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Quantum Fluctuation Theorem under Quantum Jumps with Continuous Measurement and Feedback

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While the fluctuation theorem in classical systems has been thoroughly generalized under various feedback control setups, an intriguing situation in quantum systems, namely under continuous feedback, remains to be investigated. In this work, we derive the generalized fluctuation theorem under quantum jumps with continuous measurement and feedback. The essence for the derivation is to newly introduce the operationally meaningful information, which we call *quantum-classical-transfer (QC-transfer) entropy*. QC-transfer entropy can be naturally interpreted as the quantum counterpart of transfer entropy that is commonly used in classical time series analysis. We also verify our theoretical results by numerical simulation and propose an experiment-numerics hybrid verification method. Our work reveals a fundamental connection between quantum thermodynamics and quantum information, which can be experimentally tested with artificial quantum systems such as circuit quantum electrodynamics.

Introduction.—In the last few decades, the framework of thermodynamics has been applied to small systems in which thermodynamic quantities behave stochastically due to the presence of thermal or quantum fluctuations [1–6]. A key relation that quantifies a universal behavior of such systems is the fluctuation theorem (FT)

$$\langle e^{-\sigma} \rangle = 1, \tag{1}$$

where σ is the stochastic entropy production and $\langle \cdot \rangle$ denotes the ensemble average. The FT characterizes the behavior of the entropy production even in the nonlinear nonequilibrium region, and also implies the second law of thermodynamics (SL) at the average level: $\langle \sigma \rangle \geq 0$.

In light of thermodynamics of information, originated in the gedanken experiment of Maxwell's demon [7], it has been revealed that measurement and feedback leads to generalizations of the laws of thermodynamics [8–33]. For instance, the FT has been generalized by incorporating information gain i obtained from the measurement as

$$\langle e^{-\sigma - i} \rangle = 1, \tag{2}$$

which implies the generalized SL: $\langle \sigma \rangle \geq -\langle i \rangle$. The generalized FT in the form of Eq. (2) has been derived for classical systems under single measurement and feedback [10, 11] as well as continuous measurement and feedback [12–14], and also derived for quantum systems under single measurement and feedback [21, 22] (see Fig. 1 (a)). There are also a few works about the role of continuous quantum measurement on the SL or the FT [34–39]. However, the role of continuous measurement and *feedback* in the quantum regime has not yet been elucidated, despite its significance as described below.

Continuous measurement and feedback has been of keen interest due to its capability of creating and stabilizing desired quantum states via feedback loop. It is relevant to various quantum systems [40] including its applications to thermodynamics [41–43], and is developing due to the recent advancement of experimental techniques [44–50]. It is also noteworthy that arbitrary Markovian open quantum systems described by the Lindblad master equation can be interpreted to be under continuous (non-selective) measurement [40]. These facts tells us the framework of continuous quantum measurement and feedback gives a unified description of systems under artificial control or interaction with external systems.

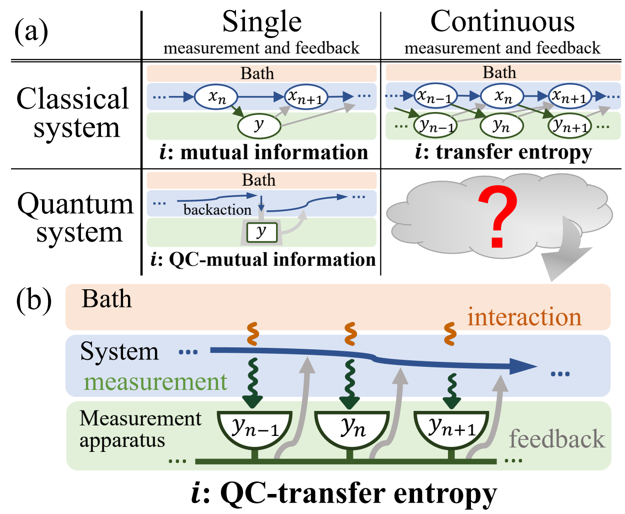

FIG. 1. (a) Summary of previous research of information thermodynamics. In all of the three cases shown here, the FT has been generalized in the form of Eq. (2) with appropriate choices of the information gain i, whereas quantum systems under continuous feedback control have not been studied. (b) Setup of the present work. We find that the QC-transfer entropy is the relevant information gain i in this setup.

In this Letter, we generalize the SL and the FT for systems under quantum jumps with continuous measurement and feedback, by introducing the *quantum-classical-transfer* (*QC-transfer*) *entropy* as a relevant information gain. The QC-transfer entropy is defined as the accumulation over time of the conditional QC-mutual information [51–54] under the past measurement outcomes. Therefore, it can naturally be interpreted as the quantum counterpart of the transfer entropy [55] that is used to derive the generalized FT for classical systems [12–14]. We also verify the generalized FT by numerical simulation in a two-level system, and propose an experiment-numerics hybrid verification method of the generalized FT.

Dynamics of the system.— Let us consider a quantum system interacting with the heat bath at inverse temperature β under continuous measurement and feedback (Fig. 1(b)). To simplify the argument, we suppose that the Born-Markov and rotating-wave approximations can be applied to the system-bath interaction [56, 57]. We discretize time as $t_n \equiv n\Delta t$, consider the time evolution from $t = 0$ to $t = \tau \equiv t_N$, and later take the continuous time limit $\Delta t \to 0, N \to \infty$ while keeping τ constant. The time evolution in $[t_n, t_{n+1})$ is described by the stochastic master equation:

$$\begin{aligned}\rho_{t_{n+1}}^{Y_{n+1}} &= \rho_{t_n}^{Y_n} + \sum_y \Delta N_y \mathcal{G}[M_y]\rho_{t_n}^{Y_n} \\ &+\Delta t\Big\{-i[H_{t_n}+h_{t_n},\rho_{t_n}^{Y_n}] + \sum_d \mathcal{D}[L_d]\rho_{t_n}^{Y_n} \\ &+\sum_y -\frac{1}{2}\{M_y^\dagger M_y, \rho_{t_n}^{Y_n}\} + \mathrm{Tr}[M_y\rho_{t_n}^{Y_n}M_y^\dagger]\rho_{t_n}^{Y_n}\Big\},\end{aligned} \tag{3}$$

where $\mathcal{G}[m]\rho \equiv (m\rho m^\dagger/\mathrm{Tr}[m\rho m^\dagger]) - \rho$ and $\mathcal{D}[c]\rho \equiv c\rho c^\dagger - 1/2\{c^\dagger c, \rho\}$. We define y_n as the newly obtained measurement result at t_n and $Y_n \equiv (y_1, y_2 \dots, y_n)$ as the outcomes until t_n. Here, $\rho_{t_n}^{Y_n}$ represents the conditional density operator at t_n when the measurement results are Y_n. We define H_t as the intrinsic system Hamiltonian and h_t as the external driving Hamiltonian. The interaction between the system and the heat bath can be described by the Lindblad operators $\{L_d\}$, where L_d represents the dissipation of energy Δ_d to the heat bath (i.e., $[L_d, H_{t_n}] = \Delta_d L_d$), and satisfies the detailed balance condition with respect to H_t (i.e., $L_{d'} = L_d^\dagger e^{-\frac{\beta}{2}\Delta_d}$ with d' being uniquely determined from $\Delta_d = -\Delta_{d'}$). The effect of continuous measurement is represented by the Lindblad operators $\{M_y\}$, and the feedback is performed by changing the Hamiltonian. In the following, we give a more detailed explanation on continuous measurement and feedback.

Continuous measurement reads out system's information via interaction with the measurement apparatus (e.g., the monitoring of an emitted photon from the system). The measurement outcome obtained at t_{n+1} is denoted as y_{n+1} with the corresponding quantum jump represented by $M_{y_{n+1}}$ (e.g., the detection of a photon). If no measurement jump is detected at t_{n+1}, y_{n+1} is defined as 0. The system's conditional dynamics is described by the set of Kraus operators which is composed of both the measurement-jump detection operators $\{M_y\sqrt{\Delta t}\}$ and no detection operator $1 - \frac{\Delta t}{2}\sum_y M_y^\dagger M_y$. In Eq. (3), the first and third lines correspond to the detection and no detection event, respectively, and the Poisson increment ΔN_y is defined as $\Delta N_y = 1$ if the jump M_y occurs, and $\Delta N_y = 0$ otherwise [40, 58]. We here emphasize that, by taking the ensemble average over the outcomes Y_{n+1}, Eq. (3) reduces to the ordinary master equation that describes dynamics interacting with an external system without post-selection of the measurement results. Such a decomposition that allows us to recover the master equation is called *unraveling*.

Continuous feedback is provided by varying H_t and/or h_t according to the measurement results. Because of the causality, the Hamiltonians in $[t_n, t_{n+1})$ are completely determined by measurement results before t_n (i.e., Y_n) while it does not depend on those after t_n. In our setup, the following types of the Hamiltonian variations are supposed: adiabatic change of the system Hamiltonian H_t [57, 59], perturbation of the external field (i.e., $h_t \ll H_t$) [60–62], and sequential pulses (i.e., $h_t = \sum_i v_i\delta(t - s_i)$). Hamiltonian variations other than those may not be given in the form of Eq. (3) [63, 64], and hence are excluded in the following argument. Note that the dependence on Y_n of some operators and variables (such as $H_t, h_t, L_d, \Delta_d, \Delta N_y$) is abbreviated for simplicity.

Generalized second law.— In this setup, the ensemble average of thermodynamic quantities such as the heat $\langle Q\rangle$ dissipated to the heat bath and the entropy change $\langle \Delta S\rangle$ between the initial and final states can be defined as follows [22, 65, 66]:

$$\begin{aligned}\langle Q\rangle &\equiv -\sum_{n=0}^{N-1}\sum_{Y_n} P[Y_n]\sum_d \mathrm{Tr}[H_{t_n}\mathcal{D}[L_d]\rho_{t_n}^{Y_n}]\Delta t, \\ \langle \Delta S\rangle &\equiv S(\rho_\tau) - S(\rho_0),\end{aligned} \tag{4}$$

where ρ_0 and $\rho_\tau \equiv \sum_{Y_N} P[Y_N]\rho_\tau^{Y_N}$ are the initial and final density operators, and $S(\rho) \equiv -\mathrm{Tr}[\rho\ln\rho]$ represents the von Neumann entropy. The average entropy production is defined as $\langle\sigma\rangle \equiv \langle\Delta S\rangle + \beta\langle Q\rangle$.

We now introduce the QC-transfer entropy as

$$\langle i_{\mathrm{QC}}\rangle = \sum_{n=0}^{N-1}\sum_{Y_n} P[Y_n]\mathcal{I}_{\mathrm{QC}}(\rho_{t_n}^{Y_n} : y_{n+1}), \tag{5}$$

where $\mathcal{I}_{\mathrm{QC}}$ represents the QC-mutual information. Here, $\mathcal{I}_{\mathrm{QC}}$ is defined as $\mathcal{I}_{\mathrm{QC}}(\rho : y) \equiv S(\rho) - \sum_y P[y]S(\rho^y)$, where $P[y]$ is the probability of the outcome y, and ρ^y denotes the conditional density operator after the measurement of y. QC-mutual information quantifies the information obtained by quantum measurement [51, 52] so that it gives the upper bound of the accessible classical information [53, 54], and also has an operational interpretation through an informational task called measurement compression [67–69]. On the basis of the foregoing definitions, the SL is generalized as

$$\langle\sigma\rangle \geq -\langle i_{\mathrm{QC}}\rangle. \tag{6}$$

This inequality gives the lower bound of $\langle\sigma\rangle$ under continuous measurement and feedback and reveals the relationship between the entropy production and quantum information at the level of ensemble average.

We here discuss the relationship between the QC-transfer entropy and the (classical) transfer entropy [55] (Fig. 1(a)). The transfer entropy is defined as

$$\langle i_{\mathrm{TE}}\rangle \equiv \sum_{n=0}^{N-1} I(x_n : y_{n+1}|Y_n), \tag{7}$$

where x_n denotes a state of a classical system at t_n. We define I as the conditional mutual information $I(x_n : y_{n+1}|Y_n) \equiv \sum_{Y_n} P[Y_n](\mathcal{H}_{Y_n}(x_n) - \sum_{y_{n+1}} P[y_{n+1}|Y_n]\mathcal{H}_{Y_n,y_{n+1}}(x_n))$, where $\mathcal{H}_{Y_n}(x_n)$ denotes the Shannon entropy of x_n when the measurement results are Y_n. From Eqs. (5) and (7), we can see that in classical systems, information transfer in $[t_n, t_{n+1})$ is described by the conditional mutual information $I(x_n : y_{n+1}|Y_n)$, whereas in quantum systems it is represented by the QC-mutual information of the conditional density operator $\rho_{t_n}^{Y_n}$. Therefore, $\langle i_{\mathrm{QC}}\rangle$ is the quantum counterpart of $\langle i_{\mathrm{TE}}\rangle$, in that they both represent the total information gain obtained by accumulating conditional information transfer over time.

We note that the QC-transfer entropy defined as Eq. (5) converges to a finite value even in the continuous time limit. This is because the information obtained in $[t_n, t_{n+1})$, i.e., $\sum_{Y_n} P[Y_n]\mathcal{I}_{\mathrm{QC}}(\rho_{t_n}^{Y_n} : y_{n+1})$, is $\mathcal{O}(\Delta t)$ as a consequence of our setup that the quantum jump is detected. Such measurement has been studied as the typical continuous quantum measurement [40, 56] and also is experimentally feasible [44–50].

Generalized fluctuation theorem.— We next introduce the generalized FT under continuous measurement and feedback, which is the main result of this Letter. Since the generalized FT is the equality concerning the stochastic entropy production σ and the stochastic information gain i_{QC}, we need to introduce proper definitions of these quantities. On the basis of a special class of stochastic decomposition of Eq. (3), which we name *fine unraveling*, both of these quantities can be defined for individual unraveled trajectories, which we call fine trajectories. In the following, we elaborate on these concepts.

As the preliminary step toward defining the fine unraveling, we first introduce another unraveling, which we call *standard unraveling*. In this unraveling, we monitor the heat-bath dissipation in addition to the original continuous measurement of y_{n+1} in Eq. (3). We here define ΔN_d as the Poisson increment of L_d and d_{n+1} as the outcome of the heat-bath monitoring in $[t_n, t_{n+1})$. We further perform two-time projective measurement at $t = 0$ and τ, just as in the standard scheme in stochastic thermodynamics [4, 54]. Here, the two-time measurement is performed in the diagonalized bases of the density operators $\rho_0 \equiv \sum_{a_0} p_0(a_0)\,|a_0\rangle\langle a_0|$ and $\rho_\tau \equiv \sum_{a_\tau} p_\tau(a_\tau)\,|a_\tau\rangle\langle a_\tau|$, and their outcomes are denoted as a_0 and a_τ, respectively. Thus, the dynamics of Eq. (3) is decomposed according to the measurement outcomes $\psi_\tau \equiv (a_0, a_\tau, \{y_n\}_{n=1}^N, \{d_n\}_{n=1}^N)$. We refer to the unraveled trajectory designated by ψ_τ as the standard trajectory. We remark that Eq. (3) can be recovered from the standard unraveling by taking the ensemble average over the results of the two-time measurement a_0, a_τ and the heat-bath monitoring $\{d_n\}$.

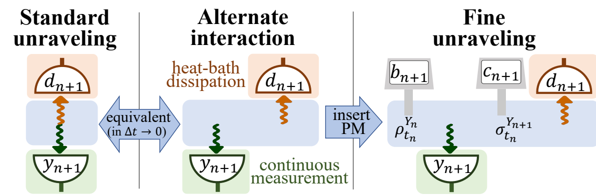

FIG. 2. The illustration of the standard unraveling, the alternate interaction situation, and the fine unraveling in $[t_n, t_{n+1})$.

The fine unraveling is introduced by the following two-step transformation from the standard unraveling (see Fig. 2). We first consider the situation that a measurement and an interaction with the heat bath occur alternately every Δt, and then insert the projective measurements in the diagonalized bases of $\rho_{t_n}^{Y_n}$ and $\sigma_{t_n}^{Y_{n+1}}$ right before and after the measurement of y_{n+1}, respectively. The diagonalizations are defined as $\rho_{t_n}^{Y_n} \equiv \sum_{b_{n+1}} p^{Y_n}(b_{n+1})\,|b_{n+1}\rangle\langle b_{n+1}|$ and $\sigma_{t_n}^{Y_{n+1}} \equiv \sum_{c_{n+1}} p^{Y_{n+1}}(c_{n+1})\,|c_{n+1}\rangle\langle c_{n+1}|$, and the outcomes of the inserted measurements before and after the monitoring of y_{n+1} are b_{n+1} and c_{n+1}, respectively. Here, $\sigma_{t_n}^{Y_{n+1}}$ represents the conditional density operator when the measurement results are Y_{n+1} in the alternate interaction situation. Thus, the fine unraveling decomposes Eq. (3) according to ψ_τ and π_τ, where $\pi_\tau \equiv (\{b_n\}_{n=1}^N, \{c_n\}_{n=1}^N)$ denotes the outcomes of the inserted projective measurements all together. By taking the ensemble average over π_τ, along with a_0, a_τ and $\{d_n\}$, the fine unraveling reproduces the original dynamics of Eq. (3). It should be emphasized that these projective measurements do not destroy the measured states at the ensemble average level, because of their choices of the bases.

For each fine trajectory, we can define both the stochastic entropy production and the QC-transfer entropy, and derive the generalized FT. The stochastic entropy production is defined as $\sigma[\psi_\tau, \pi_\tau] \equiv \Delta S[\psi_\tau, \pi_\tau] +$

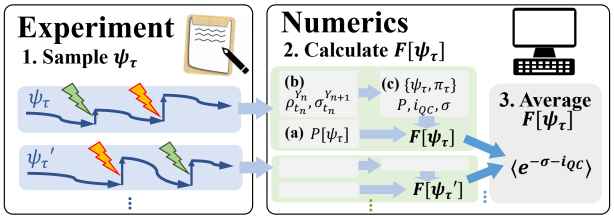

FIG. 3. The schematics for the experiment-numerics hybrid verification method. The verification protocol is composed of the experimental part, in which the standard trajectories ψ_τ are sampled, and the numerical calculation part, in which $F[\psi_\tau]$ are calculated, and then their average value is taken. The average value coincides with the left-hand side of Eq. (10).

$\beta Q[\psi_\tau, \pi_\tau]$, where the stochastic heat and stochastic entropy change are defined as follows [22, 70–74]:

$$
\begin{aligned}
Q[\psi_\tau, \pi_\tau] &\equiv \sum_{n=0}^{N-1} \sum_d \Delta N_d \Delta_d, \\
\Delta S[\psi_\tau, \pi_\tau] &\equiv -\ln p_\tau(a_\tau) + \ln p_0(a_0).
\end{aligned} \tag{8}
$$

The stochastic QC-transfer entropy is then defined as

$$
i_{\mathrm{QC}}[\psi_\tau, \pi_\tau] \equiv \sum_{n=0}^{N-1} - \ln p^{Y_n}(b_{n+1}) + \ln p^{Y_{n+1}}(c_{n+1}). \tag{9}
$$

We can confirm that the ensemble averages of the stochastic quantities (8) and (9) coincide with Eqs. (4) and (5). Based on the foregoing definitions, the generalized FT is expressed as

$$
\langle e^{-\sigma - i_{\mathrm{QC}}} \rangle = 1. \tag{10}
$$

This equality implies the generalized SL (6) and reveals the relationship between the entropy production and the QC-transfer entropy at the trajectory level. See Supplemental Material for the full proof of Eq. (10) [75].

Protocol of the hybrid method.— Although the direct realization of the fine unraveling in real experiments is difficult, the generalized FT (Eq. (10)) can be verified by an experiment-numerics hybrid verification method, in which the standard trajectories are sampled in an experiment and the auxiliary numerical calculation is performed in a classical computer (Fig. 3). The detection of the standard trajectories is feasible in real experiments such as circuit QED [41, 42, 49, 50] and cavity QED [44, 45]. Then, by the auxiliary numerical calculation, we can evaluate the left-hand side of Eq. (10). The concrete protocol of the hybrid method is as follows:

1. By a real quantum experiment, sample the standard trajectories.
2. By classical numerical simulation, calculate $F[\psi_\tau] \equiv \sum_{\pi_\tau} \frac{P[\psi_\tau, \pi_\tau]}{P[\psi_\tau]} e^{-\sigma[\psi_\tau, \pi_\tau] - i_{\mathrm{QC}}[\psi_\tau, \pi_\tau]}$ for each experimentally sampled trajectory ψ_τ:
   (a) Calculate the realization probability $P[\psi_\tau]$ by solving the stochastic Schrödinger equation of the standard unraveling.
   (b) Calculate the dynamics of conditional density operators $\rho_{t_n}^{Y_n}$ and $\sigma_{t_n}^{Y_{n+1}}$ by solving Eq. (3).
   (c) Calculate $P[\psi_\tau, \pi_\tau]$, $i_{\mathrm{QC}}[\psi_\tau, \pi_\tau]$ and $\sigma[\psi_\tau, \pi_\tau]$ for realizable fine trajectories $\{\psi_\tau, \pi_\tau\}_{\pi_\tau}$ with same ψ_τ by using the solution of (b).
3. Average $F[\psi_\tau]$ over all sampled trajectories, which gives the left-hand side of Eq. (10).

We make a remark on the first step above. In order to perform the two-time measurement of the standard unraveling, we need to know the eigenbases of the initial and final states ρ_0, ρ_τ. The simplest case is that the system is initially prepared in a Gibbs state and evolves to another Gibbs state. In this case, the projective measurements of ρ_0 and ρ_τ are essentially equivalent to the measurements of the initial and final Hamiltonians, because they share the same eigenbases. In general cases where we do not have any prior knowledge about the initial state, we need to perform quantum state tomography [76] before we start the feedback protocol. Once the initial state is known, we can obtain the final state by taking the ensemble average over the sampled trajectories in the hybrid method. We note that the general two-time measurement has been investigated and reviewed in Refs. [4, 5, 54, 77] in detail. See also Refs. [32, 33] for relevant experimental studies.

We also make some remarks on the second step. While $P[\psi_\tau]$ can be calculated solely from the standard unraveled dynamics, we have to prepare the inserted projective measurements in order to calculate the quantities for fine trajectories $P[\psi_\tau, \pi_\tau]$, $i_{\mathrm{QC}}[\psi_\tau, \pi_\tau]$ and $\sigma[\psi_\tau, \pi_\tau]$. Since the number of realizable fine trajectories with $P[\psi_\tau, \pi_\tau] \neq 0$ calculated in (c) is finite even in the limit of $\Delta t \to 0$ [75], the exact calculation of $F[\psi_\tau]$ can be performed with reasonable numerical cost.

Numerical demonstration of the generalized FT.— To further support our findings, we have numerically calculated the fine unraveled dynamics of the two-level system to verify the generalized FT. We employ the setting that the population of excited state $|1\rangle$ is reduced by continuous measurement and feedback. The system Hamiltonian is fixed as $H_t = \omega\sigma_z$, the coherent driving is applied as $h_t = \epsilon\sigma_x \cos\omega_0 t$, the heat-bath dissipation is represented by $L_\pm = \sqrt{\gamma_\pm}\sigma_\pm$, and the continuous measurement operator is defined as $M_1 = \sqrt{\gamma_m}(|1\rangle\langle 1| + \delta|0\rangle\langle 0| + \delta\sigma_x)$, where σ_i denotes the Pauli matrix. We note that if δ is negligible ($\delta \ll 1$), we can almost certainly decide that the system is in $|1\rangle$ after the detection of M_1. Thus we apply unitary gate σ_x right after the detection, in order to reduce the excited state population. If the detection occurs at t_n, the feedback protocol is realized by applying the pulse in t_{n+1}, which changes the external driv-

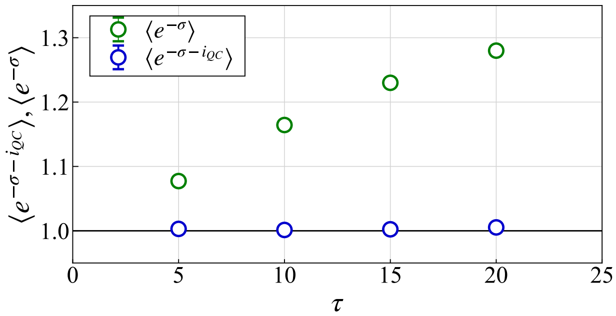

FIG. 4. Numerical verification of the generalized FT. The average values of $e^{-\sigma}$ and $e^{-\sigma-i_{\rm QC}}$ for the sampled fine trajectories are plotted. The system parameters are taken as $\omega=0.3, \epsilon=0.04, \omega_0=0.1\pi, \beta=1, \gamma_\pm=0.015\omega\{\coth\left(\frac{\beta\omega}{2}\right)\mp 1\}, \gamma_m=\gamma_++\gamma_-$ and $\delta=0.2$. Each data point denotes the average over 1.0×10^5 trajectories.

ing Hamiltonian as $h_t=\epsilon\sigma_x\cos\omega_0 t+v\delta(t-t_{n+1})$ with $e^{-iv}\equiv\sigma_x$. By computing σ and $i_{\rm QC}$ for individual trajectories, we have verified that the generalized FT holds. This is illustrated in Fig. 4, where we can see that $\langle e^{-\sigma}\rangle$ increases with τ, implying the violation of the conventional FT (i.e., $\langle e^{-\sigma}\rangle\neq 1$).

Summary and outlook.— In this Letter, we have derived the generalized SL (6) and FT (10) in the systems under quantum jumps with continuous measurement and feedback. The newly introduced information gain, the QC-transfer entropy, would play an important role in quantifying quantum information transfer by sequential quantum measurements. We have also verified the generalized FT by numerical simulation and proposed a feasible experiment-numerics hybrid verification method.

We show some future perspectives related to our work. First, it is important to verify the generalized FT (10) in real experimental systems by using the experiment-numerics hybrid verification method. Another interesting future task is to improve the tightness of the bound in the generalized SL (6) by introducing other information measures, as have been done in the classical regime [14, 15]. It is also intriguing to generalize the FT under the continuous measurement described by the Gaussian process [78], which is relevant to various experiments [33, 79–81]. The Gaussian process measurement is not directly covered by our quantum-jump detection setup, while we expect that it is obtained as an appropriate limit of quantum jump processes [40, 58].

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