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Universal Properties of Weakly Bound Two-Neutron Halo Nuclei

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We construct an effective field theory of a two-neutron halo nucleus in the limit where the two-neutron separation energy B and the neutron-neutron two-body virtual energy ϵ_n are smaller than any other energy scale in the problem, but the scattering between the core and a single neutron is not fine-tuned, and the Efimov effect does not operate. The theory has one dimensionless coupling which formally runs to a Landau pole in the ultraviolet. We show that many properties of the system are universal in the double fine-tuning limit. The ratio of the mean-square matter radius and charge radius is found to be $\langle r_m^2 \rangle / \langle r_c^2 \rangle = Af(\epsilon_n/B)$, where A is the mass number of the core and f is a function of the ratio ϵ_n/B which we find explicitly. In particular, when $B \gg \epsilon_n$, $\langle r_m^2 \rangle / \langle r_c^2 \rangle = \frac{2}{3}A$. The shape of the $E1$ dipole strength function also depends only on the ratio ϵ_n/B and is derived in explicit analytic form. We estimate that for the ^{22}C nucleus higher-order corrections to our theory are of order 20% or less if the two-neutron separation energy is less than 100 keV and the s -wave scattering length between a neutron and a ^{20}C nucleus is less than 2.8 fm.

Introduction.—Neutron-rich nuclei near the neutron drip line are at the forefront of modern nuclear physics. Some of the most exotic examples are two-neutron halo nuclei, consisting of a relatively tightly bound core and two weakly-bound neutrons, e.g., ^6He , ^{11}Li , and ^{22}C . These nuclei are called “Borromean,” i.e., bound states of three objects which would fall apart when one is removed [1].

In this Letter we develop an effective field theory (EFT) that can describe Borromean two-neutron halo nuclei in the limit of very small two-neutron separation energy B . The impetus to the construction of this theory is the observation of the halo nucleus ^{22}C with a matter radius found to be as large as 5.4(9) fm [2] which requires a small B : $B < 100$ keV [3]. A later experiment [4] yields a smaller matter radius, 3.44(8) fm, relaxing the upper limit to $B < 400$ keV, but if one incorporates the information about the neutron-core scattering [5] the upper limit is reduced to $B < 180$ keV [6]. So ^{22}C is likely the least bound among the known Borromean nuclei.

Our EFT requires two fine-tunings: we assume that the neutron-neutron s -wave scattering length a is unnaturally large and the two-neutron separation energy of the halo nucleus is unnaturally small. In other words, the n - n two-body virtual energy $\epsilon_n = \hbar^2/(m_n a^2) \approx 120$ keV (here m_n is the neutron mass) and the binding energy B of the core with two neutrons are assumed to be smaller than all other energy scales in the problem. We do not presume any hierarchy between these two energies.

Some previous attempts to apply the EFT philosophy to two-neutron halo nuclei [6, 7] rely on the existence of a near-threshold resonance in the core-neutron subsystem and the three-body Efimov effect [8]. This resonance seems to be absent in the case of ^{22}C , where experiment points to a rather small $^{20}\text{C}+n$ scattering length [5]. The theory developed in this Letter is designed to address this situation. It may also be a reasonable starting point for a description of the $^3\text{He}^4\text{He}_2$ molecular trimer [9],

whose binding energy (~ 10 – 15 mK) [10] is somewhat smaller than the energy scale set by the ^3He – ^4He scattering length (around 50 mK [11]).

The EFT, to be described, contains two relevant parameters and one dimensionless coupling. The two relevant parameters correspond to the two fine-tunings. The dimensionless coupling g can be interpreted as the probability that a halo nucleus splits into a core and a two-neutron dimer (“dineutron”), and it runs logarithmically with energy, reaching formally a Landau pole in the ultraviolet (UV) and zero in the infrared (IR) [12]. All other coupling constants are irrelevant and can be neglected in leading-order calculation.

Using the EFT one can compute many physical quantities. In particular, we compute the ratio of the mean-square matter radius and charge radius. The result is particularly simple in the limit of infinite neutron-neutron scattering length:

$$\frac{\langle r_m^2 \rangle}{\langle r_c^2 \rangle} = \frac{2}{3}A, \quad (1)$$

where A is the mass number of the core. We also obtain a fully analytic expression for the $E1$ dipole strength function, Eqs. (29) and (30).

All calculations in this Letter are performed under the assumption that the core and the neutron are point-like particles. To translate our results to the realistic nuclei, one needs account for the charge and matter distribution inside the core and the neutrons. In addition, effects from irrelevant terms in the effective Lagrangian may need to be taken into account.

The effective field theory.—We first write down the effective Lagrangian for the neutron sector. Denoting the neutron by ψ_α , $\alpha = \uparrow, \downarrow$ being the spin index,

$$\mathcal{L}_n = \sum_\sigma \psi_\sigma^\dagger \left(i\partial_t + \frac{\nabla^2}{2m_n} \right) \psi_\sigma + c_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow. \quad (2)$$

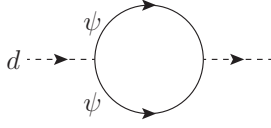


FIG. 1. The self-energy of the dimer.

From now on we set $m_n = 1$. Using a Hubbard-Stratonovich transformation, the Lagrangian can be transformed into

$$\mathcal{L}_n = \sum_{\sigma} \psi_{\sigma}^{\dagger} \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi_{\sigma} - \frac{1}{c_0} d^{\dagger} d + \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} d + d^{\dagger} \psi_{\downarrow} \psi_{\uparrow}. \quad (3)$$

Computing the self-energy of the dimer d , which, in the nonrelativistic theory, is exactly given by the one-loop diagram in Fig. 1, we find the full dimer propagator

$$D(p) = -\frac{4\pi}{\sqrt{-p_0 + \frac{\mathbf{p}^2}{4} - \frac{1}{a}}}, \quad (4)$$

where a denotes the s -wave scattering length given by

$$\frac{1}{4\pi a} = -\frac{1}{c_0} + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\mathbf{q}^2}. \quad (5)$$

The integral on the right-hand side linearly diverges in the UV and is proportional to the UV cutoff. The fine-tuning of c_0 leads to an unnaturally large scattering length a . Note that the UV behavior of the dimer propagator is different from that of a free field. In fact, the UV behavior corresponds to a field of dimension 2: $[d] = 2$ [13] [14].

To construct the EFT describing the halo nucleus, we add into the theory a field ϕ describing the core and h describing the halo nucleus. They can be either bosonic or fermionic. The effective Lagrangian is now [15]

$$\mathcal{L} = h^{\dagger} \left(i\partial_t + \frac{\nabla^2}{2m_h} + B \right) h + \phi^{\dagger} \left(i\partial_t + \frac{\nabla^2}{2m_{\phi}} \right) \phi + g(h^{\dagger} \phi d + \phi^{\dagger} d^{\dagger} h) + \mathcal{L}_n + \text{counterterms}. \quad (6)$$

where $m_{\phi} = Am_n$ and $m_h = (A+2)m_n$ are the masses of the core and the halo nucleus, respectively. As $[d] = 2$ and $[\phi] = [\psi] = \frac{3}{2}$, the dimension of the interaction $h^{\dagger} \phi d$ is 5, which means that g is dimensionless. One can check that terms not included in Eq. (6) are all irrelevant since they are accompanied by more fields or derivatives. One can compute the beta function for g ,

$$\frac{\partial g}{\partial \ln E} = \beta(g) = \frac{2}{\pi} \left(\frac{A}{A+2} \right)^{3/2} g^3. \quad (7)$$

The solution to this equation is

$$g^2(E) = \frac{\pi}{4} \left(\frac{A+2}{A} \right)^{3/2} \frac{1}{\ln \frac{E_0}{E}}, \quad (8)$$

where E_0 is the energy of the Landau pole. Due to the properties of the nonrelativistic theory, our subsequent calculations can be done to all order in g^2 .

One can arrive at the effective Lagrangian (6) by starting from a theory where the core ϕ and the resonantly interacting neutron are coupled to each other by a contact interaction $C_0 \phi^{\dagger} d^{\dagger} d \phi$, with a UV cutoff at the Landau pole scale. Through a Hubbard-Stratonovich transformation, one introduces an auxiliary field h with the coupling $h^{\dagger} d \phi + \text{h.c.}$ Integrating out degrees of freedom in a energy shell between E_0 and $E_1 < E_0$, one generates a kinetic term for h , and arrive to (6) [16].

Charge and matter radii.—We now proceed to extract physical observables from the Lagrangian (6). The mean-square (rms) charge radius (the rms of the deviation of the coordinates of the core from the center of mass [17]) can be extracted from the electric form-factor of the halo nucleus: $F(\mathbf{k}) = 1 - \frac{1}{6} k^2 \langle r_c^2 \rangle + O(k^4)$ [recall that $F(\mathbf{k})$ is a Fourier transform of the charge density]. The electric form factor is given by the Feynman diagram in Fig. 2; it is proportional to g^2 and by dimensional analysis one should have $\langle r_c^2 \rangle = g^2 B^{-1} f(\beta)$, where we introduce the dimensionless parameter

$$\beta = \frac{1}{-a\sqrt{B}} = \sqrt{\frac{\epsilon_n}{B}}, \quad (9)$$

where $\epsilon_n = 1/a^2$ (we assume $a < 0$). Computing the Feynman diagram, we find [18]

$$\langle r_c^2 \rangle = \frac{4}{\pi} \frac{A^{1/2}}{(A+2)^{5/2}} \frac{g^2}{B} f_c(\beta), \quad (10)$$

where

$$f_c(\beta) = \begin{cases} \frac{1}{1-\beta^2} - \frac{\beta \arccos \beta}{(1-\beta^2)^{3/2}}, & \beta < 1, \\ -\frac{1}{\beta^2-1} + \frac{\beta \operatorname{arccosh} \beta}{(\beta^2-1)^{3/2}}, & \beta > 1. \end{cases} \quad (11)$$

One can further define the “neutron radius” by imagining that there is a U(1) gauge boson coupled to the neutrons outside the core [19], which describes the spatial size of the dineutron distribution. The Feynman diagram determining the form-factor of the halo nucleus

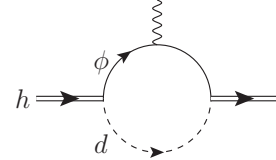


FIG. 2. The Feynman diagram determining the charge form-factor of the halo nucleus. The double line represents the halo nucleus, the single line—the core, and the dotted line—the neutron dimer, whose propagator is given in Eq. (4).

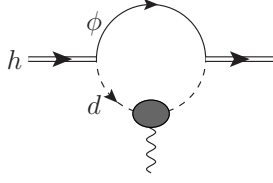


FIG. 3. The Feynman diagram determining the “neutron form-factor” of the halo nucleus.

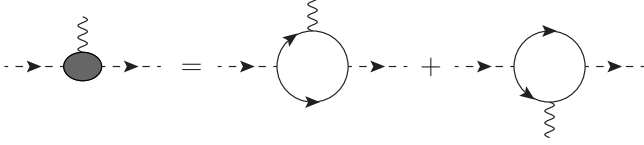


FIG. 4. The effective vertex of the dimer-photon coupling.

with respect to this “neutron-number photon” is drawn in Fig. 3, where the effective coupling of the dimer to the photon is as in Fig. 4.

The neutron radius is then calculated to be

$$\langle r_n^2 \rangle = \frac{g^2}{\pi B} \left(\frac{A}{A+2} \right)^{3/2} \left[f_n(\beta) + \frac{A}{A+2} f_c(\beta) \right], \quad (12)$$

where $f_c(\beta)$ is as in Eq. (11), and

$$f_n(\beta) = \begin{cases} \frac{1}{\beta^3} \left[\pi - 2\beta + (\beta^2 - 2) \frac{\arccos \beta}{\sqrt{1 - \beta^2}} \right], & \beta < 1, \\ \frac{1}{\beta^3} \left[\pi - 2\beta + (\beta^2 - 2) \frac{\operatorname{arccosh} \beta}{\sqrt{\beta^2 - 1}} \right], & \beta > 1. \end{cases} \quad (13)$$

Both $f_c(\beta)$ and $f_n(\beta)$ are continuous at $\beta = 1$ and have the following asymptotics at small and large values of the argument:

$$f_c(0) = 1, \quad f_n(0) = \frac{1}{3}, \quad (14)$$

$$f_c(\beta) = \frac{\ln \beta}{\beta^2}, \quad f_n(\beta) = \frac{\ln \beta}{\beta^2}, \quad \beta \rightarrow \infty. \quad (15)$$

From the charge radius and the neutron radius one can compute other radii: the mean-square matter radius $\langle r_m^2 \rangle$, the neutron-neutron distance $\langle r_{nn}^2 \rangle$, and the core-neutron distance $\langle r_{cn}^2 \rangle$ [20]:

$$\langle r_m^2 \rangle = \frac{2}{A+2} \langle r_n^2 \rangle + \frac{A}{A+2} \langle r_c^2 \rangle, \quad (16)$$

$$\langle r_{nn}^2 \rangle = 4 \langle r_n^2 \rangle - A^2 \langle r_c^2 \rangle, \quad (17)$$

$$\langle r_{cn}^2 \rangle = \langle r_n^2 \rangle + (A+1) \langle r_c^2 \rangle. \quad (18)$$

When ϵ_n is fixed, these radii depend on B in the following way. When $B \gg \epsilon_n$, the coupling g is set at the scale B , and $\langle r^2 \rangle \sim 1/[B \ln(E_0/B)]$. When $B \ll \epsilon_n$, g is frozen at

the scale ϵ_n , and the radii grow logarithmically as $B \rightarrow 0$: $\langle r^2 \rangle \sim \ln(\epsilon_n/B)$, which is a known result [21].

Note that due to the running of the coupling g the results for the radii are not truly “universal”: they cannot be expressed solely in terms of low-energy observables—the three-body binding energy B and the neutron-neutron scattering length a . Instead, they depend logarithmically on the UV cutoff through the coupling g . However, the dependence on g disappears when one computes the *ratios* of the radii. For example, the ratio of the rms matter and charge radii is

$$\frac{\langle r_m^2 \rangle}{\langle r_c^2 \rangle} = \frac{A}{2} \left[1 + \frac{f_n(\beta)}{f_c(\beta)} \right], \quad (19)$$

while the ratio of the core-neutron and neutron-neutron distances is

$$\frac{\langle r_{cn}^2 \rangle}{\langle r_{nn}^2 \rangle} = \frac{1}{4} + \frac{A+2}{4A} \frac{f_c(\beta)}{f_n(\beta)}. \quad (20)$$

In the two extreme limits, $B \gg \epsilon_n$ and $B \ll \epsilon_n$, these ratios become

$$\frac{\langle r_m^2 \rangle}{\langle r_c^2 \rangle} = \begin{cases} \frac{2}{3} A, & B \gg \epsilon_n, \\ A, & B \ll \epsilon_n, \end{cases} \quad (21)$$

$$\frac{\langle r_{cn}^2 \rangle}{\langle r_{nn}^2 \rangle} = \begin{cases} 1 + \frac{3}{2A}, & B \gg \epsilon_n, \\ \frac{A+1}{2A}, & B \ll \epsilon_n. \end{cases} \quad (22)$$

One notes, however, that f_n/f_c reaches its large- β limit very slowly. For example, the ratio of the matter and charge mean-square radii does not deviate more than 10% from its $B \gg \epsilon_n$ asymptotics unless B is less than about $\frac{1}{3}\epsilon_n$ (~ 40 keV).

The dipole strength function.—The dipole strength function can also be conveniently computed from EFT. It is defined as

$$\frac{dB(E1)}{d\omega}(\omega) = \sum_n |\langle n | \mathcal{M} | 0 \rangle|^2 \delta(E_n - E_0 - \omega), \quad (23)$$

where $|0\rangle$ is the ground state of the halo, the sum is taken over all excited states $|n\rangle$, and \mathcal{M} is the dipole operator,

$$\mathcal{M} = \sqrt{\frac{3}{4\pi}} Z e (\mathbf{r}_c - \mathbf{R}_{\text{cm}}), \quad (24)$$

where \mathbf{r}_c is the coordinates of the core, and \mathbf{R}_{cm} of the center of mass. By noting that

$$\frac{\partial}{\partial t} \mathcal{M} = \sqrt{\frac{3}{4\pi}} \mathbf{J}, \quad (25)$$

where \mathbf{J} is the total electric current, one can rewrite the dipole strength function as

$$\frac{dB(E1)}{d\omega} = \frac{3}{4\pi} \frac{1}{\omega^2} \sum_n |\langle n | \mathbf{J} | 0 \rangle|^2 \delta(E_n - E_0 - \omega), \quad (26)$$

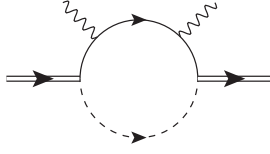


FIG. 5. The Feynman diagram for the $E1$ dipole strength function.

and express it as the imaginary part of a two-point Green's function of the current operator

$$\frac{dB(E1)}{d\omega} = -\frac{3}{4\pi} \frac{1}{\pi\omega^2} \text{Im} G_{JJ}(\omega), \quad (27)$$

where

$$iG_{JJ}(\omega) = \int dt e^{i\omega t} \langle 0 | T \mathbf{J}(t) \mathbf{J}(0) | 0 \rangle. \quad (28)$$

The problem is now similar to that of deep inelastic scattering in quantum chromodynamics [22]. Computing the Feynman diagram in Fig. 5, we find

$$\begin{aligned} \frac{dB(E1)}{d\omega} = \frac{3}{4\pi} Z^2 e^2 \frac{12g^2}{\pi} \frac{A^{1/2}}{(A+2)^{5/2}} \frac{(\omega-B)^2}{\omega^4} \\ \times f_{E1} \left(\frac{1}{-a\sqrt{\omega-B}} \right), \end{aligned} \quad (29)$$

where

$$f_{E1}(x) = 1 - \frac{8}{3}x(1+x^2)^{3/2} + 4x^2 \left(1 + \frac{2}{3}x^2 \right). \quad (30)$$

The formula is more complicated than the formula for one-neutron halo nuclei [23], but is still explicit.

One can check that the $E1$ dipole strength satisfies the sum rule

$$\int_0^\infty d\omega \frac{dB(E1)}{d\omega} = \frac{3}{4\pi} Z^2 e^2 \langle r_c^2 \rangle, \quad (31)$$

with the charge radius given by Eq. (10). The energy-weighted sum rule,

$$\int_0^\infty d\omega \omega \frac{dB(E1)}{d\omega} = \frac{3}{4\pi} Z^2 e^2 \frac{3}{A(A+2)}, \quad (32)$$

is also valid if the logarithmic divergence of the integral on the left-hand side is regularized by a UV cutoff at the energy of the Landau pole. The two sum rules are nontrivial checks of the self-consistency of our theoretical approach. The predicted shape of the $E1$ dipole strength is plotted in Fig. 6 as a function of ω/B for various values of β . One sees that the weight of the dipole strength shifts to larger ω/B as B/ϵ_n decreases.

Applicability to real systems.—The theory described above is applicable when the binding energy of the halo B

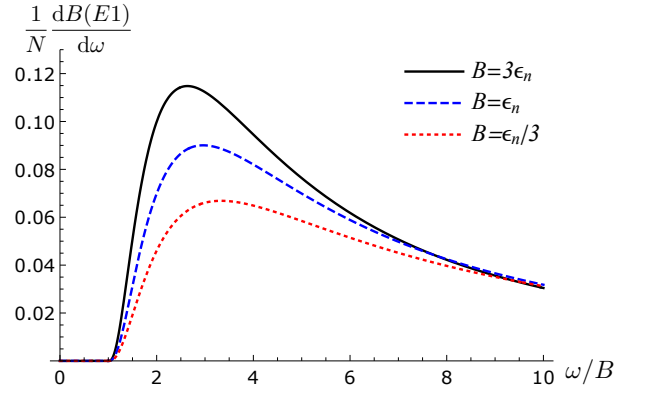


FIG. 6. The $E1$ dipole strength function, plotted as function of ω/B , for $B = 3\epsilon_n$, $B = \epsilon_n$, and $B = \frac{1}{3}\epsilon_n$. The functions are so normalized by N that the area under the theoretical curve, extended to $\omega/B = \infty$, is 1.

and the n - n two-body virtual energy ϵ_n are smaller than any other energy scales in the problem. In the real world, $\epsilon_n \approx 0.12$ MeV is indeed small. For ${}^6\text{He}$ and ${}^{11}\text{Li}$, the two-neutron separation energy somewhat larger (0.975 and 0.369 MeV, respectively); in addition, the existence of near-threshold resonances in the ${}^5\text{He}$ and ${}^{10}\text{Li}$ subsystem makes the applicability of our theory doubtful.

Nevertheless, let us try to compare our results with existing experimental data and previous theoretical calculations. For ${}^6\text{He}$, Eq. (19) predicts that $\langle r_m^2 \rangle / \langle r_c^2 \rangle \approx 0.686A$. In Ref. [24] it has been argued that the data for ${}^6\text{He}$ fits the formula $\langle r_m^2 \rangle / \langle r_c^2 \rangle = 0.862A$, which the authors derived approximately. Our value is off by about 20%. For ${}^{11}\text{Li}$ we compare our results with those of Ref. [7] where $B = 247$ keV, $\epsilon_n = 116.04$ keV were used. Setting the logarithm in Eq. (8) to 1 we find $\sqrt{\langle r_c^2 \rangle} = 0.86$ fm, $\sqrt{\langle r_n^2 \rangle} = 4.7$ fm, near the center of the error bands predicted for large energies of the ${}^{10}\text{Li}$ resonance. The opening angle θ_{nn} (defined as the vertex angle of the isosceles triangle with sides $\sqrt{\langle r_{cn}^2 \rangle}$, $\sqrt{\langle r_{cn}^2 \rangle}$, $\sqrt{\langle r_{nn}^2 \rangle}$) is close to 60° and is again within the error band. However, for the reasons listed above, it is possible that the EFT provides only a qualitative guide for ${}^{11}\text{Li}$.

The theory presented here may be quantitatively useful for the ${}^{22}\text{C}$ nucleus if its two-neutron separation energy is indeed as small as 100 keV [3]. A correction to the EFT comes from the scattering between the core and one neutron, parametrized by the irrelevant dimension-6 term $a_{cn}\phi^\dagger\psi^\dagger\psi\phi$. The contributions from this term to physical quantities should be suppressed by $a_{cn}(2m_n B)^{1/2}$ relative to the leading-order results, where a_{cn} is the core-neutron scattering length. Experiment [5] indicates that $|a_{cn}| < 2.8$ fm, so this factor is ≤ 0.2 (0.25 or 0.4 if the upper limit on B is taken as 180 keV or 400 keV, respectively). Another dimension-6 operator, $d^\dagger(i\partial_t + \frac{1}{4}\nabla^2)d$, has its coefficient fixed by the effective range of the s -wave neutron-neutron scattering; its effect is expected

to be similarly suppressed. Other terms, e.g., $\hbar^\dagger \nabla \phi \nabla d$, have dimension 7 and higher and should be more suppressed. Corrections from higher-order operators should be computable within effective field theory. The presence of the ${}^5\text{He}$ p -wave resonance can be taken into account by adding the corresponding field [25]. The present work is expected to open a potential direction to a quantitative study of the halo nuclei in addition to their universal properties. For the ${}^3\text{He}^4\text{He}_2$ trimers we expect corrections from ${}^3\text{He}$ - ${}^4\text{He}$ scattering to be relatively large: $(2\mu B)^{1/2}|a_{34}| \sim 0.5$ (where μ is the reduced mass of the ${}^3\text{He}$ - ${}^4\text{He}$ system and $a_{34} \approx -17$ Å is the ${}^3\text{He}$ - ${}^4\text{He}$ scattering length [11]). Indeed, experiment and quantum Monte Carlo simulations [26, 27] seem to imply substantially smaller values for the ratio $\langle r_{cn}^2 \rangle / \langle r_{nn}^2 \rangle$ compared the one given in Eq. (20).

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 - [12] In practice, the running is limited in the UV by the UV cutoff and in the IR by ϵ_n or B , whichever is larger.
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 - [14] In our convention, the dimension of momentum is 1 and energy is 2.
 - [15] This Lagrangian was, in essence, previously considered in Ref. [28] for the case of a three-body resonance, i.e., when the three-body binding energy B is negative.
 - [16] In this scenario the halo nucleus is bound by the three-body force. One should note, however, that the effective Lagrangian is valid irrespective of the nature of the microscopic force responsible for the binding of the halo nucleus.
 - [17] What we call here the mean-square charge radius should be understood, for real nuclei, as the difference between the mean-square point-proton radii of the halo and core nuclei. Similarly, what is later called the mean-square matter radius is for real nuclei $\langle r_m^2 \rangle_h - \frac{A-2}{A} \langle r_m^2 \rangle_c$ where $\langle r_m^2 \rangle_h$ and $\langle r_m^2 \rangle_c$ are the mean square matter radii of the halo and the core.
 - [18] In subsequent formulas, g is the renormalized coupling in the on-shell renormalization scheme.
 - [19] This can be done by coupling a gauge field to $B - \frac{A}{Z}Q$ where B is the baryon charge, Q is the electric charge, and Z and A are the atomic and mass numbers of the core.
 - [20] Using the positions of the core \mathbf{r}_c and two neutrons \mathbf{r}_1 and \mathbf{r}_2 , we define $r_m^2 \equiv \frac{1}{A+2}(A\mathbf{r}_c^2 + \mathbf{r}_1^2 + \mathbf{r}_2^2)$, $r_{nn}^2 \equiv (\mathbf{r}_1 - \mathbf{r}_2)^2$ and $r_{cn}^2 \equiv \frac{1}{2}((\mathbf{r}_1 - \mathbf{r}_c)^2 + (\mathbf{r}_2 - \mathbf{r}_c)^2)$.
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