ \hat{S}^+ + \alpha_- \hat{S}^-, \tag{6}$$

with $\alpha_\pm = -g/[\sqrt{N}(\omega_c \pm \omega_0 - i\kappa)]$. As a result of the commutator $[\hat{H}_S, \hat{\alpha}]$ the effective cavity field $\hat{\alpha}$ has two sidebands shifted by $\pm\omega_0$ from ω_c, corresponding to the excitation or de-excitation of the atoms. If we impose $\omega_0 = 0$ in Eq. (6) we recover the large-detuning result as in Ref. [21] where the Redfield master equation becomes of Lindblad form and Hermitian. Using Eq. (6) in Eq. (4) we also find co-rotating terms $[\hat{S}^\pm]^2$, dropping the latter results in the secularization approximation with the same result as in Ref. [21]. This shows that Eq. (4) is of the Lindblad form, recovers two limiting cases of the Redfield master equation [21], and does, in general, not require the large-detuning or secularization approximation which are insufficient to correctly describe the dissipative Dicke model. We now compare the spectra of Eq. (4) and the full master equation (2) for small N by diagonalizing $\mathcal{L}_{\text{eff}}$ and $\mathcal{L}$ using the symmetric states $|m\rangle$, with $\hat{S}^z|m\rangle = m|m\rangle$ for $m = -N/2, -N/2+1, \ldots, N/2$.

In Fig. 2, we show the complex eigenvalues λ of $\mathcal{L}$ and $\mathcal{L}_{\text{eff}}$ as gray circles and red crosses, respectively. Below threshold, $g < g_c$, Fig. 2(a) shows an excellent agreement of the full and effective descriptions for the eigenvalues with the largest real parts. This emphasizes that $\mathcal{L}_{\text{eff}}$ correctly describes long timescales and discards faster timescales with $\text{Re}(\lambda) < -\kappa$, thereby describing the dynamics of metastable states. Figure 2(b) shows the spectrum in the superradiant phase, $g > g_c$. Again, we find great agreement, which is remarkable since the gap between the "correctly" described modes and $\text{Re}(\lambda) \approx -\kappa$ is much smaller. This direct comparison suggests that the effective description is valid across the phase transition.

To further support this claim, we use Eq. (4) to make analytical predictions in the limit $\omega_c, \kappa \gg \omega_0$, i.e., the limit when the cavity evolves much faster than the atoms [21]. For this case, the commutator term $[\hat{H}_S, \hat{\alpha}]$ can be treated perturbatively and the coefficients in Eq. (6) can be expanded according to $\alpha_\pm = -g/[\sqrt{N}(\omega_c - i\kappa)] \pm g\omega_0/[\sqrt{N}(\omega_c - i\kappa)^2]$. In the large N limit, we can derive mean-field equations for $S^a = \langle \hat{S}^a \rangle$ with $a \in \{x, y, z\}$ that are reported in the SM [47]. The resulting equations are the same as the ones given in Ref. [21]. Consequently, we find the correct threshold, oscillation and damping rates, and critical exponents in the thermodynamic limit (see SM [47]). The steady-state values of $I = \langle \hat{a}^\dagger \hat{a} \rangle$ and S^z in the thermodynamic limit are given by $I_0 = 0$ and $S^z_0 = -N/2$ for $g < g_c$ and $I_0 = Ng^2(1 - g_c^4/g^4)/(\omega_c^2 + \kappa^2)$ and $S^z_0 = -Ng_c^2/(2g^2)$ for $g > g_c$. In Fig. 3(a) and (b), we show I_0 and S^z_0 as functions of g as black dashed lines. Furthermore, we present the values I and $\langle \hat{S}^z \rangle$ by numerically finding the steady state of Eq. (4) and then calculating $\langle \hat{a}^\dagger \hat{a} \rangle = \text{Tr}[\hat{\alpha}^\dagger \hat{\alpha} \hat{\rho}_{\text{sys}}]$ and $\langle \hat{S}^z \rangle = \text{Tr}[\hat{S}^z \hat{\rho}_{\text{sys}}]$. Since $\mathcal{L}_{\text{eff}}$ does not include the cavity degrees of freedom, we are able to diagonalize it for larger atom numbers. As can be seen in Fig. 3(a) and (b), the analytical result and the numerical results are in better agreement for larger atom numbers N. For $N = 40$, we were able to find the steady state of $\mathcal{L}$, depicted for two values of g/g_c as red crosses. This agreement indicates that Eq. (4) is also valid for finite atom numbers. Altogether, these results

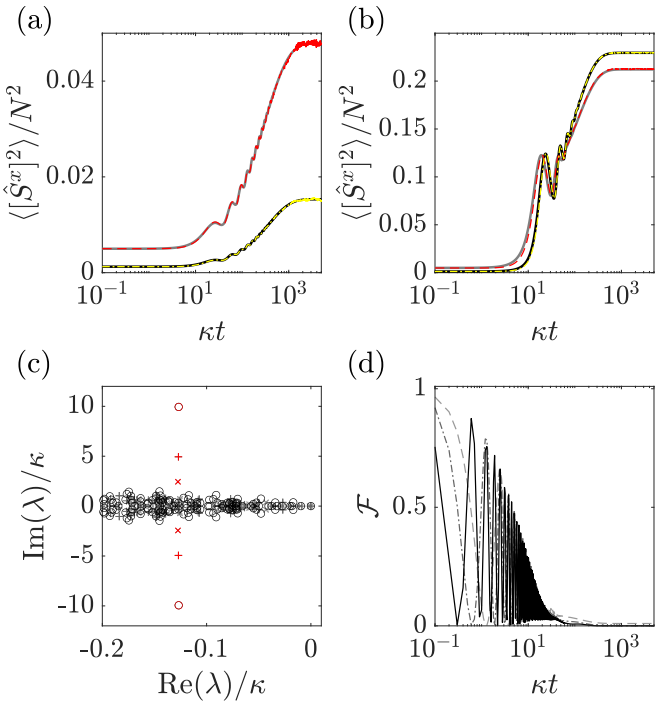

FIG. 4. The value of $\langle[\hat{S}^x]^2\rangle$ as function of time in units of $1/\kappa$ for (a) $g = 0.5g_c$ and (b) $g = 2g_c$. The gray (black) lines are obtained by simulating the effective master equation (4) with $N = 50$ ($N = 200$). The red dashed (yellow dashed-dotted) lines are simulated with the stochastic method reported in the SM [47] and averaged over 20000 simulations with $N = 50$ ($N = 200$). (c) Eigenvalues λ in units of κ of Eq. (4) for $N = 25$ ("x"), $N = 50$ ("+"), and $N = 100$ ("o"). The red symbols mark the eigenvalues discussed in the text. (d) Fidelity $\mathcal{F} = \langle\Psi|\hat{\rho}_{\mathrm{sys}}(t)|\Psi\rangle$ as function of time in units of $1/\kappa$ simulated using Eq. (4) initialized with the state $|\Psi\rangle$ discussed in the text for $N = 25$ (light gray dashed), $N = 50$ (gray dashed-dotted), and $N = 100$ (black solid) with $\omega_c = \kappa$, $\omega_0 = 0.1\kappa$.

show that $\mathcal{L}_{\mathrm{eff}}$ predicts the correct steady state, low-frequency behavior, and critical exponents.

In the remainder, we focus on out-of-equilibrium dynamics, i.e., scenarios where the system is initialized "far" away from the steady state. The dynamics and relaxation in such situations require the correct description of high and low frequency modes. Since it is difficult to simulate the full master equation (2) for large N, we use a semiclassical stochastic method to compare with simulations of Eq, (4). The stochastic method simulates the coupled dynamics of the c-number equivalents of spin components S^x, S^y, and S^z coupled to the noisy real part x and imaginary part p of the field amplitude. Details are reported in the SM [47]. In a benchmark, we initialize the system with all atoms in the ground state, $\langle\hat{S}^z\rangle = -N/2$, and evolve it according to Eq. (4). Figure 4(a) and (b) show the time evolution of $\langle[\hat{S}^x]^2\rangle$ for $g = 0.5g_c$ and $g = 2g_c$, respectively. Both simulations are in excellent agreement. Since the stochastic simulations evolve the coupled atom-cavity dynamics on equal footing, we conclude that Eq. (4) incorporates the correct retarded interaction between atoms and cavity, and is well suited for out-of-equilibrium dynamics.

Finally, we analyze a scenario with quantum features that cannot be described by semiclassical stochastic methods [47]. To achieve this we first analyze the spectrum of Eq. (4) for $g = 2g_c$, shown in Fig. 4(c). We find a mode with a growing imaginary part for increasing N (marked red). The underlying mode is related to the coherence $\hat{c} = |N/2\rangle\langle -N/2|$ that oscillates with a frequency $\sim N\omega_0$. Remarkably, its frequency exceeds the cavity resonance and linewidth while its damping is far less than κ. Therefore, it can be seen as a metastable high-frequency oscillation with a number of periods diverging with N. To find this oscillation dynamically, we initialize the system in the NOON state $|\Psi\rangle = (|N/2\rangle + |-N/2\rangle)/\sqrt{2}$ such that the coherence $\hat{c}$ is present at $t = 0$. We then evolve $|\Psi\rangle$ according to Eq. (4) and calculate the fidelity $\mathcal{F} = \langle\Psi|\hat{\rho}_{\mathrm{sys}}(t)|\Psi\rangle$, visible in Fig. 4(d). We find an oscillation frequency that increases with N, while the damping is nearly independent of N. This agrees with the behavior of the red-marked modes in Fig. 4(c) and further highlights the ability of Eq. (4) to describe out-of-equilibrium situations with entangled quantum states.

Conclusion.—We have developed a formalism for the derivation of effective master equations that describe the reduced dynamics of a QS coupled to dissipative BM. These master equations are of Lindblad form, thereby ensuring that the positivity is preserved. Furthermore, our approach includes the retarded interaction between the QS and the BM. We demonstrated this by applying the formalism to the dissipative Dicke model, where it correctly describes the steady state and dynamics for small to large atom numbers.

The method presented here may be extended to nonzero thermal occupation of the bosonic modes which would also allow the study of transport [59]. We also expect that a generalization to include higher coupling strengths is possible by modifying the displacement transformation. This might be interesting for systems with a vanishing gap, e.g., atom-cavity systems with $U(1)$ symmetry [22]. In future, it will be interesting to apply the Lindblad master equation to multi-mode systems to study many-body cooling, the formation of coherent states in the presence of dissipation, and reservoir engineering [60–62].

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