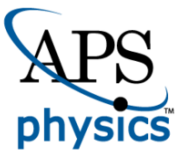

Global Stability of Fluid Flows Despite Transient Growth of Energy

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Global stability of fluid flows despite transient growth of energy

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Verifying nonlinear stability of a laminar fluid flow against all perturbations is a central challenge in fluid dynamics. Past results rely on monotonic decrease of a perturbation energy or a similar quadratic generalized energy. None show stability for the many flows that seem to be stable despite these energies growing transiently. Here a broadly applicable method to verify global stability of such flows is presented. It uses polynomial optimization computations to construct non-quadratic Lyapunov functions that decrease monotonically. The method is used to verify global stability of 2D plane Couette flow at Reynolds numbers above the energy stability threshold found by Orr in 1907. This is the first global stability result for any flow that surpasses the energy method.

A central approach to understanding fluid dynamics has been to study a handful of canonical systems in detail. Despite many discoveries over the last century, one of the simplest-seeming questions remains open for some of the most-studied systems: at given parameter values, will the flow return to its simplest (laminar) state no matter how it is perturbed? Laboratory experiments and simulations of the Navier–Stokes equations are unable to give a complete answer for all perturbations. Theoretical methods are needed to guarantee global stability.

For a steady laminar velocity field $\mathbf{U}(\mathbf{x})$ solving the incompressible Navier–Stokes equations, the velocity, $\mathbf{u}(\mathbf{x},t)$, and pressure, $p(\mathbf{x},t)$, of perturbations around the laminar state evolve according to

$$\tfrac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \tfrac{1}{\mathrm{Re}}\Delta\mathbf{u} + A(\mathbf{u}), \qquad (1)$$

$$\nabla\cdot\mathbf{u} = 0, \qquad (2)$$

where $A(\mathbf{u}) = -\mathbf{U}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{U}$ and Δ is the Laplacian operator [1]. Quantities in (1)–(2) are dimensionless, having been scaled using a length scale h, velocity scale U, and kinematic viscosity ν. Choices of h and U depend on the particular system. The dimensionless Reynolds number is $\mathrm{Re} = Uh/\nu$.

There is a critical threshold $\mathrm{Re}_G > 0$ such that the laminar state $\mathbf{U}$ is globally asymptotically stable (meaning all perturbations $\mathbf{u}$ eventually converge to zero) if and only if $\mathrm{Re} < \mathrm{Re}_G$ [1]. Loss of global stability is not sufficient for turbulence, but it is necessary, and often it is more informative than linear stability. Linear stability of the laminar state does not preclude turbulence whose onset is subcritical [2–6], nor does it ensure that the laminar state is physically realizable because the basin of attraction can be minuscule [7–9]. The value of Re_G, however, can be very hard to determine.

An upper bound on Re_G is provided by any Re at which a sustained non-laminar flow is found. A lower bound on Re_G requires finding a Re threshold below which the laminar state is globally stable. Thus far the only method applicable to all systems governed by (1)–(2) has been the energy method pioneered by Reynolds and Orr [10, 11], where one finds the threshold Re_E such that the kinetic energy, $E = \frac{1}{2}\int|\mathbf{u}|^2\mathrm{d}\mathbf{x}$, of every perturbation decreases monotonically toward zero if and only if $\mathrm{Re} < \mathrm{Re}_E$. Often the lower bound on Re_G provided by Re_E is very conservative. In systems where turbulence is driven by parallel shear, such as pressure-driven flow in a pipe or boundary-driven flow in a layer, the energy stability thresholds Re_E [12–15] are much smaller than the minimum Re at which sustained non-laminar states have been found [4, 16–18]. In other words, there is a large gap between these lower and upper bounds on Re_G.

Global stability at Re values above Re_E has been shown only in special cases where the energy method can be slightly generalized. Each such result has relied on monotonic decrease of a quadratic integral that is an inviscid invariant, meaning the nonlinear term in (1) does not contribute to the expression for the integral's evolution. For symmetric perturbations where individual components of E are conserved, for instance, one can consider various linear combinations of these components [15, 19–21]. Lacking an artificial symmetry on $\mathbf{u}$, however, E is the only nonnegative quadratic integral that can be shown to decrease globally. In this general situation there has been no method for verifying global stability above Re_E, aside from the one presented here.

The standard way to show that a solution of a dynamical system is globally asymptotically stable is to construct a Lyapunov function. Here this is a functional V that maps each spatial function $\mathbf{u}(\cdot,t)$ to a real number and satisfies $V(\mathbf{0}) = 0$. Let $\mathcal{L}V$ denote the Lie derivative of V along PDE solutions of (1)–(2), meaning $\mathcal{L}V$ is the functional such that $\mathcal{L}V(\mathbf{u}(\cdot,t)) = \frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{u}(\cdot,t))$ for all $\mathbf{u}(\mathbf{x},t)$ solving (1)–(2). The $\mathbf{u} = \mathbf{0}$ state is globally attracting if $V(\mathbf{u}) > 0$ and $\mathcal{L}V(\mathbf{u}) < 0$ for all nonzero $\mathbf{u}$ admitted by the boundary conditions [22]. The energy method uses $V = E$ or, when symmetries allow it, weighted combinations of the components of E.

Our method constructs Lyapunov functionals V with polynomial dependence on $\mathbf{u}$, in particular with

$$V(\mathbf{u}) = V(\mathbf{a}, q) = E^d + P(\mathbf{a}, q), \tag{3}$$

where $\mathbf{a}(\mathbf{u}) \in \mathbb{R}^m$, $q(\mathbf{u}) \in \mathbb{R}$, d is an integer, and P is a polynomial whose degree is at most $2d-1$. By definition, the components of $\mathbf{a}$ are projections of $\mathbf{u}$ onto an orthogonal set of spatial modes, $\{\mathbf{u}_1(\mathbf{x}), \ldots, \mathbf{u}_m(\mathbf{x})\}$, and $\frac{1}{2}q^2$ is the energy of the unprojected remainder of $\mathbf{u}$. For reasons explained shortly, we choose the $\mathbf{u}_i$ to be eigenfunctions of the energy stability operator. Constructing P and verifying that V is a valid Lyapunov functional presents major challenges beyond the quadratic case. A general way to surmount these challenges is presented below, but first we summarize stability results found by applying our method to a classic fluid flow.

To show that Lyapunov functionals of the form (3) can surpass existing methods we consider 2D plane Couette flow, which is driven by parallel relative motion of the boundaries. We have verified global stability of this flow beyond the energy stability threshold given by Orr in 1907 [11]. The reason for considering a 2D flow, aside from Orr's result being especially longstanding, is to reduce the computational cost of testing our method. The same approach is applicable to arbitrary 3D perturbations, but this is left for future work. The flow is periodic in the streamwise direction, $x \in (0, L)$, and confined in the wall-normal direction, $y \in (-\frac{1}{2}, \frac{1}{2})$. Perturbations about the laminar flow $\mathbf{U} = (y, 0)$ obey (1)–(2) and satisfy no-slip conditions $\mathbf{u}(x, \pm\frac{1}{2}) = \mathbf{0}$ at the walls. With this nondimensionalization, Re is defined using the full velocity difference and height difference between the shearing planes. Some authors use half these differences, so their Reynolds number is 1/4 of the Re shown here.

The true value of Re_G in 2D plane Couette flow is unknown. Several computational efforts have failed to find sustained non-laminar states [23–25], and the laminar state is linearly stable for all Re [3], so Re_G has no known upper bound and may be infinite. For each streamwise period L, the energy method gives a lower bound $\mathrm{Re}_E(L) \le \mathrm{Re}_G(L)$. As found by Orr [11], its minimum $\mathrm{Re}_E \approx 177.2$ occurs at integer multiples of $L_E \approx 1.659$. (In 3D, Re_G is bounded below by $\mathrm{Re}_E \approx 82.6$ [12, 14] and above by 511, the smallest Re at which traveling waves solutions have been computed numerically [17, 18].)

Here we have constructed many V of the form (3), all having quartic degree ($d = 2$) and depending explicitly on the projections a_i of $\mathbf{u}$ onto various $\mathbf{u}_i$ modes. Results are reported for four different mode sets (defined later) whose number of modes (m) are 6, 8, 12, and 13. Figure 1 shows Re values at which stability has been verified using each set of modes, along with the energy stability threshold $\mathrm{Re}_E(L)$. At each plotted point, a different Lyapunov functional was constructed to show global stability for perturbations of period L at the Re indicated. Raising the number of modes on which V depends increases the

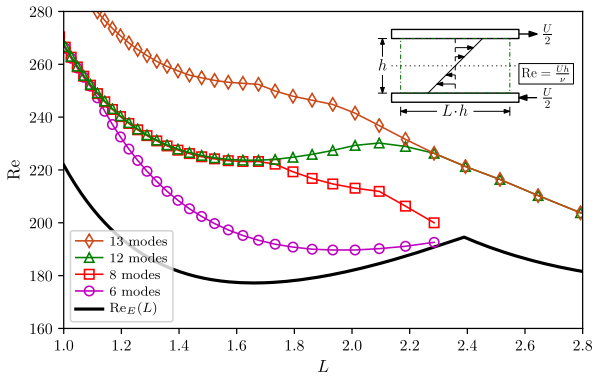

Figure 1. Reynolds numbers (Re) at which laminar plane Couette flow is globally asymptotically stable against 2D perturbations of period L. Each symbol indicates values (Re, L) where we verified stability using a quartic Lyapunov functional. Each functional depends explicitly on the flow's projection onto 6, 8, 12, or 13 energy eigenmodes, and on the unprojected energy. Lines connect symbols to guide the eye. Orr's energy stability threshold $\mathrm{Re}_E(L)$ is also shown.

Re at which stability can be verified, but it also increases the computational cost of constructing V by the method explained below, which limited us to 13 modes.

Over the full range of periods L for which computations were performed, results surpass the energy method. For instance, at the most energy-unstable period L_E where the energy method gives stability up to $\mathrm{Re}_E \approx 177.2$, our best V verified stability at $\mathrm{Re} = 252.4$. Beyond the implications for Couette flow, the greater significance of these results is the proof of concept for a broadly applicable new method—the first generalization of the energy method that is applicable to any 2D or 3D flow.

To recall the workings of the energy method, note that positivity of E is clear, so implementing the energy method amounts to determining the Re at which $\mathcal{L}E < 0$ for all admissible perturbations. In systems where $\mathbf{u}$ is periodic and/or vanishes at all boundaries,

$$\mathcal{L}E = \int \left(-\tfrac{1}{\mathrm{Re}}|\nabla \mathbf{u}|^2 - \mathbf{u} \cdot \mathsf{D} \cdot \mathbf{u}\right) \mathrm{d}\mathbf{x}, \tag{4}$$

where $\mathsf{D} = \frac{1}{2}(\nabla \mathbf{U} + \nabla^{\mathsf{T}} \mathbf{U})$ is the laminar strain-rate tensor [1]. Variational arguments imply that $\mathcal{L}E < 0$ for all divergence-free nonzero $\mathbf{u}$ if and only if all eigenvalues λ are negative for the energy eigenproblem [15, 26, 27]

$$\left(\tfrac{1}{\mathrm{Re}}\Delta - \mathsf{D}\right)\mathbf{w} - \nabla\zeta = \lambda \mathbf{w}, \qquad \nabla \cdot \mathbf{w} = 0, \tag{5}$$

where ζ is the Lagrange multiplier enforcing incompressibility of $\mathbf{w}$. The largest Re at which $\lambda \le 0$ defines the energy stability threshold Re_E. Only because $\mathcal{L}E$ is quadratic can its negativity be verified from a linear Euler–Lagrange equation (5). Going beyond quadratic V requires another way to enforce $V > 0$ and $\mathcal{L}V < 0$.

To construct new non-quadratic V, we follow the ideas

in [28] and consider a partial Galerkin expansion of $\mathbf{u}$,

$$\mathbf{u}(\mathbf{x},t) = \sum_{i=1}^{m} a_i(t)\mathbf{u}_i(\mathbf{x}) + \mathbf{v}(\mathbf{x},t)\,, \tag{6}$$

where the $\mathbf{u}_i$ are selected modes of the energy eigenproblem (5), and $a_i = \int \mathbf{u}\cdot\mathbf{u}_i\,\mathrm{d}\mathbf{x}$ is the orthogonal projection of $\mathbf{u}$ onto $\mathbf{u}_i$. Let $q = (\int |\mathbf{v}|^2\mathrm{d}\mathbf{x})^{1/2}$, so the perturbation energy is $E = \frac{1}{2}(|\mathbf{a}|^2 + q^2)$. Lyapunov functionals V will be functions of the $m+1$ scalars $(\mathbf{a}, q)$, each of which is a functional of $\mathbf{u}$.

To derive the functional $\mathcal{L}V$ that coincides with $\frac{\mathrm{d}}{\mathrm{d}t}V$ along solutions of (1)–(2), we let only even powers of q appear in V, in which case $\frac{\mathrm{d}}{\mathrm{d}t}V = \frac{\partial V}{\partial \mathbf{a}}\cdot\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} + \frac{\partial V}{\partial q^2}\frac{\mathrm{d}q^2}{\mathrm{d}t}$. Projecting the Navier–Stokes equations gives expressions of the form $\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \mathbf{f} + \mathbf{\Theta}$ and $\frac{\mathrm{d}q^2}{\mathrm{d}t} = -2\mathbf{a}\cdot\mathbf{\Theta} + 2\Gamma$ [29]. These constitute an "uncertain system" for the evolution of $(\mathbf{a}, q)$ since $\mathbf{\Theta}$ and Γ (given below) depend on the tail $\mathbf{v}$ in a way that is not uniquely determined by its energy $\frac{1}{2}q^2$. The resulting expression for $\mathcal{L}V$ is [29]

$$\begin{aligned}
\mathcal{L}V(\mathbf{a},q,\mathbf{v}) &= G(\mathbf{a},q,\mathbf{v}) + \mathbf{M}(\mathbf{a},q)\cdot\mathbf{\Theta}(\mathbf{a},\mathbf{v}),\ \text{with}\\
G(\mathbf{a},q,\mathbf{v}) &= \frac{\partial V}{\partial \mathbf{a}}\cdot\mathbf{f}(\mathbf{a}) + 2\frac{\partial V}{\partial q^2}\Gamma(\mathbf{v}),\\
\mathbf{M}(\mathbf{a},q) &= \frac{\partial V}{\partial \mathbf{a}} - 2\frac{\partial V}{\partial q^2}\mathbf{a},\\
\mathbf{\Theta}(\mathbf{a},\mathbf{v}) &= \mathbf{\Theta}_{\mathbf{AB}}(\mathbf{a},\mathbf{v}) + \mathbf{\Theta}_{\mathbf{C}}(\mathbf{v}),\\
f_i(\mathbf{a}) &= L_{ij}a_j + N_{ijk}a_j a_k,\\
L_{ij} &= \tfrac{1}{\mathrm{Re}}\langle\mathbf{u}_i\,,\Delta\mathbf{u}_j\rangle + \langle\mathbf{u}_i\,,A(\mathbf{u}_j)\rangle,\\
N_{ijk} &= -\langle\mathbf{u}_i\,,\mathbf{u}_j\cdot\nabla\mathbf{u}_k\rangle,\\
\Theta_{\mathbf{AB}i}(\mathbf{a},\mathbf{v}) &= \langle\mathbf{v}\,,\mathbf{h}_{i0}\rangle + \langle\mathbf{v}\,,\mathbf{h}_{ij}\rangle a_j,\\
\mathbf{h}_{i0} &= \tfrac{1}{\mathrm{Re}}\Delta\mathbf{u}_i + \mathbf{U}\cdot\nabla\mathbf{u}_i - \mathbf{u}_i\cdot\nabla^{\mathsf{T}}\mathbf{U},\\
\mathbf{h}_{ij} &= \mathbf{u}_j\cdot\nabla\mathbf{u}_i - \mathbf{u}_i\cdot\nabla^{\mathsf{T}}\mathbf{u}_j,\\
\Theta_{\mathbf{C}i}(\mathbf{v}) &= \langle\mathbf{v}\,,\mathbf{v}\cdot\nabla\mathbf{u}_i\rangle,\\
\Gamma(\mathbf{v}) &= \tfrac{1}{\mathrm{Re}}\langle\mathbf{v}\,,\Delta\mathbf{v}\rangle - \langle\mathbf{v}\,,\mathsf{D}\mathbf{v}\rangle,
\end{aligned} \tag{7}$$

and $\langle\mathbf{u}\,,\mathbf{v}\rangle = \int\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$.

Positivity of V is enforced by regarding $V(\mathbf{a},q)$ as a polynomial on $\mathbb{R}^{m+1}$, rather than a functional of $\mathbf{u}$. Requiring positivity of this polynomial away from the origin constrains P. Negativity of $\mathcal{L}V$ is enforced in a similar way, but since $\mathcal{L}V$ depends on the full tail $\mathbf{v}$, it first must be bounded above by a polynomial depending only on $(\mathbf{a}, q)$. The reason we choose the $\mathbf{u}_i$ to be modes of the energy eigenproblem is so that $\Gamma(\mathbf{v}) \le \kappa q^2$ [28], where κ is the largest eigenvalue from (5) not associated with any of the m modes in the sum of (6). Enough modes are included so that $\kappa < 0$, and we impose $\frac{\partial V}{\partial q^2} \ge 0$ so that

$$G(\mathbf{a},q,\mathbf{v}) \le \tilde{G}(\mathbf{a},q) = \frac{\partial V}{\partial \mathbf{a}}\cdot\mathbf{f}(\mathbf{a}) + 2\frac{\partial V}{\partial q^2}\kappa q^2. \tag{8}$$

A procedure described in the Supplement introduces a polynomial $\Xi(\mathbf{a},q)$ with auxiliary constraints that ensure

$$\mathbf{M}(\mathbf{a},q)\cdot\mathbf{\Theta}(\mathbf{a},\mathbf{v}) \le \Xi(\mathbf{a},q). \tag{9}$$

By (7)–(9), if $\tilde{G} + \Xi < 0$ for all $(\mathbf{a},q)$, then $\mathcal{L}V < 0$ for all $\mathbf{u}$. Therefore, if polynomials $P(\mathbf{a},q)$ and $\Xi(\mathbf{a},q)$ are found such that $V > 0$, $\tilde{G} + \Xi < 0$, and $\frac{\partial V}{\partial q^2} \ge 0$ for all nonzero $(\mathbf{a},q)$, and such that the inequalities in the Supplement guaranteeing (9) hold, then V is a valid Lyapunov functional. Each of these constraints amounts to nonnegativity of a polynomial expression.

Verifying that a polynomial is nonnegative is computationally intractable (NP-hard) in general [30]. A tractable sufficient condition is that the polynomial can be written as a sum of squares of other polynomials. Computational techniques for enforcing sum-of-squares (SOS) constraints, introduced two decades ago [31–33], let us search for P and Ξ in a chosen bounded-degree set of polynomials subject to SOS constraints that imply all of the inequalities described above. If such P and Ξ are found, then V defined by (3) is a valid Lyapunov functional. The tunable coefficients of P and Ξ appear linearly in the expressions that must be SOS, and the problem of choosing these coefficients subject to the SOS constraints can be reformulated [34, 35] as a semidefinite program—a type of conic optimization problem that can be solved numerically using specialized software. When $\mathrm{Re} < \mathrm{Re}_E$, a solution always exists with $P = \Xi = 0$.

The approach to fluid stability described above was proposed but not implemented in [28]. As a preliminary test, the idea was applied in [29] to an example contrived to have simple energy eigenmodes. Quartic and sextic Lyapunov functionals were successfully computed in [29], but they had no chance to improve upon the energy method; a weighted energy (which can be used due to symmetries) already gives Re_G exactly for that flow. The present work adds three contributions. First, we show that the approach of [28] can surpass quadratic Lyapunov functionals in practice. Second, we do this in a realistic context where the energy eigenproblem (5) must be solved computationally. Third, we make a crucial technical change to the way Ξ is defined and constrained in [28], as described in the Supplement, and this improves our results dramatically.

The ansatz (3) for V is not an arbitrary polynomial since some structure can be deduced *a priori*. Both V and $\mathcal{L}V$ must be sign-definite, so their highest-degree terms must be of even degree. This is possible only if the nonlinearity in (1) does not contribute to the evolution of the highest-degree term in V, in which case both expressions can have the same maximum degree. This is why the leading term in (3) takes the form E^d. Further, P can have no terms of degree less than two since V must have a unique minimum when $\mathbf{u} = \mathbf{0}$. When $d = 1$ these constraints require V to be the energy E in general, reflecting the lack of freedom in the quadratic case. When $d \ge 2$ there is significant freedom in the choice of P.

Constructing a polynomial Ξ that is guaranteed to satisfy (9) requires computing all tensors in (7). To do so one must first compute energy eigenmodes of (5) for the

chosen values of (Re, L) and then select the set of modes $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, where V will depend explicitly on projections of $\mathbf{u}$ onto these modes. It is necessary to include all modes with positive eigenvalues at the given (Re, L), so that $\kappa < 0$ in (8), and to include enough stable modes that trajectories of the truncated system $\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \mathbf{f}$ are bounded. Beyond this, there is freedom in the number and choice of modes. For a fixed number of modes, experimentation may be needed to determine which mode set gives the strongest stability results.

To apply our method to 2D plane Couette flow, we first solve the energy eigenproblem (5) as detailed in the Supplement. The eigenproblem must be solved anew for each L and Re considered, giving eigenfunctions whose streamwise wavenumbers α are multiples of $\frac{2\pi}{L}$. As an example, Fig. 2 shows eigenvalues and corresponding eigenmodes for $(\mathrm{Re}, L) = (240, 2)$, a point in the parameter regime where energy can grow transiently yet our computations verify stability.

The four nested sets of eigenmodes $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ that were used to compute the stability results of Fig. 1 are defined in the Fig. 2 caption. For each (Re, L) and set of modes, all tensors in (7) were computed numerically. We then formulated the SOS computations described above, searching for polynomials P and Ξ such that V was verified to be a Lyapunov functional. The parser YALMIP [36, 37] was used to reformulate all SOS constraints as semidefinite programs, which were then solved using MOSEK [38]. The resulting P and Ξ have many terms, so we do not report them here.

For each L and set of modes, the symbol plotted in Fig. 1 is the largest Re for which our SOS computations found a valid quartic V. We expect the stability thresholds in Fig. 1 will continue to improve with an increase to the number of eigenmodes (m) on which V explicitly depends in (3). However, our computations for 13 modes are already expensive. This prevents us from considering very large L since the number of modes that would be needed grows at least linearly with L. Thus the present version of our method cannot apply to arbitrary-L perturbations in very long domains, although it surpasses the energy method for perturbations up to whatever period is computationally tractable. Aside from adding modes, stability thresholds could be improved by raising the polynomial degree of V, but sextic V demand much larger computational cost and memory footprint.

As an independent check that the V constructed by our SOS computations decrease monotonically in time, we numerically integrated (1)–(2) for 2D Couette flow using the code Dedalus [39], starting from 10^4 random initial conditions (cf. the Supplement) in the energy-unstable case $(\mathrm{Re}, L) = (240, 2)$. In all simulations our V depending on 13 modes decreased monotonically, whereas E increased transiently in 7 simulations.

In summary, we have presented a general method for constructing polynomial Lyapunov functionals to show

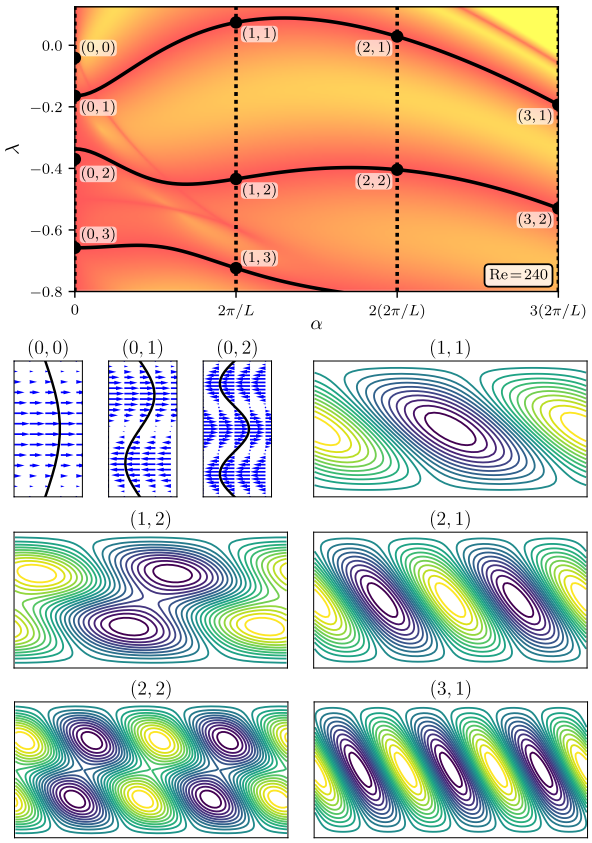

Figure 2. Energy stability eigenmodes for $(\mathrm{Re}, L) = (240, 2)$. The top panel shows eigenvalues as a function of streamwise wavenumber α. Shading indicates the minimum singular value of a boundary constraint matrix B (cf. the Supplement); black curves are zeros of this minimum, corresponding to eigenvalues. Eigenmodes consistent with $L = 2$ occur at multiples of $\frac{2\pi}{L}$, marked by black dots. The mode with the j^{th} largest eigenvalue among eigenmodes with wavenumber $\alpha = \frac{2\pi i}{L}$ is labeled (i, j). Bottom panels show streamlines for selected modes. When $i \neq 0$ we include two modes, shifted by a quarter-period in x, to span the relevant eigenspace. The 6-mode set consists of the $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 2)$ eigenmodes. The 8-mode set adds $(2, 1)$, the 12-mode set adds $(2, 2)$ and $(3, 1)$, and the 13-mode set adds $(0, 2)$.

global stability of fluid flows. It may be used to surpass the many conservative results derived using energy (or other quadratic integrals) to which past studies of fluid stability have been confined. Our approach is more technical than the energy method but can be implemented using modern computational tools of polynomial optimization. We have verified stability for 2D plane Couette in a regime where energy grows transiently. This improves on a century-old stability criterion of Orr, at least for perturbations whose streamwise periods are not too large. As far as we know, this is the first global stability result for any flow that is stronger than what can be shown using the energy method or its generalizations to other quadratic integrals. The natural next step is

to apply the same approach to 3D perturbations of plane Couette flow or another 3D flow where the energy method is overly conservative, such as pipe flow. The procedure will be the same as in the present 2D example, only with greater technicality and computational cost.

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