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The Anomalous Transport of Tracers in Active Baths

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We derive the long-time dynamics of a tracer immersed in a one-dimensional active bath. In contrast to previous studies, we find that the damping and noise correlations possess long-time tails with exponents that depend on the tracer symmetry. For generic tracers, shape asymmetry induce ratchet effects that alter fluctuations and lead to superdiffusion and friction that grows with time when the tracer is dragged at a constant speed. In the singular limit of a completely symmetric tracer, we recover normal diffusion and finite friction. Furthermore, for small symmetric tracers, the active contribution to the friction becomes negative: active particles enhance motion rather than oppose it. These results show that, in low-dimensional systems, the motion of a passive tracer in an active bath cannot be modelled as a persistent random walker with a finite correlation time.

Since Einstein and Smoluchowski, the motion of a tracer particle in a bath has been a topic of much interest [1]. The simplest textbook framework models the motion of the particle as a memoryless Brownian motion using an underdamped Langevin equation [2–4]. The momentum autocorrelation function then decays exponentially with a single time scale, signalling a transition between inertial and viscous regimes. This was, however, found to be over-simplistic: the conservation of momentum in the solvent instead leads to a power-law decay [5–7] and a host of interesting phenomena—especially in low dimensions—such as the breakdown of the Fourier law [8–10].

When compared to the equilibrium case, active fluids reveal a much richer physics, from the ratchet effects induced by asymmetric gears [11–14] and rectifiers [15–20] to the long-ranged forces and currents generated by asymmetric obstacles [20–24]. Over the past two decades, much activity has been devoted to studying passive tracers in active baths [25–63]. In the adiabatic limit in which the bath's relaxation is much faster than the tracer's response [64–71], the tracer's dynamics is described by a generalized Langevin equation (GLE). In 1D, it reads

$$\gamma_0 \dot{X}(t) + \int_0^t dt' \gamma(t-t')\dot{X}(t') = \mathcal{F}(t) + \eta(t)\ , \qquad (1)$$

where the interactions with the active particles lead to a stochastic force $\mathcal{F}(t)$ and a retarded friction $\int_0^t dt'\gamma(t-t')\dot{X}(t')$. Equation (1) also includes a memoryless viscous medium at temperature T that leads to the friction coefficient γ_0 and a Gaussian white noise $\eta(t)$ satisfying $\langle\eta(t)\eta(t')\rangle = 2\gamma_0 T\delta(t-t')$. Despite many efforts, a single unifying picture for the friction $\gamma(t)$ and the force-force correlation functions $C_{\mathcal{F}} \equiv \langle\mathcal{F}(t)\mathcal{F}(0)\rangle_c$ does not emerge from existing results.

First, a large class of experimental and numerical studies have suggested that the random, finite-duration encounters between the bath particles and the tracer lead to an exponential decay of $\gamma(t)$ over a short time scale [25–35]. Equation (1) then reduces to $(\gamma_0 + \gamma_{\rm T})\dot{X}(t) = \mathcal{F}(t)$, where $\gamma_{\rm T} \equiv \int_0^\infty dt\gamma(t)$. In this case, similarly to an underdamped Brownian particle, the large-scale motion of the tracer is diffusive. This has been justified analytically in the simple case of a tracer connected by linear springs to a bath of active Ornstein-Uhlenbeck particles [72]—an active counterpart to the celebrated work of Vernon and Feynman [73–75].

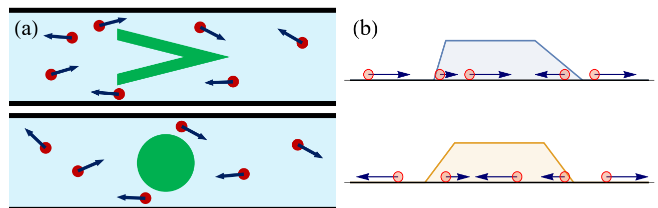

Figure 1. (a) A large tracer and a bath of small active particles are immersed in a viscous medium inside a long narrow channel. (b) The short transverse dimension allows modelling the channel as a one-dimensional system where particles can bypass each other and the tracer, even though the transverse and orientational fluctuations of the tracer are lost in this one-dimensional description. Top: asymmetric tracer. Bottom: symmetric tracer.

In contrast, a second class of experiments and models on so-called wet-active matter suggests a more complex physics [37, 41, 43, 45, 50, 55, 59]. The long-ranged decay of hydrodynamic interactions can indeed turn $\gamma(t)$ and $C_{\mathcal{F}}(t)$ into power-laws [37, 45, 59]. These may lead to anomalous diffusion on intermediate time scales but, ultimately, lead to long-time diffusion.

We note, however, that long-time tails are generic, even in the absence of hydrodynamic interactions. Indeed, the fluctuating density of active particles is a conserved quantity—and hence a slow field—so that the bath cannot have a single characteristic relaxation time. This leads to power-law memory and correlations, as already noted for equilibrium [6, 7, 76, 77] and nonequilibrium [78, 79] systems, including phoretic colloids [80] and driven tracers [81, 82]. In low-dimensional systems, these tails may result in anomalous transport over long time scales [80, 81]. Although thoroughly studied in other contexts, these effects were so far overlooked for tracers in dry active baths.

In this Letter, to resolve this issue, we consider the simplest non-trivial system in which Eq. (1) can be systematically derived: a single tracer immersed in a dry one-dimensional active bath of run-and-tumble particles. To remain as close as possible to the phenomenology of an active bath in $d > 1$ dimensions, we allow particles to overtake each other and the tracer, hence modelling the latter by a soft repulsive potential $V(x)$, see Fig. 1. Starting from the coupled dynamics of the bath particles and tracer positions, $\{x_i(t), X(t)\}$, we determine explicitly the long-time behaviors of $\gamma(t)$ and $C_{\mathcal{F}}(t)$ as functions of the tracer shape and of the microscopic parameters of our model. To do so, we employ a *controlled* adiabatic expansion [83, 84] valid in the large γ_0 limit in which the tracer dynamics can be described by Eq. (1). Our results show the emergence of long-time tails that lead to interesting and qualitatively different behaviors for symmetric and asymmetric tracers. For generic, *asymmetric* tracers, ratchet effects make $\gamma(t)$ and $C_{\mathcal{F}}(t)$ scale as $\sim t^{-1/2}$ in the long-time limit, leading to *superdiffusive* behavior around their mean displacements:

$$\left\langle X^2(t)\right\rangle_{\rm c} \equiv \langle X^2(t)\rangle - \langle X(t)\rangle^2 \sim Kt^{3/2} \,. \tag{2}$$

When the tracer is towed at a constant velocity U, it experiences a friction force from the active particles that grows as:

$$\frac{f_{\rm fric}(t)}{U} \sim -\Gamma_{\rm T} t^{1/2} \,. \tag{3}$$

We provide below explicit expressions for K and $\Gamma_{\rm T}$ in the presence of a soft asymmetric potential in a dilute active bath. In the singular limit of a *symmetric* tracer, $C_{\mathcal{F}}(t)$ and $\gamma(t)$ scale as $\sim t^{-3/2}$, similar to a tracer in a bath of equilibrium Brownian particles [76, 85], which yields a *diffusive* behavior:

$$\left\langle X^2(t)\right\rangle_{\rm c} \sim 2Dt \,. \tag{4}$$

Towing the tracer at constant velocity U, the active particles exert a *finite* friction force:

$$\frac{f_{\rm fric}(t)}{U} = -\gamma_{\rm T} - \gamma_1 t^{-1/2} + \mathcal{O}(t^{-3/2}) \,, \tag{5}$$

where $\gamma_{\rm T} \equiv \int_0^\infty dt\gamma(t)$. Interestingly, for small tracer sizes, $\gamma_{\rm T}$ and γ_1 are negative: the active bath pushes the tracer in the towing direction. We provide perturbative expressions for D and $\gamma_{\rm T}$ and defer their systematic derivations for later work [86]. All our results are confirmed by microscopic simulations shown in Fig. 2. The derivation presented below suggests that the exponents are *universal* to any bath with long-time diffusive statistics. We confirm that they hold in the presence of soft repulsive interparticle forces in Section I of [87].

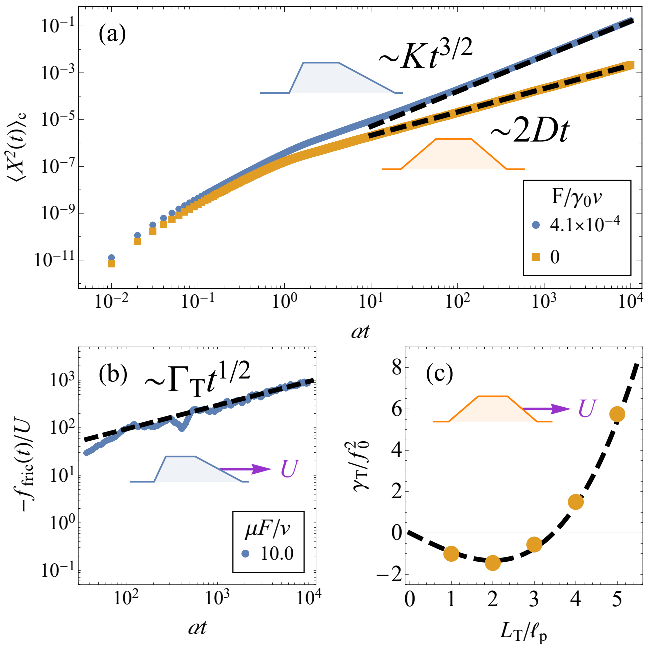

Figure 2. Simulation results (symbols) compared with our theoretical predictions for the long-time limit, without any fitting parameters, (dashed black lines): (a) MSD for symmetric and asymmetric tracers; (b) friction force exerted on an asymmetric tracer; (c) symmetric-tracer friction coefficient vs tracer size $L_{\rm T}$. Simulation details and results for soft repulsive interactions are given in Section I of [87].

Model. We consider bath particles moving with speed v and randomly switching their orientations with rate $\alpha/2$, leading to a persistence length $\ell_{\rm p} = v/\alpha$. The tracer interacts with the active bath via a short-range potential V which vanishes outside $[0, L_{\rm T}]$, such that the force on bath particle i is $f(x_i - X) = -\partial_{x_i} V(x_i - X)$ and the tracer size is $L_{\rm T}$. We take $|\mu f(x)| < v$ so that particles are able to cross the tracer, which emulates the channel in Fig. 1a. The tracer and bath-particle dynamics thus read

$$\gamma_0 \dot{X}(t) = F_{\rm tot}(t) \equiv -\sum_i f[x_i(t) - X(t)] \,, \tag{6}$$

$$\dot{x}_i(t) = v\sigma_i(t) + \mu f[x_i(t) - X(t)] \,, \tag{7}$$

where the $\sigma_i(t) \in \{\pm 1\}$ flip independently with rate $\alpha/2$ and μ is the bath-particle mobility. In Eqs. (6) and (7) we neglected the thermal noises acting on the tracer and bath particles, which are typically much weaker than the active and viscous forces [25, 26, 34, 38, 88]. (See section V of [87] for a discussion of the $T \neq 0$ case.) In the analytical derivations below we consider a dilute bath of active particles, without interparticle forces, in either infinite systems or periodic ones of size $L \gg L_{\rm T}, \ell_{\rm p}$.

Theory. The fluctuating force $F_{\rm tot}(t)$ differs from the average force F exerted on a tracer held *fixed*. This is due to both the tracer's motion and the stochasticity of the active bath. The average correction due to the tracer motion is characterised by $\gamma(t)$ in Eq. (1). Within an adiabatic perturbation theory $\gamma(t)$ is defined as

$$\langle F_{\rm tot}(t)\rangle - F \equiv -\int_0^t dt' \gamma(t-t')\dot{X}(t') \,, \tag{8}$$

where the average is conditioned on a given realization of

$\dot{X}(t)$. The fluctuations of $F_{\rm tot}$ are then characterized through

$$\mathcal{F}(t) \equiv F_{\rm tot}(t) + \int_0^t dt' \gamma(t-t')\dot{X}(t') \ . \tag{9}$$

Adiabatic perturbation theory tells us that, when γ_0 is large, the statistics of $\mathcal{F}(t)$ are identical to those of the force exerted on a tracer held fixed [84]. Furthermore, it relates $\gamma(t)$ and $\mathcal{F}(t)$ through an Agarwal-Kubo-type formula [83]

$$\gamma(t-t') = \left\langle \mathcal{F}(t)\partial_{X_0} \ln \rho_{\rm s}\big[x(t')-X_0,\sigma(t')\big]\right\rangle^{\rm s} \ . \tag{10}$$

Here, $\rho_{\rm s}(x-X_0,\sigma)$ is the steady-state density of bath particles with orientation σ and displacement $x-X_0$ from a tracer held fixed at X_0. The brackets $\langle\cdot\rangle^{\rm s}$ represent an average with respect to $\rho_{\rm s}$. In the following, we set $X_0=0$ without loss of generality. For an equilibrium bath at temperature T, $\langle\mathcal{F}(t)\rangle^{\rm s}=0$ and Eq. (10) reduces to the fluctuation-dissipation theorem (FDT) $\gamma(t)=C_{\mathcal{F}}(t)/T$ where $C_{\mathcal{F}}(t)=\langle\mathcal{F}(t)\mathcal{F}(0)\rangle^{\rm s}$. Outside equilibrium, these constraints need not hold.

To characterize the tracer dynamics, we compute independently $F=\langle\mathcal{F}(t)\rangle^{\rm s}$, $C_{\mathcal{F}}(t)$ and $\gamma(t-t')$. To do so, we start from the expression for the steady state of noninteracting run-and-tumble particles in the presence of an external force $f(x)$ [89, 90]:

$$\rho_{\rm s}(x,\sigma) = \frac{\frac{1}{2}\rho_{\rm L}}{1+\sigma\frac{\mu}{v}f(x)} \exp\left\{\beta_{\rm eff}\int_0^x dy \frac{f(y)}{1-\left[\frac{\mu}{v}f(y)\right]^2}\right\}, \tag{11}$$

where $\rho_{\rm L}$ is the particle density at $x=0^-$, $T_{\rm eff}=v^2/\mu\alpha$ is the effective temperature, and $\beta_{\rm eff}=1/T_{\rm eff}$. The steady-state density is $\rho_{\rm s}(x)=\sum_\sigma \rho_{\rm s}(x,\sigma)$.

Asymmetric tracer. For an asymmetric tracer, the densities of active particles $\rho_{\rm R}$ and $\rho_{\rm L}$ at the right and left ends of the tracer differ and are given by $\rho_{\rm R}=2\rho_0/\left[1+\exp\left(\beta_{\rm eff}\varepsilon\right)\right]$ and $\rho_{\rm L}=2\rho_0/\left[1+\exp\left(-\beta_{\rm eff}\varepsilon\right)\right]$, where $\varepsilon\equiv-\int dx f(x)/\{1-[\mu f(x)/v]^2\}$. The density difference then leads to a nonvanishing average force $F=-\int dx f(x)\rho_{\rm s}(x)$ exerted on the tracer [21, 91, 92], which is given by

$$F=-T_{\rm eff}(\rho_{\rm R}-\rho_{\rm L})=2T_{\rm eff}\rho_0\tanh\left(\frac{\varepsilon}{2T_{\rm eff}}\right) \ , \tag{12}$$

where we have introduced the average background density $\rho_0=(\rho_{\rm R}+\rho_{\rm L})/2$. Note that Eq. (12) is consistent with the ideal gas law applied to the left and right sides of the tracer.

The long-time behavior of $C_{\mathcal{F}}(t)$ and $\gamma(t)$ can be derived from the knowledge of the propagator $p(x,\sigma,t|x',\sigma',0)$. In the long-time limit, the dynamics of the active particles are diffusive so that the support of $p(x,\sigma,t|x',\sigma',0)$ spreads over a region of length $2b(t)$ around x', where $b(t)\sim(\pi D_{\rm eff}t)^{1/2}$ is a diffusive propagating front. For any $x-x'\ll b(t)$, and to leading order in $b(t)$, $p(x,\sigma,t|x',\sigma',0)$ has relaxed to the normalized steady-state distribution $\rho_{\rm s}(x,\sigma)/\sum_\sigma\int_{-b(t)}^{b(t)}dx\rho_{\rm s}(x,\sigma)$. For $L_{\rm T}\ll 2b(t)$, one can neglect the region inside the tracer in the integral so that $\sum_\sigma\int_{-b(t)}^{b(t)}dx\rho_{\rm s}(x,\sigma)\sim(\rho_{\rm R}+\rho_{\rm L})b(t)$, up to corrections of order $\mathcal{O}(L^{-1})$. Since $b(t)\sim(\pi D_{\rm eff}t)^{1/2}$ we get

$$p(x,\sigma,t|x',\sigma',0)\sim\frac{\rho_{\rm s}(x,\sigma)}{\rho_{\rm R}+\rho_{\rm L}}(\pi D_{\rm eff}t)^{-1/2} \ . \tag{13}$$

This heuristic result can be derived exactly, within the adiabatic limit, and its sub-leading correction can be shown to scale as $\mathcal{O}(t^{-3/2})$ (See Section II of [87]).

On long times, $p(x,\sigma,t|x',\sigma',0)$ is independent of the initial coordinate (x',σ'). Therefore, two-point correlations are factorized in this limit. Furthermore, for N noninteracting particles, the forces exerted by different particles on the tracer are uncorrelated so that $C_{\mathcal{F}}(t)=N\{\langle f(t)f(0)\rangle^{\rm s}-[\langle f(t)\rangle^{\rm s}]^2\}$, where $f(t)$ is the force due to a single bath particle. Since $\langle f(t)\rangle^{\rm s}=F/N$, $N[\langle f(t)\rangle^{\rm s}]^2$ only contributes a correction of order $\mathcal{O}(L^{-1})$ to $C_{\mathcal{F}}(t)$. Using Eq. (13), $C_{\mathcal{F}}(t)$ can then be evaluated as:

$$C_{\mathcal{F}}(t)=\sum_{\sigma\sigma'}\int dxdx' f(x)p(x,\sigma,t|x',\sigma',0)f(x')\rho_{\rm s}(x',\sigma') \tag{14}$$

$$=\frac{F^2}{\rho_{\rm R}+\rho_{\rm L}}(\pi D_{\rm eff}t)^{-1/2}+\mathcal{O}\left(t^{-3/2}\right). \tag{15}$$

Similarly, we obtain from Eqs. (10) and (12)

$$\gamma(t)=\sum_{\sigma\sigma'}\int dxdx' f(x)p(x,\sigma,t|x',\sigma',0)\partial_{x'}\rho_{\rm s}(x',\sigma') \tag{16}$$

$$=\beta_{\rm eff}\frac{F^2}{\rho_{\rm R}+\rho_{\rm L}}(\pi D_{\rm eff}t)^{-1/2}+\mathcal{O}\left(t^{-3/2}\right) \ . \tag{17}$$

Remarkably, the long-time regime satisfies an effective FDT $\gamma(t)=\beta_{\rm eff}C_{\mathcal{F}}(t)+\mathcal{O}(t^{-3/2})$. We also note that Eqs (11)-(17) hold in the infinite-system-size limit. For large-but-finite systems, they are complemented by $\mathcal{O}(L^{-1})$ corrections, as discussed in Section III of [87].

Equations (15) and (17) immediately show that the asymmetric tracer undergoes anomalous dynamics on long times. Indeed, the noise and friction intensities, defined as $I=\int_0^\infty dt C_{\mathcal{F}}(t)$ and $\gamma_{\rm T}=\int_0^\infty dt\,\gamma(t)$ are infinite, hence leading to an ill-defined diffusivity $D\equiv I/(\gamma_0+\gamma_{\rm T})^2$. To characterize the anomalous dynamics of the tracer we first consider its free motion. We define the tracer's mobility $B(t)$ through $X(t)=\int_0^t dt' B(t-t')\mathcal{F}(t')$, which leads to

$$\left\langle X(t)^2\right\rangle_{\rm c}=2\int_0^t dt_1\int_0^{t_1}dt_2 B(t_1)B(t_2)C_{\mathcal{F}}(t_1-t_2) \ . \tag{18}$$

Since we are working in the large γ_0 limit, $B(t)\sim1/\gamma_0$ [93]. Using Eq. (15) for $C_{\mathcal{F}}(t)$ then gives Eq. (2), hence implying *superdiffusion*, with

$$K=\frac{4F^2}{3\rho_0\gamma_0^2\sqrt{\pi D_{\rm eff}}} \ . \tag{19}$$

In addition to anomalous diffusion, the asymmetric tracer experiences friction that grows with time, as shown by the following towing experiment. Setting a constant velocity $\dot{X} = U$ in Eq. (1), the friction exerted by the active particles on the tracer can be measured as $f_{\rm fric}(t) \equiv \langle F_{\rm tot}\rangle - F$. From Eqs. (8) and (17), we get

$$f_{\rm fric}(t) = -U\int_0^t dt'\gamma(t') \sim -U\frac{F^2}{T_{\rm eff}\rho_0}\left(\frac{t}{\pi D_{\rm eff}}\right)^{1/2}, \quad (20)$$

which yields Eq. (5) with $\Gamma_{\rm T} = F^2(\pi D_{\rm eff})^{-1/2}/T_{\rm eff}\rho_0$.

Symmetric tracer. For a symmetric tracer, $F = 0$. Equations (15) and (17) then imply that $\gamma(t)$, $C_{\mathcal{F}}(t) = \mathcal{O}\left(t^{-3/2}\right)$. In this case, I and $\gamma_{\rm T}$ remain finite so that $D = I/(\gamma_0+\gamma_{\rm T})^2$ is well defined and Eq. (4) holds. We now present heuristic discussions of $C_{\mathcal{F}}(t)$ and $\gamma(t)$ that account for two important features: their scaling as $t^{-3/2}$ and their sign changes for small tracers. These results can be derived exactly, within the adiabatic limit, for piecewise linear potentials [86].

Consider a symmetric tracer of length $L_{\rm T}$ whose potential is depicted in Fig. 3. While our results can be derived exactly [86], we present here a simple argument which holds in the limit in which the edges of the tracer have a small width d and small slopes $\pm f_0$. Consider first a single particle located at the left end of the tracer, at $\hat{x} \simeq 0$, moving in the direction $\hat{\sigma}$. At long times, the probability distribution of its position x is a Gaussian centered around $\hat{\sigma}\ell_{\rm p}$, of variance $2D_{\rm eff}t$ (See Fig. 3). The force-force correlation of this particle can be computed as

$$c(\hat{\sigma},t) = \frac{f_0^2 d}{\sqrt{4\pi D_{\rm eff}t}}\left[e^{-\frac{\ell_{\rm p}^2}{4D_{\rm eff}t}} - e^{-\frac{(L_{\rm T}-\hat{\sigma}\ell_{\rm p})^2}{4D_{\rm eff}t}}\right], \quad (21)$$

as can be inferred from Eq. (14) using $\rho_s(x',\sigma') = \delta(x')\delta_{\sigma',\hat{\sigma}}$. Note that the factor d comes from the integration over x in Eq. (14), which also leads to the two exponentials corresponding to $x \simeq 0$ and $x \simeq L_T$, respectively. This amounts to summing the contribution due to particles returning to the left end, such that $f(x)f(x') = f_0^2$, and that of particles crossing the tracer, such that $f(x)f(x') = -f_0^2$.

Let us return to the case of an active bath of density ρ_0. We denote by m the polarization of particles around $x' = 0$ so that the local density of particles with orientation σ is $\rho_0\frac{1+\sigma m}{2}$. The force-force correlation is then obtained from the single-particle result through $C_{\mathcal{F}}(t) = 2\rho_0[\frac{1+m}{2}c(1,t) + \frac{1-m}{2}c(-1,t)]$, where the factor 2 stems from the contributions of particles starting at $x' \simeq L_{\rm T}$. Expanding the exponentials in (21) in the long-time limit, one finds the leading orders to cancel, yielding the $t^{-3/2}$ scaling of $C_{\mathcal{F}}(t)$. Using Eq. (11) leads to $m = \mu f_0/v$, which is consistent with the fact that active particles polarize against external potentials [94]. Straightforward algebra then gives

$$C_{\mathcal{F}}(t) \sim \frac{\rho_0(f_0 d L_{\rm T})^2}{4\pi^{1/2}(D_{\rm eff}t)^{3/2}}G(\ell_{\rm p}/L_{\rm T}) \quad (22)$$

where $G(y) = 1 - \frac{2\mu f_0}{v}y$. Importantly, $C_{\mathcal{F}}(t)$ becomes negative when the size of the tracer is small, $L_{\rm T} \leq 2\mu\ell_{\rm p}f_0/v$. In the discussion above, we neglected $\mathcal{O}(f_0)$ corrections to the propagator and to the steady-state density due to the edges of the tracer. Including the f_0 corrections to all orders confirms the scaling (22), to order $\mathcal{O}(d^2)$, albeit with $G(y) = [1-(\frac{2\mu f_0 y}{v})^2]/[1-(\frac{\mu f_0}{v})^2]^2$ (See Section IV of [87]). This does not change the leading order estimate for the crossover length $\sim 2\mu f_0\ell_{\rm p}/v$. Negative autocorrelations have been reported in other contexts, in [7] and out [80] of equilibrium. Here, it is a direct consequence of the polarization against the potential. Setting $m = 0$ in the computation above always leads to $C_{\mathcal{F}}(t) > 0$.

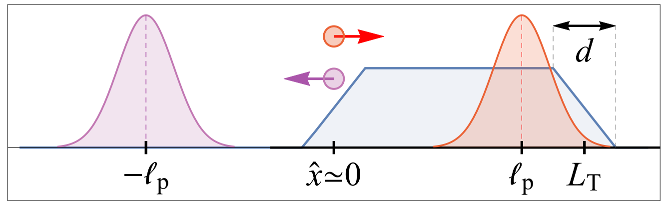

Figure 3. Consider a symmetric tracer (blue potential) and an active particle located at its left end at position $\hat{x}$ at $t = 0$. The particle is shown in orange and magenta for $\hat{\sigma} = \pm 1$ respectively. At late times, the particle position is distributed as a Gaussian centered around $x_{\rm c} = \hat{x} + \hat{\sigma}\ell_{\rm p}$. For $\hat{\sigma} = 1$, when $\ell_{\rm p} \gg L_{\rm T}$, the anticorrelation between $f(x')$ and $f(x)$ leads to a negative contribution to $C_{\mathcal{F}}(t)$. Conversely, a $\hat{\sigma} = -1$ particle leads to a positive contribution to $C_{\mathcal{F}}$. Due to the polarization against the potential, $\hat{\sigma} = \pm 1$ occur with different probabilities. This leads to an overall negative $C_{\mathcal{F}}(t)$ for small $L_{\rm T}$ and a positive one for large sizes.

We now turn to the long-time behavior of $\gamma(t)$. Inserting Eq. (11) in Eq. (10) leads to $\gamma = \gamma_{\rm p} - \gamma_{\rm a}$, with

$$\gamma_{\rm p}(t-t') \equiv \beta_{\rm eff}\left\langle \mathcal{F}(t)\frac{\mathcal{F}(t')}{1-\left[\frac{\mu}{v}\mathcal{F}(t')\right]^2}\right\rangle_{\rm c}, \quad (23)$$

$$\gamma_{\rm a}(t-t') \equiv \beta_{\rm eff}\left\langle \mathcal{F}(t)\frac{\sigma(t')\ell_{\rm p}\partial_{x(t')}\mathcal{F}(t')}{1-\sigma(t')\frac{\mu}{v}\mathcal{F}(t')}\right\rangle_{\rm c}. \quad (24)$$

The heuristic argument developed above for $C_{\mathcal{F}}(t)$ directly extends to the correlators (23) and (24), showing that $\gamma_{\rm a}$ and $\gamma_{\rm p}$ both inherit the $t^{-3/2}$ scaling of $C_{\mathcal{F}}(t)$ at long times. Inspecting Eq. (23) shows that, to leading order in f_0,

$$\gamma_{\rm p}(t) \sim \beta_{\rm eff}C_{\mathcal{F}}(t) = \frac{\beta_{\rm eff}\rho_0(f_0 d L_{\rm T})^2}{4\pi^{1/2}(D_{\rm eff}t)^{3/2}}G(\ell_{\rm p}/L_{\rm T}). \quad (25)$$

Equation (25) is nothing but an effective FDT for the passive tracer. Our results show that the FDT is only expected to hold for small f_0 and should be generically violated when γ_a is not negligible compared to $\gamma_{\rm p}$.

The presence of $\sigma(t')$ in Eq. (24) makes the contributions of $\sigma' = \pm 1$ particle add up, instead of cancelling, leading to $\gamma_{\rm a}(t) > 0$ for all $L_{\rm T}$ and a long-time scaling $\gamma_{\rm a} \sim \mathcal{O}(f_0^3)t^{-3/2}$. Therefore, to leading order in f_0, $\gamma \sim \beta_{\rm eff}C_{\mathcal{F}}(t)$. This suggests that $\gamma_{\rm T} = \int_0^\infty dt\gamma(t)$ can also change sign and become *negative* for small tracers. Indeed, a perturbative calculation finds that

$$\gamma_{\mathrm{T}} \sim \beta_{\mathrm{eff}} v^{-1} \rho_0 (f_0 d)^2 \frac{L_{\mathrm{T}}}{\ell_{\mathrm{p}}} \left(1 - \frac{d^2 + 6\ell_{\mathrm{p}}^2}{3 d L_{\mathrm{T}}}\right) . \qquad (26)$$

The derivations of this result and of the asymptotics of γ_{a} are not particularly illuminating; they are deferred to Section IV of [87]. Importantly, Eq. (26) implies that when a small symmetric tracer is dragged at velocity U, the active bath *enhances* its motion rather than resisting it.

Adiabatic limit. Although Eq. (1) is a common framework to describe tracer's dynamics, it relies on the assumption that their motion is slow. An important—but rarely debated—question is thus its range of validity. Here, this is set by the requirement that the tracer's response is much slower than the diffusive relaxation of the bath, *i.e.* $\langle X(t)\rangle$, $\langle X^2(t)\rangle_{\mathrm{c}}^{1/2} \ll (D_{\mathrm{eff}} t)^{1/2}$. For an asymmetric tracer, using $\langle X(t)\rangle \sim Ft/\gamma_0$ and Eq. (2), we find $t \ll \tau_1 \equiv D_{\mathrm{eff}}(\gamma_0/F)^2$ and $t \ll \tau_2 \equiv (D_{\mathrm{eff}}/K)^2$. Equation (19) implies $\tau_1 \ll \tau_2$ so that the adiabatic limit holds up to $t \ll \tau_1$. Beyond this time scale, which can be arbitrarily large, an asymmetric tracer in an active bath cannot be described by Eq. (1). Considering a finite system of size L, the diffusive relaxation time is $t = \tau_{\mathrm{rel}} \sim L^2/D_{\mathrm{eff}}$. Thus, the adiabatic limit for an asymmetric tracer in a finite system is valid for $FL \ll D_{\mathrm{eff}}\gamma_0$, which can be achieved by designing the tracer shape to bound F or by using a small enough system. For a symmetric tracer, there is no temporal restriction and the only requirement is $D \ll D_{\mathrm{eff}}$, which can be fulfilled by setting $\gamma_0 \gg (I/D_{\mathrm{eff}})^{1/2}$. For towing both asymmetric tracers and symmetric tracers at constant velocity U, the only requirement is $U \ll D_{\mathrm{eff}}/L$.

Conclusion. In this Letter, we have derived the long-time dynamics of a passive tracer in a dilute active bath under the sole assumption of an adiabatic evolution. We have revealed new regimes for both asymmetric and symmetric tracers. First, ratchet effects generically lead to the superdiffusion of asymmetric tracers, which also experience friction that grows with time when they are dragged at constant velocity U. For symmetric tracers, the long-time tail preserves the diffusive behavior, but negative active friction is observed for small tracers. The latter solely follows from the persistent motion of active particles and their polarization by external potentials, a mechanism that differs from previously studied cases with negative mobility [72, 95, 96]. We expect the tails for asymmetric and symmetric tracers to become $t^{-d/2}$ and $t^{-(d/2+1)}$ in d dimensions, respectively. This suggests, in two dimensions, that $\langle X(t)^2\rangle_{\mathrm{c}} \sim t \ln t$ for an asymmetric tracer, which remains to be verified. Our results stem from generic features of dry active particles and should thus hold generically. The exponents are expected to be universal, but the transport coefficients can be dressed, for instance, by interactions. Moreover, the mechanisms should lead to even richer behaviours for active suspensions in momentum-conserving fluids [37, 45, 50, 59], or in the presence of phoresis [80].

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