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Magic-Angle Twisted Bilayer Graphene as a Topological Heavy Fermion Problem

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Magic-angle ($\theta = 1.05^\circ$) twisted bilayer graphene (MATBG) has shown two seemingly contradictory characters: the localization and quantum-dot-like behavior in STM experiments, and delocalization in transport experiments. We construct a model, which naturally captures the two aspects, from the Bistritzer-MacDonald (BM) model in a *first principle spirit*. A set of local flat-band orbitals (f) centered at the AA-stacking regions are responsible to the localization. A set of extended *topological* semi-metallic conduction bands (c), which are at small energetic separation from the local orbitals, are responsible to the delocalization and transport. The topological flat bands of the BM model appear as a result of the hybridization of f- and c-electrons. This model then provides a new perspective for the strong correlation physics, which is now described as strongly correlated f-electrons coupled to nearly free c-electrons - we hence name our model as the *topological heavy fermion model*. Using this model, we obtain the U(4) and U(4)×U(4) symmetries of Refs. [1–5] as well as the correlated insulator phases and their energies. Simple rules for the ground states and their Chern numbers are derived. Moreover, features such as the large dispersion of the charge ±1 excitations [2, 6, 7], and the minima of the charge gap at the Γ_M point can now, for the first time, be understood both qualitatively and quantitatively in a simple physical picture. Our mapping opens the prospect of using heavy-fermion physics machinery to the superconducting physics of MATBG.

Introduction — Since the initial experimental discovery of the correlated insulator phases [8] and superconductivity [9] in MATBG [10], extensive experimental [11–35] and theoretical [1–7, 36–113] efforts have been made to understand the nature of these exotic phases. Theoretical challenges for understanding the correlation physics come from both the strong interaction compared to relatively small band width as well as from the topology [36, 38, 41–43, 104], which forbids a symmetric lattice description of the problem. The two flat bands of MATBG posses strong topology in the presence of $C_{2z}T$ (time-reversal followed by C_{2z} rotation) and particle-hole (P) symmetries [104], which supersedes the earlier, $C_{2z}T$ symmetry-protected fragile topology [41, 42]. This strong topology extends to the entire continuum BM model, and implies the absence of a lattice model for any number of bands. The topology is also responsible to exotic phases such as quantum anomalous Hall states [2, 5, 55, 60, 82, 84] and fractional Chern states [96, 98, 99, 109].

Two types of complementary strategies have been proposed to resolve the problem of the lattice description. One is to construct extended Hubbard models [1, 7, 37, 40, 42, 49, 51, 67, 71], where either $C_{2z}T$ [1, 7, 40, 49, 67] or P [42] becomes non-local in real space. The other is to adopt a full momentum-space formalism [2, 5, 6, 85, 86, 105, 106, 111], where locality becomes hidden. (Besides the two strategies, some phenomenological models are also proposed [39, 48, 63, 64, 90, 92, 97, 100].) The real and momentum space strong coupling models elucidated the nature of the correlated insulator states: they are ferromagnets - sometimes carrying Chern numbers - in a large U(4) or U(4)×U(4) symmetry space that contains spin, valley and band quantum number [1, 2, 4]. The dispersion of the excitations above the correlated insulators [2, 6, 7] - where superconductivity appears upon doping - is, despite being exact - not physically understood.

In the current manuscript, nevertheless, we find it possible to write down a fully symmetric model that has a simple real space picture, which, remarkably and elegantly, solves the aforementioned puzzles. We reformulate and map the interacting MATBG as an effective topological heavy fermion system, which consists of local orbitals (f) centered at the AA-stacking regions and delocalized topological conduction bands (c). The f-electrons are so localized that they have an almost zero kinetic energy (~ 0.1meV) but a strong on-site Coulomb repulsion that we compute to be ~ 60meV. The c-electrons carry the symmetry anomaly and have unbounded kinetic energies. The actual flat bands of the BM model are from a hybridization ($\sim$20meV) between the f- and c-bands. The interacting Hamiltonian also couples the f and c electrons through the presence of several types of interactions. Using this model, the ground states [1, 2, 44, 50, 53, 65, 68, 71, 77, 84–86, 114] and their topologies can be understood in a simple, physical picture. The quasi-particle excitation bandwidth can even be analytically determined.

Topological heavy fermion model — The single-valley BM model has the symmetry of the magnetic space group $P6'2'2$, generated by C_{3z}, C_{2x}, $C_{2z}T$, and moiré translations. (See Refs. [41, 115] for this group and its irreducible representations - irreps.) The energy bands in the valley $\eta = +$ of the BM model are shown in Fig. 1(b), where the bands are labeled by their irreps. Refs. [41, 42]

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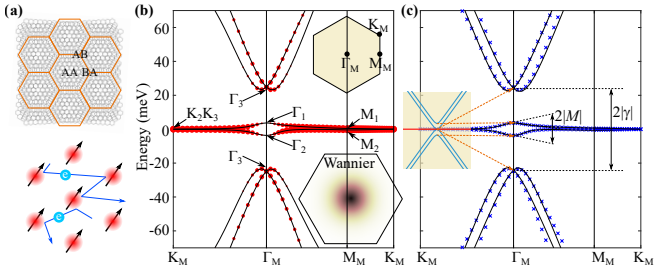

FIG. 1. Topological heavy fermion model. (a) A sketch of the moiré unit cell of MATBG and its heavy fermion analog, where the local moments and itinerant electrons are formed by the effective f-orbitals at the AA-stacking regions and topological conduction bands (c), respectively. (b) The band structure of the BM model at the magic angle $\theta = 1.05°$, where the moiré BZ and high symmetry momenta are illustrated in the upper inset panel. The overlaps between the Bloch states and the trial WFs are represented by the red circles. The density profile of the constructed maximally localized WFs (f-orbitals) is shown in the lower inset panel. (c) Bands given by the topological heavy fermion model (black lines) compared to the BM bands (blue crosses). The c- (blue) and f-bands (red) in the decoupled limit, where $\gamma = v'_\star = 0$, are shown in the inset. Orange dashed lines indicate evolution of energy levels as f-c coupling is turned on.

showed that the irreps formed by the two flat bands, *i.e.*, $\Gamma_1 \oplus \Gamma_2$; $M_1 \oplus M_2$; K_2K_3, are not consistent with any local orbitals (band representations [116]) and indicate a fragile [117–120] topological obstruction to a two-band lattice model. Here we resolve the fragile topology by involving higher energy bands. Suppose we can "borrow" a Γ_3 irrep from higher (∼20meV) energy bands and use it to replace the $\Gamma_1 \oplus \Gamma_2$ states; then the replaced irreps - Γ_3, $M_1 \oplus M_2$, K_2K_3 - are consistent with $p_x \pm ip_y$ orbitals located at the triangular lattice. We hence introduce two trial Gaussian-type Wannier functions (WFs) that transform as $p_x \pm ip_y$ orbitals under the crystalline symmetries. As indicated by the overlaps between the trial WFs and the Bloch bands (Fig. 1(a)), the trial WFs are supported by the flat band states at $\mathbf{k}$ away from Γ_M and by the lowest higher energy band states around Γ_M. Feeding the overlaps into the program Wannier90 [121–123], we obtain the corresponding maximally localized WFs, density profile of which is shown in Fig. 1(b) [115]. (Similar local states are also discussed using different methods in Refs. [38, 112].) These WFs are extremely localized - their nearest neighbor hoppings are about 0.1meV - and span 96% percent of the flat bands.

To recover the irreps and topology of the middle two bands, we have to take into account the remaining 4% states, without which the localized electrons could not form a superconductor. To do this, we define the projector into the WFs as $\mathbb{P}$, the projector into the lowest six bands (per spin valley) as $\mathbb{I}$, and divide the low energy BM Hamiltonian H_{BM} into four parts: $H^{(f)} = \mathbb{P}H_{BM}\mathbb{P}$, $H^{(c)} = \mathbb{Q}H_{BM}\mathbb{Q}$, $H^{(fc)} = \mathbb{P}H_{BM}\mathbb{Q}$, $H^{(cf)} = H^{(fc)\dagger}$, where $\mathbb{Q} = \mathbb{I} - \mathbb{P}$, $H^{(c)}$ is the remaining Hamiltonian, and $H^{(fc)} + h.c.$ is the coupling between WFs and the remaining states. As the couplings between WFs are extremely weak (∼0.1meV) we find $H^{(f)} \approx 0$. Since the two states in $\mathbb{P}$ form Γ_3 at Γ_M, the four states in $\mathbb{Q}$ must form $\Gamma_3 \oplus \Gamma_1 \oplus \Gamma_2$ at Γ_M due to the irrep counting. Due to the crystalline and P symmetries, $H^{(c)}$ in the valley η takes the form [115]

$$H^{(c,\eta)}(\mathbf{k}) = \begin{pmatrix} 0_{2\times2} & v_\star(\eta k_x\sigma_0 + ik_y\sigma_z) \\ v_\star(\eta k_x\sigma_0 - ik_y\sigma_z) & M\sigma_x \end{pmatrix} \quad (1)$$

to linear order of $\mathbf{k}$, where the first two-by-two block is spanned by the Γ_3 states and the second two-by-two block is spanned by the $\Gamma_1 \oplus \Gamma_2$ states. The Γ_1 and Γ_2 states are split by the M term (blue bands in Fig. 1(c)), while the Γ_3 states form a quadratic touching at $\mathbf{k} = 0$, which is shown in Ref. [115] responsible to the symmetry anomaly [104] jointly protected by $C_{2z}T$ and P. The coupling $H^{(fc)}$ in the valley η has the form

$$H^{(fc,\eta)}(\mathbf{k}) = \left(\gamma\sigma_0 + v'_\star(\eta k_x\sigma_x + k_y\sigma_y),\ 0_{2\times2}\right) , \quad (2)$$

where the second block is computed to be extremely small and hence is omitted and written as $0_{2\times2}$. $H^{(fc,\eta)}$ will gap $H^{(c,\eta)}$, and hence provides for both the single particle gap and for the flat band topology of the BM model. Using a set of usually adopt parameters for MATBG, we find $v_\star = -4.303\text{eV}\cdot\text{Å}$, $M = 3.697\text{meV}$, $\gamma = -24.75\text{meV}$, $v'_\star = 1.622\text{eV}\cdot\text{Å}$.

Since the WFs and the remaining "c" degrees of freedom have localized and plane-wave-like wave functions, respectively, we make the analogy with local orbitals and conduction bands in heavy fermion systems. We refer to them as local f-orbitals and (topological) conduction c-bands, respectively. We use $f_{\mathbf{R}\alpha\eta s}$ ($\alpha = 1, 2$, $\eta = \pm$, $s = \uparrow, \downarrow$) to represent the annihilation operator of the α-th WF of the valley η and spin s at the moiré unit cell $\mathbf{R}$. We use $c_{\mathbf{k}a\eta s}$ ($a = 1, 2, 3, 4$) to represent the annihilation operator of the a-th conduction band basis of the valley η and spin s at the moiré momentum $\mathbf{k}$. The single-particle Hamiltonian can be written as

$$\hat{H}_0 = \sum_{|\mathbf{k}|<\Lambda_c}\sum_{aa'\eta s} H^{(c,\eta)}_{aa'}(\mathbf{k})c^\dagger_{\mathbf{k}a\eta s}c_{\mathbf{k}a'\eta s} + \frac{1}{\sqrt{N}}\sum_{|\mathbf{k}|\le\Lambda_c}\sum_{\mathbf{R}}\sum_{\alpha a\eta s}\Big(e^{i\mathbf{k}\cdot\mathbf{R}-\frac{|\mathbf{k}|^2\lambda^2}{2}} H^{(fc,\eta)}_{\alpha a}(\mathbf{k})f^\dagger_{\mathbf{R}\alpha\eta s}c_{\mathbf{k}a\eta s} + h.c.\Big) , \quad (3)$$

where Λ_c is the momentum cutoff for the c-electrons, a_M is the moiré lattice constant, N is the number of moiré unit cell in the system, and λ, which is found to be $0.3375a_M$, is a damping factor proportional to the size of WFs. We plot the band structure of $\hat{H}_0$ in Fig. 1(c), where the splitting of the two Γ_3 states is given by $2|\gamma|$ and the bandwidth of the two flat bands is given by $2M \approx 7.4\text{meV}$. The spectrum of $\hat{H}_0$ matches very well with the BM model (Fig. 1(a)) in the energy range [-70meV, 70meV].

The U(4) symmetry — The projected model of MATBG [1, 2, 4] is found to possess a U(4) symmetry if the kinetic energy of the flat bands is omitted. In the heavy fermion basis, this U(4) symmetry can be realized by imposing the flat band condition, *i.e.*, $M = 0$. (Note

that $2|M|$ is the bandwidth of the flat bands.) The U(4) moments of the f-electrons, Γ_3 c-electrons, and $\Gamma_1 \oplus \Gamma_2$ c-electrons are given by [115]

$$\begin{aligned}
\hat{\Sigma}^{(f,\xi)}_{\mu\nu}(\mathbf{R}) =& \frac{\delta_{\xi,(-1)^{\alpha-1}\eta}}{2} A^{\mu\nu}_{\alpha\eta s,\alpha'\eta' s'} f^\dagger_{\mathbf{R}\alpha\eta s} f_{\mathbf{R}\alpha'\eta' s'} \\
\hat{\Sigma}^{(c\prime,\xi)}_{\mu\nu}(\mathbf{q}) =& \frac{\delta_{\xi,(-1)^{a-1}\eta}}{2N} A^{\mu\nu}_{\alpha\eta s,a'\eta' s'} c^\dagger_{\mathbf{k}+\mathbf{q}a\eta s} c_{\mathbf{k}a'\eta' s'},\ (a=1,2) \\
\hat{\Sigma}^{(c\prime\prime,\xi)}_{\mu\nu}(\mathbf{q}) =& \frac{\delta_{\xi,(-1)^{a-1}\eta}}{2N} B^{\mu\nu}_{\alpha\eta s,a'\eta' s'} c^\dagger_{\mathbf{k}+\mathbf{q}a\eta s} c_{\mathbf{k}a'\eta' s'},\ (a=3,4)
\end{aligned} \tag{4}$$

respectively, where repeated indices should be summed over and $A^{\mu\nu}, B^{\mu\nu}$ $(\mu,\nu = 0,x,y,z)$ are eight-by-eight matrices

$$\begin{aligned}
A^{\mu\nu} =& \{\sigma_0\tau_0\varsigma_\nu, \sigma_y\tau_x\varsigma_\nu, \sigma_y\tau_y\varsigma_\nu, \sigma_0\tau_z\varsigma_\nu\} \\
B^{\mu\nu} =& \{\sigma_0\tau_0\varsigma_\nu, -\sigma_y\tau_x\varsigma_\nu, -\sigma_y\tau_y\varsigma_\nu, \sigma_0\tau_z\varsigma_\nu\}\ ,
\end{aligned} \tag{5}$$

with $\sigma_{0,x,y,z}$, $\tau_{0,x,y,z}$, $\varsigma_{0,x,y,z}$ being the Pauli or identity matrices for the orbital, valley, and spin degrees of freedom, respectively. The ±1 valued index ξ, equal to $(-1)^{\alpha-1}\eta$ or $(-1)^{a-1}\eta$ in the moments, labels different fundamental representations of the U(4) group. The global U(4) rotations are generated by $\hat{\Sigma}_{\mu\nu} = \sum_{\xi=\pm1} \hat{\Sigma}^{(f,\xi)}_{\mu\nu} + \hat{\Sigma}^{(c\prime,\xi)}_{\mu\nu} + \hat{\Sigma}^{(c\prime\prime,\xi)}_{\mu\nu}$. Unlike the U(4) rotations found in Refs. [1, 2, 4], which only commute the projected Hamiltonian into the flat bands, the U(4) rotations here commute with the full Hamiltonian. (Generators of the U(4) or U(4)×U(4) symmetry in the first chiral limit [36, 102] is also given in Ref. [115].)

Interaction Hamiltonian — To obtain the interaction Hamiltonian in the heavy fermion basis, we can first express the density operator $\rho(\mathbf{r})$ of the BM model in terms of $f_{\mathbf{R}\alpha\eta s}$ and $c_{\mathbf{k}a\eta s}$, and then substitute it into the Coulomb interaction, $\rho(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')$. By evaluating the Coulomb integrals, we obtain the interaction Hamiltonian resembling a periodic Anderson model with extra f-c exchange interactions [115],

$$\hat{H}_I = \hat{H}_{U_1} + \hat{H}_J + \hat{H}_{U_2} + \hat{H}_V + \hat{H}_W\ . \tag{6}$$

$\hat{H}_{U_1} = \frac{U_1}{2}\sum_{\mathbf{R}} :\rho^f_{\mathbf{R}}::\rho^f_{\mathbf{R}}:$ are the on-site interactions of f-electrons, where $\rho^f_{\mathbf{R}} = \sum_{\alpha\eta s} f^\dagger_{\mathbf{R}\alpha\eta s} f_{\mathbf{R}\alpha\eta s}$ is the f-electrons density and the colon symbols represent the normal ordered operator with respect to the normal state.

$$\hat{H}_J = -J\sum_{\mathbf{Rq}}\sum_{\mu\nu}\sum_{\xi=\pm} e^{-i\mathbf{q}\cdot\mathbf{R}} :\hat{\Sigma}^{(f,\xi)}_{\mu\nu}(\mathbf{R})::\hat{\Sigma}^{(c\prime\prime,\xi)}_{\mu\nu}(\mathbf{q}):\tag{7}$$

is a ferromagnetic exchange coupling between U(4) moments of f-electrons and $\Gamma_1\oplus\Gamma_2$ c-electrons. Using practical parameters for MATBG, we obtain $U_1 = 57.95$meV and $J = 16.38$meV. The other three terms in $\hat{H}_I$ are: H_{U_2} - repulsion (~ 2.3meV) between nearest neighbor f-electrons, H_V - repulsion (~ 48meV) between c-electrons, H_W - repulsion (~ 47meV) between c- and f-electrons.

As a widely adopted approximation in heavy fermion materials, $\hat{H}_{U_2} + \hat{H}_V + \hat{H}_W$ can be decoupled in the Hartree channel due to the delocalized and localized natures of c- and f-electrons. Hence these terms only effectively shift the band energies of f- and c-bands. Then,

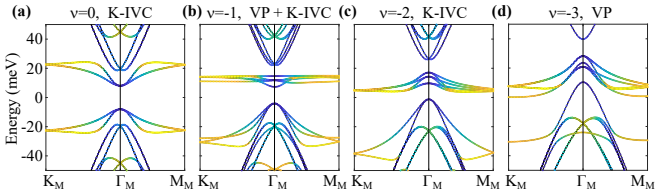

FIG. 2. The self-consistent HF bands upon the ground states at the fillings $\nu = 0, -1, -2, -3$. The color of the bands represent the contributing components, wherein yellow represents the f-electron states and blue represents the c-electron states.

U_1 - the on-site repulsion of the f-electrons - is by far the largest energy scale of the problem - more than twice the hybridization (γ) and three times the exchange (J). In Hartree-Fock (HF) calculations U_1 is found to be the source of spontaneous symmetry-breakings.

Ground states — Since U_1 is much larger than the couplings (γ, J, $v'_\star(\eta k_x\sigma_x + k_y\sigma_y)$) between f- and c-electrons, a reasonable guess of the ground states would be product states of f-multiplets and the (gapless point) Fermi liquid state ($|\mathrm{FS}\rangle$) of the half-filled c-electrons. We call such product states "the parent states". *E.g.*, the parent valley-polarized (VP) state at the charge neutrality ($\nu = 0$) is

$$|\mathrm{VP}^{\nu=0}_0\rangle = \prod_{\mathbf{R}}\prod_{\alpha=1,2}\prod_{s=\uparrow\downarrow} f^\dagger_{\mathbf{R},\alpha,+,s}|\mathrm{FS}\rangle\ . \tag{8}$$

The parent Kramers inter-valley-coherent (K-IVC) state is a U(4)-rotation of $|\mathrm{VP}^{\nu=0}_0\rangle$ along the τ_x-direction

$$\begin{aligned}
&|\text{K-IVC}^{\nu=0}_0\rangle = e^{-i\frac{\pi}{2}\hat{\Sigma}_{x0}}|\mathrm{VP}^{\nu=0}_0\rangle \\
=&\prod_{\mathbf{R}}\prod_{s=\uparrow\downarrow}\frac{1}{2}(f^\dagger_{\mathbf{R},1,+,s} + f^\dagger_{\mathbf{R},2,-,s})(-f^\dagger_{\mathbf{R},1,-,s} + f^\dagger_{\mathbf{R},2,+,s})|\mathrm{FS}\rangle\ .
\end{aligned} \tag{9}$$

Parent states at other integer fillings ($\nu = 0, \pm1, \pm2, \pm3$) can be similarly constructed [115]. They would be ground states of the Hamiltonian if γ, J, $v'_\star$ terms vanished; hybridization of f- and c-electrons will develop, *i.e.*, $\langle f^\dagger c\rangle \neq 0$, otherwise. The determination of ground states by self-consistent HF calculation with initial states given by the parent states is given in Ref. [115]. The numerically found HF ground states at the integer fillings (Fig. 2) are fully consistent with those in Ref. [5].

The parent states are so good initial states for the HF calculations that the one-shot HF is already qualitatively same as the self-consistent HF (see Fig. 3). Thanks to the simplicity of the heavy fermion model, the one-shot energies can be analytically estimated and we are able to derive two rules for the ground states [115]. *First*, in the parent state, f-electrons at each site tend to be symmetric under permutation of U(4) indices to save the Coulomb energy (Hunds' rule). Both Eqs. (8) and (9) satisfy the first rule. *Second*, for U(4)-related states at a given integer filling ν, the state that minimizes $\hat{H}_M + \hat{H}_J$ is the ground state, where $\hat{H}_M$ is the U(4)-breaking M

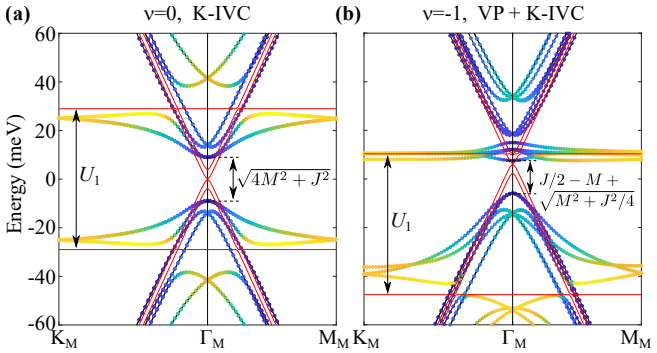

FIG. 3. The one-shot HF bands of the ground states at the fillings $\nu = 0, -1$. The red solid bands are the quasi-particle bands of the decoupled Hamiltonian, where $\gamma = v'_\star = J = 0$. The horizontal and dispersive red bands are of the f- and c-electrons, respectively. The touching point of the dispersive red bands at Γ_M is quadratic, while since M is small, it may look like linear. The one-shot bands can be understood as a result of hybridization between f- and c-electrons.

term in $\hat{H}_0$ (Eq. (1)). This energy can be estimated by the lowest $\nu + 4$ levels of the mean field Hamiltonian $H^{(\Gamma_1\oplus\Gamma_2)}$ spanned by the $\Gamma_1 \oplus \Gamma_2$ basis of the c-bands at $\mathbf{k} = 0$, which reads (up to constants)

$$H^{(\Gamma_1\oplus\Gamma_2)} = M\sigma_x\tau_0\varsigma_0 - \frac{J}{2}(\tau_z\overline{O}^{fT}\tau_z + \sigma_z\overline{O}^{fT}\sigma_z) \ . \quad (10)$$

Here $\overline{O}^f_{\alpha\eta s,\alpha'\eta's'} = \langle f^\dagger_{\mathbf{R}\alpha\eta s} f_{\mathbf{R}\alpha'\eta's'}\rangle - \frac{1}{2}\delta_{\alpha\alpha'}\delta_{\eta\eta'}\delta_{ss'}$ is the density matrix of the local f-orbitals with homogeneity assumed. We have assumed that, for all the integer fillings, the eight lowest energy levels (closest to the gap) are always contributed by the $\Gamma_1 \oplus \Gamma_2$ c-band basis - which are part of the flat bands - hence we only need to look at the $\Gamma_1 \oplus \Gamma_2$ subspace. This assumption is fully justified by previous studies based on two-band projected Hamiltonian, where only $\Gamma_1 \oplus \Gamma_2$ basis exists, and will become clearer after we discuss the charge ± 1 excitations.

We now apply the second rule to Eq. (8) and (9) to determine the one-shot state of lowest energy. $\overline{O}^f$ matrices given by Eqs. (8) and (9) are $\frac{1}{2}\sigma_0\tau_z\varsigma_0$ and $-\frac{1}{2}\sigma_y\tau_y\varsigma_0$, respectively; the resulted lowest four levels of $H^{(\Gamma_1\oplus\Gamma_2)}$ are $\pm M - J/2$ (each 2-fold) and $-\sqrt{M^2 + J^2/4}$ (4-fold), respectively. It is direct to verify that the latter (K-IVC) has a lower energy. Applying the two rules to parent states at other fillings, we obtain consistent results with the full numerical calculations in Refs. [5, 105]. We also obtain an analytical expression for the Chern numbers of ground states [115].

Charge ± 1 *excitations* — As shown in Figs. 2 and 3 and in Refs. [2, 6, 7, 85, 86, 110], at $\mathbf{k}$ away from Γ_M, the quasi-particle bands have a large gap ($\sim U_1$) and are relatively flat; at $\mathbf{k}$ around Γ_M, the bands have significant dip. Such features are found related to the topology of the two flat-bands [6, 7] but have not yet been quantitatively understood. The heavy fermion model provides a natural explanation to these features. We first consider the decoupled limit ($\gamma = v'_\star = J = 0$) at $\nu = 0$, where the f-electron bands are flat and have a (charge) gap U_1, and the c-electron bands are given by $H^{(c,\eta)}$ (Fig. 3(a)). Tuning on $\gamma, v'_\star, J$ then yields the one-shot quasi-particle bands. At $\mathbf{k} = 0$, γ gaps out the Γ_3 c-bands, and J further gaps out the $\Gamma_1 \oplus \Gamma_2$ c-bands. As the splitting of $\Gamma_1 - \Gamma_2$ is smaller than that of the Γ_3, the lowest excitations will carry Γ_1, Γ_2 representations, matching Refs. [2, 6, 7] and, according to the discussion after Eq. (10), equals to $2\sqrt{M^2 + J^2/4}$ and $|J - 2M|$ for K-IVC and VP states, respectively. At $\mathbf{k} \neq 0$, the $v'_\star$ term hybridizes the flat f-bands and dispersive c-bands. For large $\mathbf{k}$, where the c-bands have very high kinetic energies, the hybridization is relatively weak and the gap is still approximately U_1. Thus the shape of the quasi-particle bands is explained, and its bandwidth is approximately given by $(U_1 - J)/2$ when M is small. As discussed in Ref. [115], the feature that the larger ($\sim U_1$) and smaller ($\sim J$) gaps are contributed by f- and c-electrons, respectively, is reflected in the STM spectra and Landau levels at different regions (AA or AB sites) of MATBG.

At nonzero fillings, the quasi-particle bands can also be understood as hybridized flat f-bands and dispersive c-bands, except that the f- and c-bands may feel different effective chemical potentials due to the density-density interactions between them. For example, at $\nu = -1$, the upper branch of the f-bands is shifted to an energy close to the quadratic touching of the c-bands (Fig. 3(b)) [115]. Thus one of the hybridized bands is extremely flat.

Discussion — The coexistence of quantum-dot-like behavior [21, 25] and superconductivity [9, 11, 12, 14, 15] may now be understood - they come from two different types (f and c) of carriers. In fact, inspired by the pomeranchuk effect experiments [33, 34] and strange metal behavior [19, 20], authors of Refs. [33, 34, 107] also conjecture the possibility of coexistence of local momenta and itinerant electrons. (The heavy fermion theory analog may also exist in other twisted materials [124].) Our paper *derives* and shows the exact mapping of MATBG to such a heavy-fermion type model. As such, the machinery of heavy fermions [125–139] can now be applied, for the first time, to MATBG. We speculate that it will lead to pairing [52, 54, 56–59, 66, 70, 78–80, 91, 94, 95, 108] in nontrivial gap channels.

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