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Phys. Rev. Lett. **128**, 180503 — Published 4 May 2022

DOI: 10.1103/PhysRevLett.128.180503

Quantum Metrological Power of Continuous-Variable Quantum Networks

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We investigate the quantum metrological power of typical continuous-variable (CV) quantum networks. Particularly, we show that most CV quantum networks provide an entanglement to quantum states in distant nodes that enables one to achieve the Heisenberg scaling in the number of modes for distributed quantum displacement sensing, which cannot be attained using an unentangled probe state. Notably, our scheme only requires local operations and measurements after generating an entangled probe using the quantum network. In addition, we find a tolerable photon-loss rate that maintains the quantum enhancement. Finally, we numerically demonstrate that even when CV quantum networks are composed of local beam splitters, the quantum enhancement can be attained when the depth is sufficiently large.

Quantum metrology is a study on advantages of quantum resources for parameter estimation [1–6]. In many years, nonclassical features of quantum probes have been shown to achieve a better sensitivity than any classical means. Especially in continuous-variable (CV) systems, a squeezed state, one of the most representative nonclassical states, elevates the sensitivity of optical interferometers [7, 8] including gravitational wave detectors [9–11]. In addition, enhanced phase estimation using a squeezed state has been implemented in many experiments [12–14].

More recently, besides quantum enhancement from a local system, much attention has been paid to employ a metrological advantage from entanglement between distant sites. Particularly, distributed quantum sensing has been proposed and extensively studied to enhance the sensitivity by exploiting quantum entanglement constituted by a quantum network for estimating parameters in distant nodes [15–25]. For example, a single-mode squeezed vacuum state distributed by a balanced beam splitter network (BSN) was shown to enable estimating the quadrature displacement with a precision up to the Heisenberg scaling in the number of modes, which cannot be achieved without entanglement [18]. Such an enhancement has also been found in distributed quantum phase sensing [17, 19, 21, 25]. Furthermore, the enhancement from entanglement between nodes has been experimentally demonstrated in various tasks [19, 20, 22, 26].

While particular CV quantum networks provide an enhancement for distributed sensing, it is unclear whether general quantum networks are beneficial. Since quantum entanglement between distant nodes is the key to improving the sensitivity in many cases, investigating what kinds of quantum networks are advantageous for distributed sensing is crucial fundamentally and practically. To answer similar questions such as the usefulness of general quantum states, Ref. [27] has initiated a study for quantum enhancement from typical quantum states by considering the role of interparticle entanglement for quantum phase estimation and shown advantages of typical bosonic random states for quantum phase estimation.

In this Letter, motivated by Ref. [27], we study global random CV networks and show that typical CV quantum networks provide quantum metrological enhancement. More specifically, we prove that most CV quantum networks except for an exponentially small fraction in the number of modes enable us to achieve the Heisenberg scaling in the number of modes for a distributed quantum displacement sensing scheme. Since we focus on the Heisenberg scaling in the number of sensor nodes, the intermode entanglement is the key resource. On the other hand, Ref. [27] investigates the Heisenberg scaling in the number of particles for quantum phase estimation with interparticle entanglement. In addition, we show that local operations after an input quantum state undergoes a CV quantum network are essential for the enhancement because the Heisenberg scaling cannot be attained without them with a high probability. We then study the effect of photon loss and find tolerable loss amount that maintains the Heisenberg scaling. Furthermore, we numerically demonstrate that quantum networks composed of local-random beam splitters also render the Heisenberg scaling for distributed displacement sensing on average within a depth proportional to M^2 with M being the number of modes.

Distributed quantum displacement sensing.— For distributed displacement sensing (see Fig. 1), we first prepare a product state with a total mean photon number $\bar{N}$. The state is then injected into a BSN $\hat{U}$ to generate an entangled probe between M modes. Here, a BSN is described by an $M \times M$ unitary matrix U, which transforms input annihilation operators $\{\hat{a}_i\}_{i=1}^{M}$ as $\hat{a}_i \to \hat{U}^\dagger \hat{a}_i \hat{U} = \sum_{j=1}^{M} U_{ij}\hat{a}_j$. After the BSN, we perform local phase shift operations, $\hat{R}(\boldsymbol{\phi}) \equiv \otimes_{j=1}^{M} \hat{R}_j(\phi_j)$ with $\hat{R}_j(\phi_j) \equiv e^{i\phi_j \hat{a}_j^\dagger \hat{a}_j}$ being a phase-shift operator on jth mode for ϕ_j. Thus, for a given BSN, a local-phase optimization is implemented by manipulating ϕ_j's. The entangled probe then encodes a displacement parameter x of interest. We assume that the same displacement oc-

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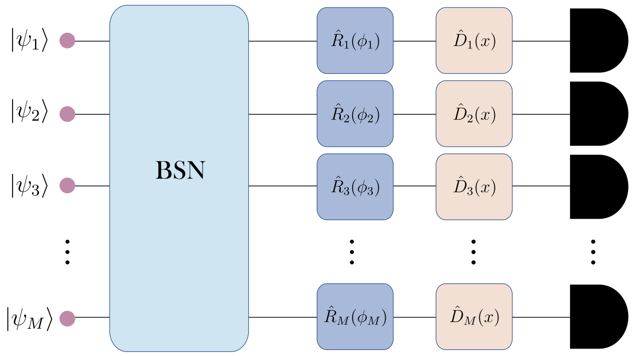

FIG. 1. Schematic diagram of distributed quantum displacement sensing (see the main text).

curs in all M modes, the operator of which is written as $\otimes_{j=1}^{M}\hat{D}_j(x)$ with $\hat{D}_j(x) \equiv e^{-i\hat{p}_j x}$ being a displacement operator for jth mode along x-direction. Here, quadrature operators of jth mode are defined as $\hat{x}_j \equiv (\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}$, $\hat{p}_j \equiv (\hat{a}_j - \hat{a}_j^\dagger)/\sqrt{2}i$ for x and p directions in phase space, respectively. Finally, we locally measure the output state on each site using homodyne detection and estimate the parameter x using the measurement outcomes. We emphasize that our scheme has tensor product inputs and local measurements, while only the BSN can generate entanglement. Note that the proposed scheme is similar to the one in Ref. [18] except that we employ an arbitrary BSN instead of a balanced one. Also, such a distributed sensing scheme can offer advantages for many quantum metrological applications [18, 28–30].

Meanwhile, when we estimate a parameter θ of interest using a quantum state probe $\hat{\rho}$, the estimation error of any unbiased estimator, $\Delta^2\theta$, is bounded by the quantum Cramér-Rao lower bound as $\Delta^2\theta \geq 1/H$, where H is the quantum Fisher information (QFI) for a given system and a probe state $\hat{\rho}$ [31, 32]. Therefore, QFI quantifies the ultimate achievable estimation error using a given quantum state. Especially for a pure state probe $|\psi\rangle$ and a unitary dynamics with a Hamiltonian operator $\hat{h}$, the QFI can be simplified as $H = 4(\Delta^2\hat{h})_\psi$.

For a distributed displacement sensing, the attainable QFI without an entangled probe scales at most linear in $\bar{N}$ and M (e.g., a product of identical states for M modes such as squeezed states) [18, 33]. Remarkably, if one employs the optimal entangled scheme (see Eq. (2)), the corresponding QFI scales as $\bar{N}M$ [18, 33]. Therefore, an entanglement provides an advantage for distributed quantum displacement sensing if one prepares a suitable CV quantum network, and the advantage is apparent from the scaling of $\bar{N}M$. For the purpose of the Letter that is to study the scaling of QFI in terms of the number of sensor nodes, we inspect the behavior of QFI as the number of modes M grows with fixing the mean photon number per mode $\bar{n} \equiv \bar{N}/M$. It is worth emphasizing that since random quantum networks do not evenly allocate the input energy, the number of photons occupying a single mode fluctuates and can be much larger than $\bar{n}$.

Results.— We first derive the QFI for distributed displacement sensing for a given CV quantum network, characterized by an $M \times M$ unitary matrix U, with a squeezed state input. After a BSN and phase shifters, the probe state can be written as $|\psi\rangle = \hat{R}(\boldsymbol{\phi})\hat{U}|\psi_{\text{in}}\rangle$, where $|\psi_{\text{in}}\rangle$ is a product state of a squeezed state in the first mode and $(M-1)$ vacua in other modes. Since the Hamiltonian operator is $\hat{h} = \sum_{j=1}^{M}\hat{p}_j$, the QFI for distributed displacement estimation can be obtained as

$$H_{LO}(U) = \max_{\boldsymbol{\phi}} 4(\Delta^2\hat{h})_\psi = 2M + 4\left(\sum_{a=1}^{M}|U_{a1}|\right)^2 f_+(\bar{n}M), \tag{1}$$

where we have defined $f_+(\bar{n}M) \equiv \bar{n}M + \sqrt{\bar{n}^2M^2 + \bar{n}M}$. Here, the optimality condition of local phases for a given U is written as $e^{-i\phi_a^*} = U_{a1}/|U_{a1}|$, which depends only on the first column of U. It is worth emphasizing that the optimality condition is immediately obtained from U. The derivation of the QFI and the optimality condition is provided in Ref. [33].

Since the factor $f_+(\bar{n}M)$ in Eq. (1) is order of M for fixed $\bar{n}$, whether the Heisenberg scaling can be achieved, i.e., $H_{LO}(U) \propto M^2$, is determined by the property of BSN U. Particularly, for a trivial BSN, $U = \mathbb{1}_M$, we do not attain any entanglement and the QFI is linear in M. Thus, it fails to achieve the Heisenberg scaling without entanglement. Meanwhile, one may easily show that the QFI is maximized by a balanced BSN, i.e., $|U_{a1}| = 1/\sqrt{M}$ for all a's, which leads to the QFI as

$$H_{\max} \equiv \max_U H_{LO}(U) = 2M + 4Mf_+(\bar{n}M). \tag{2}$$

It clearly shows the quantum enhancement from an optimal CV quantum network and the entanglement generated from it. One can also prove that $H_{\max}$ is the maximum QFI not only in our scheme but also over any quantum states [33].

Since our goal is to show a quantum metrological enhancement of typical CV quantum networks, we now compute the average QFI over random CV quantum networks using Eq. (1), i.e., random unitary matrices drawn from the Haar measure μ on $U(M)$ group, and prove the following lemma (See Ref. [33] for a proof):

Lemma 1. *The average QFI over random U for distributed quantum displacement sensing using a single-mode squeezed state is*

$$\mathop{\mathbb{E}}_{U\sim\mu}[H_{LO}(U)] = 2M + 4\left[\frac{\pi}{4}(M-1) + 1\right] f_+(\bar{n}M). \tag{3}$$

First of all, Lemma 1 shows that the average QFI over random CV quantum networks follows the Heisenberg scaling. Also, note that for large M, the ratio of the average QFI to the maximum QFI $H_{\max}$ approaches to $\pi/4$. Therefore, one may expect that typical CV quantum networks render a quantum metrological advantage.

We prove that indeed, most CV quantum networks offer a quantum enhancement for estimating displacement.

Theorem 1. *For an M-mode CV quantum network, characterized by an $M \times M$ unitary matrix drawn from the Haar measure μ on the $M \times M$ unitary matrix group, the Heisenberg scaling of QFI can be achieved with a fraction of BSNs such that*

$$\Pr_{U\sim\mu}\left[H_{LO}(U) = \Theta(M^2)\right] \geq 1 - \exp\left[-\Theta(M)\right]. \quad (4)$$

Proof sketch. (See Ref. [33] for a formal proof.) From the concentration of measure inequality [27, 34], we have

$$\Pr_{U\sim\mu}\left[\left|f(U) - \mathop{\mathbb{E}}_{U\sim\mu}\left[f(U)\right]\right| \geq \epsilon\right] \leq 2\exp\left(-\frac{M\epsilon^2}{4L^2}\right), \quad (5)$$

where $f : U \mapsto \mathbb{R}$ is a real function and L is its Lipschitz constant. If we let $f(U) \equiv H_{LO}(U)$, the average $H_{LO}(U)$ is given by Lemma 1. We then show that L is upper-bounded by $8Mf_+(\bar{n}M)$. Finally, setting $\epsilon = \Theta(M^2)$ leads to Eq. (4) [35].

Since a product state renders QFI at most linear in M, Theorem 1 indicates that typical CV quantum networks with a squeezed vacuum state are beneficial for quantum metrology. In other words, for a randomly chosen CV quantum network except for an exponentially small fraction, the proposed scheme achieves the Heisenberg scaling of QFI for the distributed displacement estimation. It also implies that most CV quantum networks enable one to construct an entanglement using a single-mode squeezed vacuum state since the Heisenberg scaling can only be achieved using entanglement in our scheme. Moreover, we prove that the QFIs can always be attained by performing homodyne detection along x-axis without an additional network [33]. Since the input state is product and additional operations, such as local optimization and measurement, are local, the entanglement is constituted only from CV quantum networks.

While our scheme with a squeezed vacuum state at a fixed mode is sufficient for our goal, the input state can be further optimized in principle. For example, one may use an optimal input mode for a squeezed vacuum state for a given BSN or a product of squeezed vacuum states as an input.

Furthermore, since we can achieve the Heisenberg scaling using the optimal local phase shifts $\boldsymbol{\phi}^*$, Theorem 1 can be interpreted from a different aspect. From the perspective of active transformation, the local phase shift for ith mode $\hat{R}_i(\phi_i^*)$ transforms the quadrature operator $\hat{p}_i$ into $\hat{R}_i^\dagger(\phi_i^*)\hat{p}_i\hat{R}_i(\phi_i^*) = \hat{x}_i \sin\phi_i^* + \hat{p}_i\cos\phi_i^*$. Thus, if we absorb the local phase shifters into displacement operators by the above transformation, Theorem 1 implies that the QFI of the state right after a BSN mostly follows the Heisenberg scaling with respect to a parameter x generated by operators $\sum_{i=1}^M(\hat{x}_i\sin\phi_i^* + \hat{p}_i\cos\phi_i^*)$. Consequently, we obtain the following corollary:

Corollary 1. *When a single-mode squeezed vacuum state undergoes a random BSN, most of the output states are beneficial for distributed quantum displacement sensing with a specific direction of displacement.*

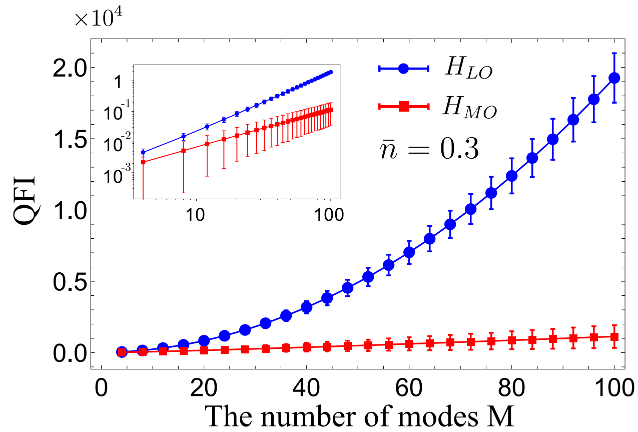

FIG. 2. QFI averaged over 20000 different Haar-random BSNs with a squeezed state input (inset: log-log scale). The error bars represent three times of the standard deviation of QFIs.

Therefore, most CV quantum networks render an entanglement that enables one to attain the Heisenberg scaling for particular metrological tasks. Nevertheless, if we fix the direction of displacement of interest, we find that local optimization is essential for our protocol. In fact, without local operation, i.e., $\phi_a = 0$ for all a's, we cannot attain the Heisenberg scaling even if the input state is chosen to be the optimal state that maximizes QFI for a given U.

Theorem 2. *Without local operation, the fraction of random BSNs for which QFI attains Heisenberg scaling is almost zero even if we choose the optimal input state for a given U,*

$$\Pr_{U\sim\mu}\left[\mathcal{H}(U) = \Theta(M^2)\right] \leq \exp\left[-\Theta(M)\right]. \quad (6)$$

where $\mathcal{H}(U)$ is the QFI of the optimal state.

Proof sketch. First, we find an upper bound of the QFI of the optimal state for a given U without local optimization. We then show that the upper bound scales as M except for an exponential small fraction of U's in M, which implies that the QFI scales at most linearly in M except for an exponentially small fraction of U's. The detailed proof is provided in Ref. [33].

We now numerically demonstrate our results. We sample random unitary matrices by following the standard method that first generates Gaussian random matrix and orthogonalizes its column vectors [33, 34]. Figure 2 exhibits average QFIs over different Haar-random BSNs with a squeezed vacuum state input. As implied by Theorems 1 and 2, it clearly shows that when we optimize the local phase shifts, we obtain QFIs following the Heisenberg scaling as the number of modes M grows, while if we do not control the local phases, the Heisenberg scaling cannot be achieved [36]. Here, the QFI for a single-mode

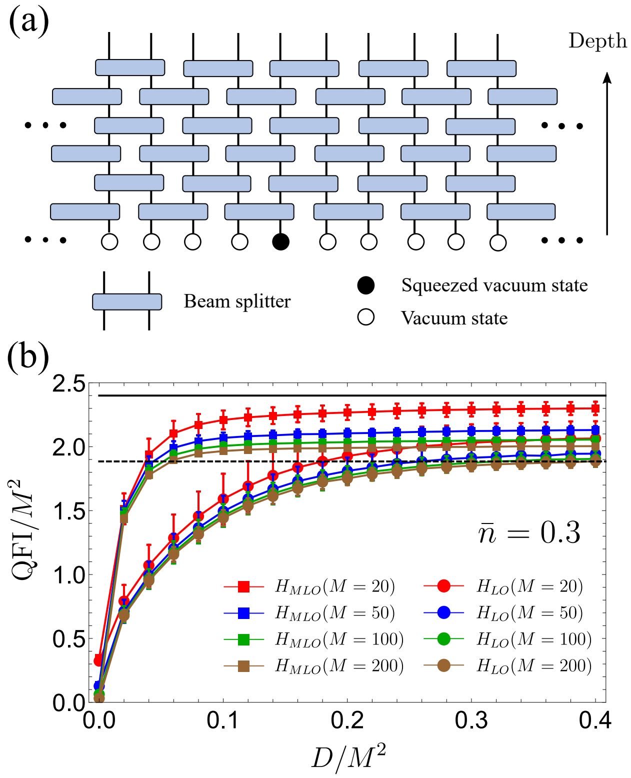

FIG. 3. (a) CV quantum network composed of depth D local beam splitters with a squeezed vacuum input. (b) Average QFIs over 1000 different local Haar-random beam splitters with (H_{MLO}) and without (H_{LO}) optimizing the input mode. The error bars represent the standard deviation of QFIs over samples. Black dashed (solid) line represents the asymptotic average (maximum) QFI divided by M^2, obtained by a random (balanced) BSN, which is equal to $2\pi\bar{n}$ ($8\bar{n}$).

squeezed state input injected into an optimal input mode without local optimization is given by [33]

$$H_{MO} \equiv \max_{1\leq b\leq M}\left[2M + 4\left|\sum_{a=1}^{M} U_{ab}\right|^2 f_+(\bar{n}M)\right]. \qquad (7)$$

Although we have used a single-mode squeezed state instead of the optimal input state, the overall scalings of H_{MO} and $\mathcal{H}$ are equal when M is large [33]. Furthermore, the standard deviation of QFIs are small for both cases, indicating that most BSNs with local-phase optimization allow the Heisenberg scaling using our scheme, while those without local-phase optimization does not.

Effect of loss.— We analyze the effect of photon loss on the Heisenberg scaling with typical BSNs and find a tolerable loss rate that maintains the Heisenberg scaling. Photon loss can be modeled by a beam splitter with its transmittivity η, which transforms an annihilation operator as $\hat{a}_j \to \sqrt{\eta}\hat{a}_j + \sqrt{1-\eta}\hat{e}_j$, where $\hat{e}_j$ is an annihilation operator for environment mode for all j's [37]; thus, we assume that a photon-loss rate is constant over all modes. Since a photon-loss channel of the uniform loss rate commutes with beam splitters, our analysis includes photon loss occurring either before or after a BSN. One can easily find that in the presence of photon loss, the corresponding QFI and its expectation value over random U are degraded and their analytical expression can be written by merely replacing $f_+(\bar{n}M)$ in Eqs. (1) and (3) by $\eta f_+(\bar{n}M)/\left[2(1-\eta)f_+(\bar{n}M)+1\right]$, which are shown in Ref. [33]. Using these results we can show that Theorem 1 is still valid as long as a loss rate $1-\eta$ is smaller than a threshold $\beta = \Theta(1/\bar{n}M)$ [33], i.e., as M increases, a threshold of the loss rate has to decrease at least as $1/\bar{n}M$ to maintain the Heisenberg scaling. We note that CV error correction scheme [38, 39] and quantum repeater [40] can be considered to alleviate the effect of loss.

Local beam splitter network.— While a global random BSN is suitable to model a sufficiently complex CV network, it is also crucial to investigate how complicated the network has to be to attain a metrological enhancement from a practical perspective. To do that, we study a CV quantum network composed of local Haar-random beam splitters instead of a global random BSN (See Fig. 3(a)) [41–44]. We numerically show that the Heisenberg scaling can also be achieved by using CV quantum networks consisting of local beam splitters. Figure 3(b) shows the averaged local-phase-optimized QFIs with and without optimizing the input mode for a squeezed vacuum state. The QFI of the latter is given by [33]

$$H_{MLO} \equiv \max_{1\leq b\leq M}\left[2M + 4\left(\sum_{a=1}^{M} |U_{ab}|\right)^2 f_+(\bar{n}M)\right], \quad (8)$$

which is obviously equal or greater than $H_{LO}(U)$. Most importantly, the QFI divided by M^2 is almost constant for a given D/M^2 and different M's. It implies that the Heisenberg-scaling can be achieved on average with a depth proportional to M^2, independent of input-mode optimization, which is consistent with the result in Ref. [42]. Nevertheless, by optimizing the input mode, the Heisenberg scaling is achieved much faster. Moreover, the figure shows that the standard deviation of QFIs is very small, indicating that most local BSNs are beneficial for distributed displacement estimation, and that the standard deviation decreases as M grows. Since they achieve the Heisenberg scaling on average, the quantum networks of local beam splitters constitute sufficient entanglement on average as expected in Ref. [42]; namely, large entanglement can be obtained for a depth $D \propto M^2$.

Discussion.— From a theoretical perspective, our results imply that most CV quantum networks have the same scaling of estimation error for distributed displacement sensing as the optimal one, i.e., from a balanced BSN. Thus, for quantum enhancement in practice, one may not necessarily implement a very special structure such as a balanced BSN because most CV networks provide the same quantum enhancement when it comes to scaling. Such an experimental generalization would be particularly useful when one needs a large scale of networks. For example, if we already have a CV quantum network for various purposes, which is not necessarily balanced but complex enough, we can immediately exploit the network for quantum-enhanced displacement sensing. Furthermore, although we have focused on distributed displacement sensing, future research could continue to investigate if similar results hold for different metrological tasks, such as multiparameter displacement estimation [18, 20] or phase estimation [17, 19, 21]. It is also worth mentioning that since our scheme only employs a squeezed state, beam splitters, and homodyne detection, the current technology can already benefit from our results.

We finally emphasize the major differences of our study from Ref. [27]. While both consider random bosonic states from the quantum metrological perspective, the two schemes benefit from different kinds of entanglement. Ref. [27] studies phase sensing that exploits interparticle entanglement, while we study distributed displacement sensing which benefits from intermode entanglement. The difference is apparent from the following example. The random state $\hat{R}(\boldsymbol{\phi}^*)\hat{U}\,|N,0,\cdots,0\rangle$ has mode entanglement and typically brings quantum enhancement for the distributed displacement sensing task whereas it has no particle entanglement regardless of BSN U, so it does not lead to an enhancement for phase estimation [33]. Besides, our study considers a task where a photon number fluctuates, which is unclear to interpret by particle formalism, typically assuming a definite photon number. It would be an interesting future work to find a class of probes that is useful for both sensing schemes and to identify the relation between interparticle entanglement and intermode entanglement more rigorously.

We acknowledge useful discussions with Quntao Zhuang. H.K. and H.J. are supported by the National Research Foundation of Korea (NRF-2019M3E4A1080074, NRF-2020R1A2C1008609, NRF-2020K2A9A1A06102946) via the Institute of Applied Physics at Seoul National University and by the Ministry of Science and ICT, Korea, under the ITRC (Information Technology Research Center) support program (IITP-2021-2020-0-01606) supervised by the IITP (Institute of Information & Communications Technology Planning & Evaluation). Y.L. acknowledges National Research Foundation of Korea a grant funded by the Ministry of Science and ICT (NRF-2020M3E4A1077861) and KIAS Individual Grant (CG073301) at Korea Institute for Advanced Study. L.J. and C.O. acknowledge support from the ARO (W911NF-18-1-0020, W911NF-18-1-0212), ARO MURI (W911NF-16-1-0349, W911NF-21-1-0325), AFOSR MURI (FA9550-19-1-0399), DoE Q-NEXT, NSF (EFMA-1640959, OMA-1936118, EEC-1941583, OMA-2137642), NTT Research, and the Packard Foundation (2013-39273). We also acknowledge the University of Chicagos Research Computing Center for their support of this work.

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