

This is the accepted manuscript made available via CHORUS. The article has been published as:

Quantum-Critical Resistivity of Strange Metals in a Magnetic Field

Chandra M. Varma

Phys. Rev. Lett. **128**, 206601 — Published 17 May 2022

DOI: [10.1103/PhysRevLett.128.206601](https://doi.org/10.1103/PhysRevLett.128.206601)

Quantum-critical resistivity of strange metals in a magnetic field

Chandra M. Varma

^a*Physics Department, University of California, Berkeley, Ca.*

^b*Physics Department, University of California, Riverside, Ca.*

Abstract

Resistivity in the quantum-critical fluctuation region of several metallic compounds such as the cuprates, the heavy-fermions, Fe-chalogenides and pnictides, Moiré bi-layer graphene (MBLG) and WSe₂, is linear in temperature T as well as in the magnetic field H_z perpendicular to the planes. Scattering of fermions by the excitations of a time-reversal odd polar vector field $\mathbf{\Omega}$ has been shown to give a linear in T resistivity and other marginal Fermi liquid properties. An extension of this theory to an applied magnetic field is presented. Magnetic field is shown to generate a density of vortices in the field $\mathbf{\Omega}$ proportional to H_z . The elastic scattering of fermions from the vortices gives a resistivity linear in H_z with the coefficient varying as the marginal fermi-liquid susceptibility $\ln(\frac{\omega_c}{T})$. Quantitative comparison with experiments is presented for cuprates and MBLG .

^a Recalled Professor

^b Emeritus Distinguished Professor

High temperature cuprates [1] [2] have a linear in T resistivity for doping in the region above T_c which is bounded by a phase with a "pseudo-gap" on one side and cross-over to a Fermi-liquid on the other. This and related anomalies [3] in this region suggested a quantum-critical origin for the anomalies [4] [5] and the prediction that the pseudogap phase breaks time-reversal and inversion symmetries. Linear in T resistivity and other anomalies, similar to those in the cuprates, are also found in several Fe-based compounds in the fluctuation region of their antiferromagnetic (AFM) quantum-critical point [6–9], in heavy fermion compounds [10, 11], as well as more recently in twisted bi-layer graphene [12, 13] and in the twisted bi-layer compound WSe_2 [14] with hitherto un-discovered order parameters. An important recent discovery [15] [16] [17] [14] is that in all of them the resistivity is linear also in an applied magnetic field $|H|$. The magnitude of the magneto-resistivity is similar to the zero-field resistivity at temperature T for $\mu_B H$ of $O(k_B T)$. Where investigated [18], [19] the linear in $|H|$ resistivity is found only for the component H_z applied perpendicular to the planes.

Three important general points should be noted: First, a transport scattering rate linear in $|H|$ and independent of temperature can only be due to elastic scattering of fermions from time-reversal odd axial objects induced by the magnetic field. Second, the fact that only the component of the field orthogonal to the high conducting plane in all these metals is effective excludes magnetic moment due to spins in favor of magnetic moments due to orbital loop currents. Third, the magnitude mentioned above implies that the theory of the linear in $|H|$ resistivity must be closely related to the theory which gives linear in T resistivity.

A theory that gives linear in T resistivity and other anomalies in cuprates rests on the theory of quantum critical fluctuations [20, 21] which are prelude to a state of loop current order. The new experiments invite extension of this theory to the effects of a magnetic field. The occurrence of the linear in T and in H resistivity as well as the associated $T \ln(\omega_c/\pi T)$ entropy in the quantum-critical regions in at least all the other compounds where results are available [7, 10] is to be expected if their quantum-criticality is described by a model which maps to the quantum-xy model coupled to fermions (QXY-F). The mapping has been shown [22] for the planar ferro or antiferro-magnetic model or an incommensurate Ising model. Here, I will first present a theory for the magnetic field dependence of the resistivity in the cuprate compounds for which more quantitative information is available than the other compounds and briefly comment on the other cases.

Loop-current order in cuprates can be represented as a time-reversal odd polar vector $\mathbf{\Omega}$ on a lattice, sketched in Fig. (1 - A). Using conservation laws alone Else and Senthil [23] have recently shown that to get resistivity proportional to T for $T \rightarrow 0$ in the pure limit, the critical fluctuations must be of an order parameter of such a symmetry. Such an order parameter has indeed been found to be consistent with experiments using a variety of different techniques [24–28].

The orbital magnetic susceptibility of the model is obtained from the fluctuations already derived in Refs. [20, 29–31]. The model at $H = 0$ is specified by the interaction energy of the angles $\theta_{i,\tau}$ of $\mathbf{\Omega}_i$ at neighboring sites, by the kinetic energy due to their angular momentum \mathbf{L}_{zi} , and the coupling of spatial and temporal fluctuations in $\theta_{i,\tau}$ to the fermions. The QXY-F model, just as the classical XY model, does not belong to the universality class of the Ginzburg-Landau-Wilson theories and their quantum extensions. The quantum-critical fluctuations are driven by proliferation of topological defects, 2D spatial vortices, and warps which are spatially local events interacting logarithmically in imaginary time [20]. The critical correlations, $C(\mathbf{r}, \tau) \equiv \langle e^{-i\theta(\mathbf{r},\tau)} e^{i\theta(0,0)} \rangle$, have been obtained by quantum-monte-carlo calculations [29, 30] as well as derived by renormalization group [31]. It is shown in an Appendix in Ref. [32] that the orbital magnetic susceptibility $\chi_{LL}(\mathbf{r}, \tau)$ defined by (1), are proportional to those of $C(\mathbf{r}, \tau)$. Near criticality the (dimensionless) dynamic orbital magnetic susceptibility is

$$\chi_{LL}(\mathbf{r}, \tau) \equiv \mu_B^2 \langle \mathbf{L}_z^+(\mathbf{r}, \tau) \mathbf{L}_z(\mathbf{0}, 0) \rangle = \mu_B^2 \langle L_z^2 \rangle \frac{\tau_c^2}{\tau} e^{-(\tau/\xi_\tau)^{1/2}} \ln \frac{r}{a} e^{-r/\xi_r}. \quad (1)$$

$\mu_B^2 \langle L_z^2 \rangle$ is the expectation value of the square of the magnitude of the orbital magnetic moment per unit-cell volume. We take this to be given by the amplitude of the measured [25] ordered staggered moment per unit-cell $(\ell_z \mu_B)^2$. The amplitude $\langle L_z^2 \rangle$ is nearly temperature independent in the region of interest. τ_c is the short time cut-off obtainable from experiments. The spectral function (1) is of the form proposed phenomenologically [33] to give the fluctuations of marginal Fermi-liquid, rather than the $1/\tau^2$ of the Landau Fermi-liquid. In terms of the frequency ω and temperature T ,

$$\chi_{LL}(\omega, T) = \frac{\mu_B^2 \langle L_z^2 \rangle}{\omega_c} \left(\ln \left| \frac{\omega_c}{\max(\omega, \pi T)} \right| - i \tanh \frac{\omega}{2T} \right), \quad (2)$$

at criticality. $\omega_c = 1/\tau_c$ is the ultra-violet cut-off. This functional form is also the principal result of theories on interesting models of mathematical interest such as the SYK model [34],

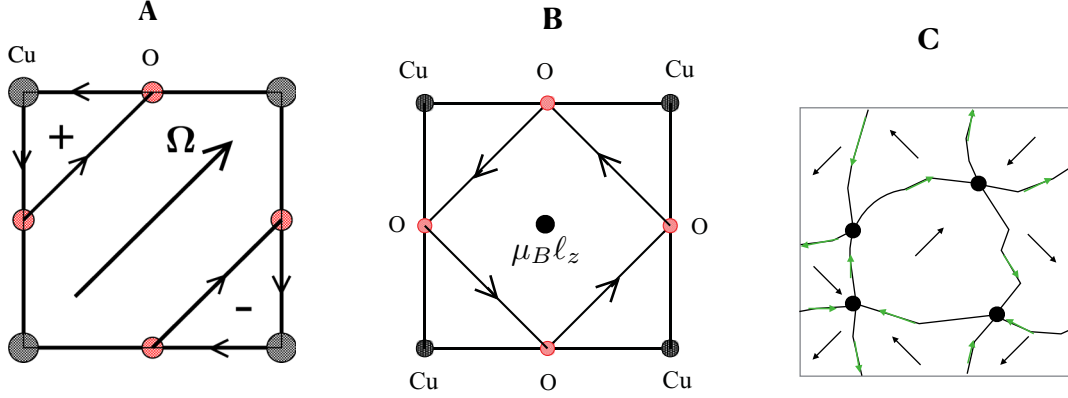


Figure 1. Representation of the current distribution in the Cu-O unit cell of the current distributions for, A: the vector field Ω which has one of four possible angles θ_i in the unit-cell i , B: the angular momentum ℓ_z which is a generator of rotations of Ω and has a magnetization at its core. C: Represents a fluctuation of Ω over regions of many cells. A current represented by the green arrows runs at the boundary between any two orientations of Ω . At all corners of the variations in Ω a vortex or ℓ_z , represented by the black dot, is required to exist. At $H = 0$, the vortices are of equally up and down orientations. But an applied finite H leads to a net orbital angular momentum due to unequal density of vortices of different orientation.

holographic models [35] and of other models [36], [3, 37, 38]. The magnetic field couples to the angular momentum as $-\mu_B \sum_i \mathbf{H} \cdot \mathbf{L}_{iz}(\tau)$. In the quantum-critical regime H_z induces a static macroscopic $\langle L_z \rangle$ given by

$$\mu_B \langle L_z \rangle = \chi'_{LL} H_z, \quad \chi'_{LL}(T) = \frac{\mu_B^2 \ell_z^2}{\omega_c} \log\left(\frac{\omega_c}{\pi T}\right). \quad (3)$$

From the experimental observations [25] that the ordered staggered moment per-cell is about $0.1\mu_B$, and $\omega_c \approx 2000K$ [2, 32], χ'_{LL} is estimated to be about $10^{-5}\mu_B^2/(Kelvin - cell)$. So a magnetic field of 50 Tesla can be estimated to produce a static magnetization $\approx 5 \times 10^{-4}\mu_B$'s, not including the numerical factor due to the logarithmic temperature dependence in χ_{LL} . An important question in the present context is how such a moment would be distributed. To think of this, it is useful to know the physical description of ℓ_z , the quasi-quantized unit of orbital angular momentum in the present problem. A loop-current carrying the lattice representation of angular momentum is shown in Fig. (1 - B) [39]. It has been shown [32, 40, 41] to be the generator of rotations of the magneto-electric vector Ω in the plane,

from one of its four orientations to the clockwise or anti-clockwise orientation:

$$e^{i(\pi/4)\ell_z}|\hat{\Omega}\rangle = |\hat{\Omega} + \pi/4\rangle. \quad (4)$$

The pictorial representation in Fig. (1 - c) of L_z corresponds to a vortex in the vector field Ω with quantized angle but a magnetic moment given by the area and the current carried by the core cell around which the four-orientations of Ω meet. Over long wave-lengths, one may ignore the granularity of the lattice so that L_z is similar to the vortex in more familiar $U(1)$ fields such as superconductors in a magnetic field or superfluids in rotation. Instead of quantization of the magnetic moment in terms of fundamental constants, it is non-universal and given by the magnitude of the vectors Ω which have very weak temperature dependence. From the estimates given above, the density of the moments n_L is about $5 \times 10^{-3}/\text{unit-cell}$ for a field of 50 Tesla so that their separation is about 50 unit-cells. In an ordered state of Ω , such moments would crystallize at low enough temperature due to their long-range mutual interactions. But we are considering the region in which they live in a bath of Ω 's quantum-fluctuating in time and space. Therefore such moments would remain disordered at the temperatures of interest and diffuse at a very slow rate because of their enormous effective mass. If the motion of $\langle L_z \rangle$ is very slow compared to the motion of fermions with which they scatter, the scattering should be considered elastic.

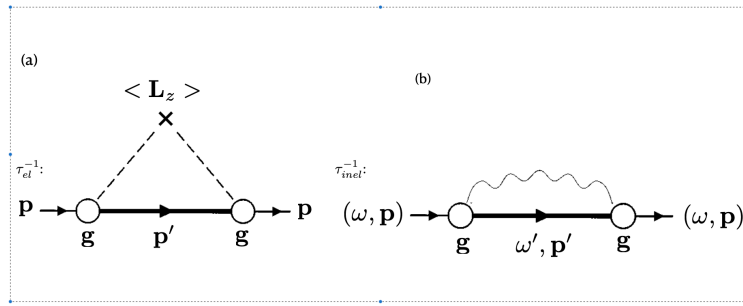


Figure 2. (a) Elastic scattering of fermions by vortices of angular momentum $\langle L_z \rangle$, (b) Inelastic scattering of fermions by fluctuations $\chi''(\omega, T, \mathbf{q})$.

The s-wave scattering rate $1/\tau_L$ of such local magnetic field generated by density n_L of the defects can be easily estimated, See Fig. (2-a).

$$1/\tau_L = 2\pi n_L (g_0 \mu_B \ell_z)^2 N(0), \quad n_L = \frac{\chi'_{LL} H_z}{\mu_B \ell_z} \approx \ell_z \frac{\mu_B H_z}{\omega_c} \ln(\omega_c/\pi T). \quad (5)$$

$N(0)$ is the density of states of fermions at the chemical potential and g_0 is the coupling energy [32] of the fermions to a vortex with orbital moment $\mu_B \ell_z$. This is to be compared with the inelastic scattering of fermions by the fluctuations $\chi''(\omega, T)$, See Fig. (2-b). This is calculated from the analytic continuation of the imaginary part of the self-energy at zero-frequency which has been often derived, [33, 42, 43]

$$1/\tau(T) = 2U \text{Im} \Sigma''(0, T), \quad \Sigma(i\omega_n) = g_0^2 \sum_{\omega_m, \mathbf{k}} G(\mathbf{k}, i\omega_m) \chi(i\omega_n - i\omega_m), \quad (6)$$

U is a dimensionless Umklapp factor, which is necessary for finite resistivity. Recently, in an asymptotically exact theory for resistivity due to fluctuations of the QXY-CF model, it has been shown that U is temperature independent [44]. A way to estimate U is to compare the transport scattering rate with the imaginary part of the self-energy in the direction on the Fermi-surface of maximum velocity. This gives U of $O(1)$ [2] for the cuprates where both have been measured. Eq. (6) gives

$$1/\tau(T) \approx \pi U (g_0 \mu_B \ell_z)^2 N(0) \frac{k_B T}{\omega_c}. \quad (7)$$

$1/\tau_L$ and $1/\tau(T)$ are of similar magnitude at $\mu_B H/k_B T$ of $O(1)$ for $\ln(\omega_c/\pi T) \approx 1$. They are similar because the inelastic scattering rate comes from the imaginary part of the same fluctuations whose real part gives n_L to give the elastic scattering rate and the coupling energy to fermions is identical. Specifically the ratio of the scattering rates is

$$(1/\tau_L) \div (1/\tau(T)) \approx \frac{2\ell_z \mu_B H}{U k_B T} \ln(\omega_c/\pi T). \quad (8)$$

The result (8) is subject to a cut-off at low temperatures if one is not at critical parameters, (the critical point is also expected to shift in a magnetic field if the usual magnetic susceptibility of the system is different on the two sides of critical point) and a high temperature cut-off on the scale of the upper cut-off ω_c .

We can compare the result in Eq. (8) quantitatively with experiments. The data for the resistivity in the most extensively investigated case, for a cuprate near criticality, is represented in Ref. [16] by $\rho(T, H) = \alpha k_B T + \beta(T) \mu_B H$. We can write using Eqs. (7) and

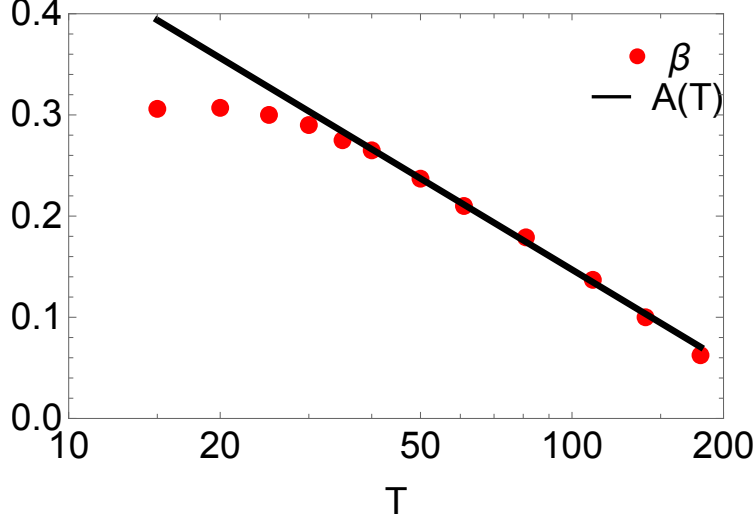


Figure 3. The temperature dependence of the linear in H resistivity in $La_{2-x}Sr_xCuO_4$, for $x = 0.19$. The data shown as red dots is taken from Fig. 1 b of Ref. ([16]). $\beta(T)$ is obtained from the fit to the resistivity (after subtracting a small residual value) $\rho(T, H) - \rho_0 = \alpha T + \beta(T) H$ at $H = 70$ Tesla. $A(T) = 0.14 \ln \left| \frac{\omega_c}{\pi T} \right|$, with $\omega_c \approx 1600K$.

(8) that $\beta(T) = \alpha \frac{2\ell_z}{U} \ln(\omega_c/\pi T)$. $\beta(T)$ from low T to the highest available temperature, $180K$, and a logarithmic fit to it by $0.14 \ln(1500/\pi T)$ are given in Fig. (3). The coefficient 0.14 should be compared with 0.19 that is estimated from parameters above and the value of $\alpha \approx 1.1$ deduced in the experiment [16]. A logarithmic fit appears reasonable for $T \gtrsim 30K$ below which the data saturates. The parameter ω_c is about $1600K$, which may be compared with the $O(3000)K$ deduced [2] from the fit to the logarithmic C_v/T measured [47] between $0.3K$ and $10K$. The peril of deducing a number from a logarithm in a range far above the data should be kept in mind. The data in Fig. 1a in [16] shows systematic rounding towards zero below about $30 K$ even in a field of 70 Tesla. One may be tempted to ascribe it to not being very close to criticality, but a closer look at all the data at various fields suggests a more mundane reason. The data shows a large region of rounding from the zero-field transition temperature ($\approx 41K$) towards zero resistivity at low temperatures even in large fields. This is generally the rule in 2d strongly type superconductors or superconducting films due to an enhanced region of phase fluctuations in a field.

An independent way to test the prediction made here is to see if a direct measurement of magnetization in the range in which the resistivity satisfies Eqn. (8) shows the same

logarithmic enhancement.

I now briefly discuss the other compounds, to begin with those for which the quantum-criticality is that of antiferromagnetism. Significantly, the important critical fluctuations for planar and incommensurate Ising ferromagnets or anti-ferromagnets (or charge density waves) are of the phase variable given by the xy model [22, 48]. It is very interesting to note that the measured spectral functions for the quantum-critical fluctuations for planar antiferromagnetism in $BaFe_{1.85}Co_{0.15}As_2$ [49] or incommensurate anti-ferromagnetism in the heavy fermion $CeCu_6$ [50, 51] are consistent with the product form in momentum and energy [52] as in Eq. (1) for the QXY-SF model.

The data [15] in $BaFe_2(As_{1-x}P_x)_2$ with $T_c \approx 30K$, is available only to 60 K with fields up to 59 Tesla has a severe rounding of resistivity towards zero at low temperatures for fields less than 50 Tesla so that linearity of H above this field is observed only in a narrow range of temperatures. We therefore cannot usefully compare the data in $BaFe_2(As_{1-x}P_x)_2$. The fit of the resistivity data made as $\propto \sqrt{(\mu_B H)^2 + (k_B T)^2}$ made earlier [15] is not good under closer examination of the detailed data kindly received from the authors (I. Hayes - private communication - Dec. 2021). That fit also does not work for the cuprate or for the twisted bi-layer graphene [17], as stated by the authors. However, an H^2 dependence of magneto-resistance at low fields is conventional and well understood and there is no reason why it should be completely absent in the metals under discussion.

The relevant order parameter for Twisted bi-layer (TB)-graphene and TB- WSe_2 is not known yet from experiments, although there are theoretical calculations suggestive of loop-current ordered states [53, 54] in TB-graphene. TB- WSe_2 is similar except for the large spin-orbit coupling. Their structure has a triangular motif and it is expected that the nearest neighbor repulsion is comparable to the kinetic energy. In this situation, loop-current order is a likely instability [55–57]. It should be ascertained if only the component of the magnetic field perpendicular to the plane is responsible for the resistance linear in the field. If this holds, more experiments to test the time-reversal, inversion and possible chirality given by loop-currents are suggested to decipher their long-range order. The data on WSe_2 is not yet detailed enough to compare with theory for $\beta(T)$, but it is for TB-graphene [17]. This is plotted in (4). The logarithmic fit is found with a coefficient 0.7 and an upper cut-off $\omega_c \approx 150K$, obtained from the quantity πT where the logarithm extrapolates to 0. There are no independent numbers from other experiments to compare. But the scale of the

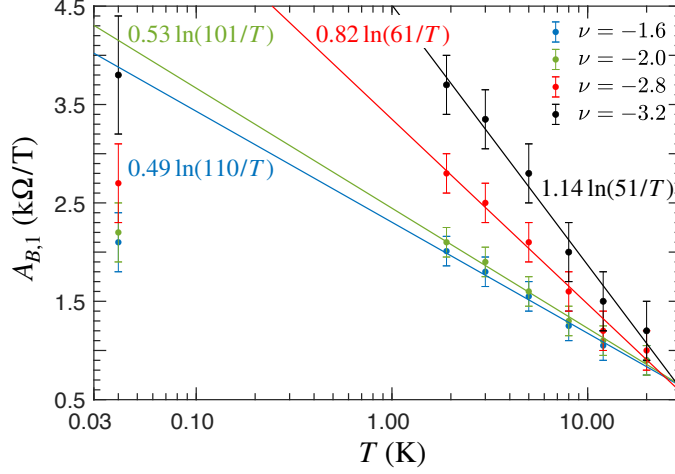


Figure 4. The temperature dependence of the linear in H resistivity in twisted bi-layer graphene. The data is from Ref. ([17]) and replotted by the authors. $A_{B,1} \equiv \beta(T)$ is obtained just as for the cuprate compound in Ref. (3). The fit to experiments is given at various dopings; the critical dopings in the experiment are not definitely known because only at $\nu = 2.8$, the resistivity is available down to very low temperatures.

fluctuation energies an order of magnitude smaller than the cuprates appears reasonable. The saturation at the lowest point at $40mK$ is almost certainly due to rounding of resistivity due to impending superconductivity, while at the highest temperature πT is essentially the upper-cutoff and so a saturation is inevitable.

The experiments in a magnetic field test an important aspect of the theory of quantum-fluctuations of the xy-model because magnetic field couples to the generator of rotations of the vector characterizing in-plane order. It is already understood that d-wave superconductivity is not possible if the self-energy of the fermions is angle-independent as it is in cuprates without the fermions coupling to the fluctuations of angular momentum [2, 32][58]. To conclude, one might also add that the mechanism of superconductivity in all these systems is inevitably related to the fluctuations which give resistivity linear in T and in H .

Acknowledgements: I have benefitted enormously from discussions with Arkady Shekhter both of the experimental data and the theory. Discussions of the theory with Lijun Zhu and Vivek Aji have been invaluable. I have also benefitted by discussions of the state of experiments with Greg Boebinger on cuprates, James Analytis and Ian Hayes on the Fe-compound, with Augusto Ghiotto and Abhay Pasupathy on WSe_2 , with A. Jaoui on MBLG

who also provided me with Fig. (4), and with Erez Berg on the theory for TB-graphene. This work was partially performed at the Aspen Center for Physics, which is supported by National Science Foundation grant PHY-1607611.

-
- [1] D. Ginsburg, ed., *Physical Properties of High Temperature Superconductors* (World Scientific, Singapore, Vol I (1989), Vol II (1990), Vol III (1992), Vol. Iv (1994)).
 - [2] C. M. Varma, Rev. Mod. Phys. **92**, 031001 (2020).
 - [3] C. Varma, "Theoretical framework for the normal state of copper oxide metal" in *Strongly Correlated Electronic Systems, the Los Alamos Symposium 1991* (Addison-Wesley, Reading MA, 1994, 1994) pp. 573–603.
 - [4] C. M. Varma, Phys. Rev. B **55**, 14554 (1997).
 - [5] M. E. Simon and C. M. Varma, Phys. Rev. Lett. **89**, 247003 (2002).
 - [6] K. Ishida, Y. Nakai, and H. Hosono, J. Phys. Soc. Jpn. **78**, 062001 (2009).
 - [7] T. Shibauchi, A. Carrington, and Y. Matsuda, Annual Review of Condensed Matter Physics **5**, 113 (2014), <https://doi.org/10.1146/annurev-conmatphys-031113-133921>.
 - [8] M. Kim, M. Wang, G. Tucker, P. Valdivia, D. Abernathy, and e. a. Chi, Songxue, arXiv:1510.04167 (2015).
 - [9] B. Lv, M. Gooch, B. Lorenz, F. Chen, A. Guloy, and C. Chu, New J. Phys. **11**, 025013 (2009).
 - [10] H. Löhneysen, A. Rosch, M. Vojta, , and P. Wölfle, Rev. Mod. Phys. **79**, 1015 (2007).
 - [11] P. Gegenwart, T. Westerkamp, C. Krellner, Y. Tokiwa, S. Paschen, C. Geibel, F. Steglich, E. Abrahams, and Q. Si, Science **315**, 969 (2007), <https://www.science.org/doi/pdf/10.1126/science.1136020>.
 - [12] Y. Cao, D. Chowdhury, D. Rodan-Legrain, O. Rubies-Bigorda, K. Watanabe, T. Taniguchi, T. Senthil, and P. Jarillo-Herrero, arXiv:1901.03710 (2019).
 - [13] X. Lu, P. Stepanov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, T. Watanabe, Kenji and-Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, arXiv:1903.06513 (2019).
 - [14] A. Ghiotto, E.-M. Shih, G. S. S. G. Pereira, D. A. Rhodes, B. Kim, J. Zang, A. J. Millis, K. Watanabe, T. Taniguchi, J. C. Hone, L. Wang, C. R. Dean, and A. N. Pasupathy, (2021), arXiv:Preprint available at <http://arxiv.org/abs/2103.09796>.

- [15] I. Hayes, N. Breznay, T. Helm, P. Moll, B. Wartenbe, R. McDonald, A. Shekhter, and J. G. Analytis, arXiv:1412.6484 [cond-mat.str-el] (2014).
- [16] P. Giraldo-Gallo, J. A. Galvis, Z. Stegen, K. A. Modic, F. F. Balakirev, J. B. Betts, X. Lian, C. Moir, S. C. Riggs, J. Wu, A. T. Bollinger, X. He, I. Božović, B. J. Ramshaw, R. D. McDonald, G. S. Boebinger, and A. Shekhter, *Science* **361**, 479 (2018), <https://science.sciencemag.org/content/361/6401/479.full.pdf>.
- [17] A. Jaoui, I. Das, G. Di Battista, J. Díez-Mérida, X. Lu, K. Watanabe, T. Taniguchi, H. Ishizuka, L. Levitov, and D. K. Efetov, (2021), arXiv:Preprint available at <http://arxiv.org/abs/2108.07753>.
- [18] I. M. Hayes, Z. Hao, N. Maksimovic, S. K. Lewin, M. K. Chan, R. D. McDonald, B. J. Ramshaw, J. E. Moore, and J. G. Analytis, *Phys. Rev. Lett.* **121**, 197002 (2018).
- [19] G. Boebinger and A. Shekhter - private communication 2020.
- [20] V. Aji and C. M. Varma, *Phys. Rev. Lett.* **99**, 067003 (2007).
- [21] V. Aji and C. M. Varma, *Phys. Rev. B* **82**, 174501 (2010).
- [22] C. M. Varma, *Rep. Prog. Phys.* **75**, 052501 (2012).
- [23] D. V. Else and T. Senthil, *Phys. Rev. Lett.* **127**, 086601 (2021).
- [24] A. Kaminski, S. Rosenkranz, H. M. Fretwell, J. C. Campuzano, Z. Li, H. Raffy, W. G. Cullen, H. You, C. G. Olson, C. M. Varma, and H. Höchst, *Nature* **416**, 610 (2002).
- [25] P. Bourges and Y. Sidis, *Comptes Rendus Physique* **12**, 461 (2011).
- [26] L. Zhao, C. A. Belvin, R. Liang, D. A. Bonn, W. N. Hardy, N. P. Armitage, and D. Hsieh, *Nature Physics* **13**, 250 EP (2016).
- [27] Y. Lubashevsky, L. Pan, T. Kirzhner, G. Koren, and N. P. Armitage, *Phys. Rev. Lett.* **112**, 147001 (2014).
- [28] J. Zhang, Z. Ding, C. Tan, K. Huang, O. O. Bernal, P.-C. Ho, G. D. Morris, A. D. Hillier, P. K. Biswas, S. P. Cottrell, H. Xiang, X. Yao, D. E. MacLaughlin, and L. Shu, *Science Advances* **4** (2018), 10.1126/sciadv.aao5235, <https://advances.sciencemag.org/content/4/1/eaao5235.full.pdf>.
- [29] L. Zhu, Y. Chen, and C. M. Varma, *Phys. Rev. B* **91**, 205129 (2015).
- [30] L. Zhu, C. Hou, and C. M. Varma, *Phys. Rev. B* **94**, 235156 (2016).
- [31] C. Hou and C. M. Varma, *Phys. Rev. B* **94**, 201101 (2016).
- [32] V. Aji, A. Shekhter, and C. M. Varma, *Phys. Rev. B* **81**, 064515 (2010).

- [33] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein, Phys. Rev. Lett. **63**, 1996 (1989).
- [34] D. Chowdhury, A. Georges, O. Parcollet, and S. Sachdev, (2021), arXiv:Preprint available at <http://arxiv.org/abs/2109.05037>.
- [35] T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, Phys. Rev. D **83**, 125002 (2011).
- [36] P. T. Dumitrescu, N. Wentzell, A. Georges, and O. Parcollet, (2021), arXiv:Preprint available at <http://arxiv.org/abs/2103.08607>.
- [37] A. E. Ruckenstein and C. M. Varma, Physica C Superconductivity **185**, 134 (1991).
- [38] C. H. Mousatov, E. Berg, and S. A. Hartnoll, Proceedings of the National Academy of Sciences **117**, 2852 (2020), <https://www.pnas.org/content/117/6/2852.full.pdf>.
- [39] K. Sun and E. Fradkin, Phys. Rev. B **78**, 245122 (2008).
- [40] Y. He and C. M. Varma, Phys. Rev. B **86**, 035124 (2012).
- [41] Y. He, J. Moore, and C. M. Varma, Phys. Rev. B **85**, 155106 (2012).
- [42] F. Pelzer, Phys. Rev. B **44**, 293 (1991).
- [43] C. Varma, Z. Nussinov, and W. van Saarloos, Phys. Reports **361**, 267 (2002).
- [44] H. Maebashi, C.M. Varma - unpublished.
- [45] A. Shekhter - private communication (Dec. 2021).
- [46] E. Abrahams and C. M. Varma, Phys. Rev. B **68**, 094502 (2003).
- [47] B. Michon, C. Girod, S. Badoux, J. Kačmarčík, Q. Ma, M. Dragomir, H. A. Dabkowska, B. D. Gaulin, J.-S. Zhou, S. Pyon, T. Takayama, H. Takagi, S. Verret, N. Doiron-Leyraud, C. Marcenat, L. Taillefer, and T. Klein, Nature **567**, 218 (2019).
- [48] W. McMillan, Phys. Rev. B **12**, 1187 (1975).
- [49] D. S. Inosov, J. T. Park, P. Bourges, D. L. Sun, Y. Sidis, A. Schneidewind, K. Hradil, D. Haug, C. T. Lin, B. Keimer, and V. Hinkov, Nature Phys. **6**, 178 (2010).
- [50] A. Schröder and et al., Phys. Rev. Lett. **80**, 5623 (1998).
- [51] A. Schröder and et al., Nature (London) **407**, 351 (2000).
- [52] C. M. Varma, L. Zhu, and A. Schröder, Phys. Rev. B **92**, 155150 (2015).
- [53] N. Bultinck, E. Khalaf, S. Liu, S. Chatterjee, A. Vishwanath, and M. P. Zaletel, Phys. Rev. X **10**, 031034 (2020).
- [54] J. S. Hofmann, E. Khalaf, A. Vishwanath, E. Berg, and J. Y. Lee, (2021), arXiv:Preprint available at <http://arxiv.org/abs/2105.12112>.

- [55] C. Weber, T. Giamarchi, and C. M. Varma, Phys. Rev. Lett. **112**, 117001 (2014).
- [56] R. Nandkishore and L. Levitov, Phys. Rev. Lett. **107**, 097402 (2011).
- [57] L. Zhu, V. Aji, and C. M. Varma, Phys. Rev. B **87**, 035427 (2013).
- [58] J. M. Bok, J. J. Bae, H.-Y. Choi, C. M. Varma, W. Zhang, J. He, Y. Zhang, L. Yu, and X. J. Zhou, Science Advances **2** (2016), 10.1126/sciadv.1501329, <https://advances.sciencemag.org/content/2/3/e1501329.full.pdf>.