Degeneracy-Projected Polarization Formulas for Hall-Type Conductivities

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Degeneracy-projected polarization formulas for Hall-type conductivities

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Kubo formulas for Hall, transverse thermoelectric and thermal Hall conductivities are simplified into on-shell commutators of degeneracy projected polarizations. The new expressions are computationally economical, and apply to general Hamiltonians without a gap restriction. We show that Hall currents in open boundaries are carried by gapless chiral excitations. Extrapolation of finite lattice calculations to the DC-thermodynamic limit is demonstrated for a disordered metal.

Electric, thermoelectric, and thermal Hall conductivities, a.k.a. σ_{xy}, α_{xy}, and κ_{xy}, respectively, characterize charge and thermal carriers of condensed matter phases, and identify their topology [1–3]. Anomalous Hall and thermal Hall behavior have been reported in strongly interacting systems e.g. cuprates [4, 5] and correlated insulators [6, 7]. In principle, they might be explained by computing Kubo formulas [8–10].

Unfortunately, DC Hall-type Kubo formulas are computationally costly. Their *off-shell* (energy non-conserving) matrix elements of the currents, require full diagonalization of the Hamiltonian on large systems. In addition, divergent magnetization subtractions [10] for thermal Hall coefficients require careful cancellation [11, 12].

Berry curvature (Chern) integrals [1, 13] and Streda (equilibrium) formulas [14, 15], approximate the Kubo formula by *reversing* the DC order of limits (i.e. setting frequency to zero before taking the large volume limit [16]). Thus, they only apply to bulk-gapped phases with vanishing longitudinal conductivities, e.g. Quantum Hall (QH) and topological insulators (TI).

This paper simplifies the Kubo formulas in the proper DC order of limits. The new formulas are compact and valid for general Hamiltonians, including gapless phases with disorder and interactions. Physical insight is gained by expressing σ_{xy}, α_{xy} and κ_{xy} as commutators of degeneracy-projected polarizations (DPPs). The DPPs generalize the role of Landau guiding centers to gapless phases. The formulas imply that Hall and thermal Hall currents are carried by extended chiral excitations which may be supported on the sample edges (for e.g. QH and TI) or may percolate through the bulk.

The conductivities are expressed by a smaller sum over *on shell* matrix elements, which is computationally economical. Problematic magnetization subtractions in the thermal conductivities are eliminated. At low temperatures, the relevant eigenstates are confined to low energies, which allows one to replace the microscopic model by a simpler low energy effective Hamiltonian.

Finite lattice calculations require extrapolation to the DC-thermodynamic limit. A finite size scaling scheme is demonstrated for the metallic phase of disordered electrons at weak magnetic fields. The numerical results recover Drude-Boltzmann (DB) theory, and Wiedemann-Franz law for that model.

Kubo formulas – The DC-thermodynamic limit of transport coefficients is defined by,

$$S^{xy-\mathrm{dc}}_{\mathrm{OO}'} \equiv \lim_{\substack{\varepsilon\to 0\\ V\to\infty}} S^{xy}_{\mathrm{OO}'}(\varepsilon, V), \quad \mathrm{O},\mathrm{O}' = \mathrm{C},\mathrm{T}, \tag{1}$$

where the charge (C) and thermal (T) Hall-type conductivities are $\sigma_{xy} \equiv S^{xy}_{\mathrm{CC}}$, $\alpha_{xy} \equiv S^{xy}_{\mathrm{CT}}/T$, and $\kappa_{xy} \equiv S^{xy}_{\mathrm{TT}}/T$. ε, V are finite imaginary frequency and volume respectively. The DC order of limits is taken from bottom to top [17].

Here we consider a general many-body lattice Hamiltonian H on open boundary conditions (OBC) [18], C4 symmetry in the xy plane [19], and a magnetic field B in the z direction. Its spectrum and eigenstates are $\{E_n, |n\rangle\}$. The Hall-type Kubo formulas in the Lehmann representation are,

$$S^{xy}_{\mathrm{OO}'} = \frac{\hbar}{V}\mathrm{Im}\sum_{n,m}\frac{(\rho_m-\rho_n)\langle m|j^x_{\mathrm{O}}|n\rangle\langle n|j^y_{\mathrm{O}'}|m\rangle}{(E_n-E_m)(E_n-E_m-i\varepsilon)} - \frac{\langle M_{\mathrm{OO}'}\rangle}{V}. \tag{2}$$

$\rho_n(T)$ are Boltzmann weights at temperature T. The magnetization terms $\propto \langle M_{\mathrm{OO}'}\rangle$ eliminate circulating magnetization currents from the first term [10].

The currents j^α_{O} and magnetizations $M_{\mathrm{OO}'}$ are defined as follows. The Hamiltonian is spatially decomposed on the lattice $H = \sum_i h_i$ [20]. The charge and thermal polarizations are,

$$P^\alpha_{\mathrm{C}} \equiv e\sum_i n_i x^\alpha_i, \quad P^\alpha_{\mathrm{T}} = \sum_i h_i x^\alpha_i, \quad \alpha = x, y, \tag{3}$$

where en_i is the local charge density, and $\boldsymbol{x}_i$ is the position of lattice site i. The electric and thermal currents are,

$$j^\alpha_{\mathrm{O}} = \frac{i}{\hbar}[H, P^\alpha_{\mathrm{O}}], \quad \mathrm{O} = \mathrm{C},\mathrm{T}. \tag{4}$$

In the literature one often finds first quantized expressions for the magnetizations [21]. Here we use more general definitions which apply to any form of the hamiltonian,

$$M_{\mathrm{CT}} = -\frac{i}{\hbar}[P^x_{\mathrm{C}}, P^y_{\mathrm{T}}], \quad M_{\mathrm{TT}} = -\frac{i}{\hbar}[P^x_{\mathrm{T}}, P^y_{\mathrm{T}}]. \tag{5}$$

Note that $M_{\mathrm{CC}} = 0$, since the two charge polarizations commute. For anomalous bosonic Hamiltonians $\langle M_{\mathrm{TT}}\rangle/T$ may diverge as $\lim_{T\to 0}$. Such divergence must be precisely cancelled by the current correlators, as shown

for non-interacting QH systems [11, 22]. Such cancellations could be problematic if one applies separate approximations to the two terms in Eqs. (2).

DPP formulas – Eqs. (2) are simplified as follows. The real part of the summands' numerator vanishes by C4 symmetry,

$$\mathrm{Re}\langle m|j^x_{\mathrm{O}}|n\rangle\langle n|j^y_{\mathrm{O}'}|m\rangle = 0. \tag{6}$$

The real part of the denominator is written as two terms,

$$\mathrm{Re}\frac{1}{\Delta_{nm}(\Delta_{nm}-i\varepsilon)} = \frac{1}{\Delta^2_{nm}} - \frac{\varepsilon^2}{\Delta^2_{nm}(\Delta^2_{nm}+\varepsilon^2)}, \tag{7}$$

where $\Delta_{nm} \equiv E_n - E_m$. The matrix elements of Eq. (4) in the eigenstates basis are,

$$\frac{\langle n|j^\alpha_{\mathrm{O}}|m\rangle}{\Delta_{nm}} = \frac{i}{\hbar}\langle n|P^\alpha_{\mathrm{O}}|m\rangle, \tag{8}$$

which we insert into Eq. (2) to yield,

$$S^{xy}_{\mathrm{OO}'} = \frac{1}{\hbar V}\sum_{nm}(\rho_n-\rho_m)\,\mathrm{Im}\Bigg(\langle m|P^x_{\mathrm{O}}|n\rangle\langle n|P^y_{\mathrm{O}'}|m\rangle - \frac{\varepsilon^2\langle m|P^x_{\mathrm{O}}|n\rangle\langle n|P^y_{\mathrm{O}'}|m\rangle}{(\Delta^2_{nm}+\varepsilon^2)}\Bigg) - \frac{\langle M_{\mathrm{OO}'}\rangle}{V}. \tag{9}$$

The top row, which is an off-shell sum, can be rewritten as the thermodynamic average of the polarizations' commutator $\langle[P^\alpha_{\mathrm{O}}, P^\beta_{\mathrm{O}'}]\rangle$. Therefore it vanishes for σ_{xy}, and precisely cancels with the magnetization corrections (5) for α_{xy} and κ_{xy} (good riddance!).

Surprisingly, it is the seemingly negligible ε^2-term which fully determines $S^{xy-\mathrm{dc}}$! The Kubo formulas reduce to a purely on-shell expression,

$$S^{xy-\mathrm{dc}}_{\mathrm{OO}'} = -\lim_{\substack{\varepsilon\to 0\\ V\to\infty}}\frac{1}{\hbar V}\mathrm{Im}\sum_n \rho_n\langle n|\left[\tilde{P}^x_{\mathrm{O}}, \tilde{P}^y_{\mathrm{O}'}\right]|n\rangle, \tag{10}$$

where $\tilde{P}^\alpha_{\mathrm{O}}$ is the DPP in the α direction,

$$\langle n|\tilde{P}^\alpha_{\mathrm{O}}|m\rangle = \langle n|P^\alpha_{\mathrm{O}}|m\rangle\Theta_\varepsilon(|E_n - E_m|), \tag{11}$$

and the Lorentzian $\Theta_\varepsilon(x) = \frac{\varepsilon^2}{x^2+\varepsilon^2}$ can be replaced by a projector Heaviside function $\Theta_\varepsilon(x) \to \Theta(\pi\varepsilon/2 - |x|)$ in the limit $\varepsilon \to 0$.

Reduction to single particle (SP) Hamiltonians – For non-interacting fermions or bosons,

$$H^{\mathrm{sp}} = \sum_{ij} h_{ij}(B)a^\dagger_i a_j = \sum_\alpha \epsilon_\alpha(B)a^\dagger_\alpha a_\alpha, \tag{12}$$

Eq. (10) reduces to

$$S^{xy-\mathrm{sp}}_{\mathrm{OO}'} = -\lim_{\substack{\varepsilon\to 0\\ V\to\infty}}\frac{1}{\hbar V}\mathrm{Im}\sum_\alpha n_\alpha\,\left[\tilde{P}^x_{\mathrm{O}}, \tilde{P}^y_{\mathrm{O}'}\right]_{\alpha\alpha}, \tag{13}$$

where n_α is the Fermi-Dirac or Bose-Einstein occupation of SP state $a^\dagger_\alpha|0\rangle$. The DPPs are [23],

$$\tilde{P}^\gamma_{\mathrm{O}} = \sum_{\alpha\beta}\left(\tilde{P}^\gamma_{\mathrm{O}}\right)_{\alpha\beta} a^\dagger_\alpha a_\beta,$$
$$\left(\tilde{P}^\gamma_{\mathrm{O}}\right)_{\alpha\beta} = \langle\alpha|P^\gamma_{\mathrm{O}}|\beta\rangle\Theta_\varepsilon(|\epsilon_\alpha - \epsilon_\beta|). \tag{14}$$

A version of Eq. (13) was derived by Bradlyn and Read [22] for integer QH states without disorder.

DPP's in clean Landau levels – Eigenstates of electrons of effective mass m in a strong magnetic field are described by degenerate Landau levels. The charge polarizations (whose components commute) can be decomposed as

$$P^\gamma_{\mathrm{C}} = eR^\gamma + el_B(\boldsymbol{\pi}\times\hat{\boldsymbol{z}})^\gamma, \tag{15}$$

where $l_B = \sqrt{\frac{\hbar c}{eB}}$. $\boldsymbol{\pi}$ connect between adjacent Landau levels. $\boldsymbol{R}$ are guiding center coordinates which satisfy $[R^x, R^y] = -il_B^2\mathbb{I}$, and $[R^\alpha, \pi^\beta] = 0$.

On OBC, H^{sp} includes a confining potential $V(\boldsymbol{x})$ on its edges. A smooth potential [24] with $|\nabla \log V|^{-1} \ll l_B$ can be approximated by an intra-Landau level operator $V^{\mathrm{eff}}(\boldsymbol{R})$. $V^{\mathrm{eff}}(\boldsymbol{R})$ (which commutes $\boldsymbol{\pi}$) acts only within a single Landau level labelled by ν. One can choose the eigenstate basis of say $R^y|\nu,k\rangle = kl_B|\nu,k\rangle$, in which $\langle\nu,k|V^{\mathrm{eff}}|\nu,k'\rangle$ is generally not diagonal. $U_{\alpha,k}(V^{\mathrm{eff}})$ is the unitary matrix which diagonalizes $V^{\mathrm{eff}}(\boldsymbol{R})$, and defines the energy eigenbasis $|\nu,\alpha\rangle$. Since $[R^x, R^y] = -il_B^2\mathbb{I}$, we know that $[U^\dagger R^x U, U^\dagger R^y U] = -il_B^2\mathbb{I}$. The expectation value of the second commutator is used in Eq. (13) to obtain $\sigma_{xy} = \frac{nec}{B}$, where n is the electron density. This result also holds in the presence of translationally invariant many body interactions.

Disordered metals in weak magnetic fields.– This regime can be described by DB theory [25] at small Hall angles $\omega_c\tau \ll 1$, where τ is the transport scattering time and $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The DB Hall conductivity yields,

$$\sigma^{\mathrm{DB}}_{xy} = \frac{ne^2\omega_c\tau^2}{m}. \tag{16}$$

In the weak field regime, disorder strongly mixes the Landau levels and severs the relation between the DPPs and the guiding centers. Eq. (16) can be recovered by a multiplicative renormalization of the DPPs, i.e. $\tilde{P}^\gamma_{\mathrm{C}} \simeq e(\omega_c\tau)R^\gamma$, in Eq. (13).

Numerical calculations – Eq. (10) are significantly less costly than the off-shell formulas Eq. (2). Having eliminated $-\frac{M_{\mathrm{CT}}}{T}, -\frac{M_{\mathrm{TT}}}{T}$ in α_{xy} and κ_{xy} respectively, one may apply controlled approximations without worrying about precise cancellations of divergent corrections. While Eq. (2) requires full diagnalization of H and calculations of many current matrix elements, Eq. (10) includes only matrix elements between nearly degenerate eigenstates in the spectrum below temperature T. These states may be numerically accessible by Lanczos algorithms [26] or approximated by variational methods [27].

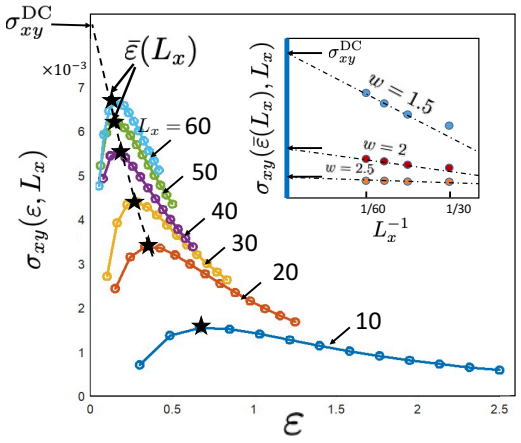

FIG. 1. Extrapolation of numerical Hall conductivities of the square lattice Hamiltonian, Eq. (18). Disorder averaged σ_{xy} are plotted versus ε, for a sequence of linear dimensions L_x. Stars mark the values of $\bar{\varepsilon}$ as defined in Eq. (17). The disorder strength is fixed at $w = 3$. The temperature, Fermi energy, and magnetic field are $T = 0.3, \epsilon_F = -1$ and $B = 0.025$ respectively. **Inset:** The DC limit $\sigma_{xy}^{\rm dc}$ (marked by black arrows) for three values of disorder strength.

Contrary to the initial off-shell formulation of Eq. (2), H in (10) may be replaced by its low energy effective Hamiltonian in the spectrum range $E_n - E_0 \leq k_B T$. For example, Fermi liquid theory can be used for interacting fermions, or continuum field theories for magnets and superconductors.

Numerical calculations are mostly performed on finite lattices which require extrapolation to the DC-thermodynamic limit, of Eq. (1). If $S_{\rm OO'}(\varepsilon, L_x)$ is computed for a sequence of linear dimensions $\{L_x^i\}$, "optimal" values of $\bar{\varepsilon}(L_x^i)$ can be extracted by the extrema conditions,

$$\partial_\varepsilon S^{xy}_{\rm OO'}(\varepsilon, L_x^i) = 0 \Rightarrow \bar{\varepsilon}(L_x^i). \tag{17}$$

The DC limit is obtained by extrapolating the extrema, $S^{\rm dc} = \lim_{i\to\infty} S_{\rm OO'}(\bar{\varepsilon}(L_x^i), L_x^i)$. This scheme is demonstrated in Fig. 1 for the square lattice Hamiltonian,

$$H^{\rm SL} = -\sum_{\langle ij\rangle} \left(e^{-iA_{ij}} c_i^\dagger c_j + {\rm h.c.}\right) + \sum_i (w_i - \epsilon_F) c_i^\dagger c_i, \tag{18}$$

where $w_i \in [-w/2, w/2]$ is a uniformly distributed random number and $B = \sum_\square A_{ij}$ is the magnetic field.

$\bar{\varepsilon}$ are marked by black stars which apparently can be fit to a power law $\bar{\varepsilon} \simeq 7(L_x)^{-1}$. This scaling is consistent with level spacings of one dimensional extended states. In the inset of Figure 1, the Hall conductivities at different disorder strengths extrapolate linearly in L_x^{-1}, to their respective DC limits.

In Ref. [28], we show that $\sigma_{xy}^{\rm dc} \sim w^{-4}$, which is consistent with DB result (16), since by Fermi's golden rule $\frac{1}{\tau} \propto w^2$. The zero field Hall coefficient $R_{\rm H} = \frac{d}{dB}\sigma_{xy}\sigma_{xx}^{-2}$ varies weakly with w in the moderate disorder regime, as expected by DB theory in the constant life-time approximation. $R_{\rm H}$ is approximated fairly well by the equilibrium Hall coefficient formula derived in Refs. [16, 29]. In addition, Wiedemann-Franz law $\frac{\kappa_{xy}}{T\sigma_{xy}}$ reaches close proximity to the DB result of $\frac{\pi^2}{3}$ at low temperatures.

Discussion – Since Eq. (10) applies to any Hamiltonian with OBC, we can draw general conclusions concerning Hall effects even in strong disorder and interactions regimes: (i) Quasi-degenerate manifolds of eigenstates are created by the magnetic field. (ii) These manifolds are subjected to *non commutative geometry* by the DPPs, i.e. $\langle \left[\tilde{P}^\alpha_{\rm O}, \tilde{P}^\beta_{\rm O'}\right]\rangle = i\epsilon_{\alpha\beta} c$. In a sense, $c^{-1}\epsilon_{\alpha\beta}\tilde{P}^\beta_{\rm O} = \Pi^\alpha_{\rm O}$ generates translations and acts as a conjugate momentum to $\tilde{P}^\alpha_{\rm O'}$. Thus, the DPP's generalize the algebra of guiding centers to regimes of strong Landau level mixing. (iii) The quasi-degenerate eigenstates are *chiral* as defined by their nonzero vorticity as $\langle \nabla_{\tilde{P}_{\rm O}} \times \vec{\Pi}_{\rm O'}\rangle = 2/c \neq 0$.

These gapless chiral wavefunctions may be supported exclusively on the sample edges, as in QH and TI phases, or in the bulk. Bulk chiral states have been derived semiclassically by Chalker and Coddington [30], who described the transition between incompressible plateaux using a percolating network of extended chiral states. Finally, we can also infer that thermal Hall currents in insulators [5–7] are also carried by extended chiral modes.

Summary – Microscopic computations of charge and thermal Hall conductivities are made easier by Eqs. (10, 13), which is especially needed in gapless phases. The new formulas unveil the essential role of non-commuting DPPs and associated quasi-degenerate chiral eigenstates. We expect these formulas to allow better connection between model Hamiltonians and Hall-type measurements in regimes of strong scattering.

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[1] David J Thouless, Mahito Kohmoto, M Peter Nightingale, and Marcel den Nijs. Quantized Hall conductance in a two-dimensional periodic potential. *Physical review letters*, 49(6):405, 1982.

[2] Joji Nasu, Junki Yoshitake, and Yukitoshi Motome. Thermal transport in the Kitaev model. *Physical review letters*,

119(12):127204, 2017.

[3] N. Read and Dmitry Green. Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect. *Phys. Rev. B*, 61:10267–10297, Apr 2000.

[4] S. Badoux, W. Tabis, F. Laliberté, G. Grissonnanche, B. Vignolle, D. Vignolles, J. Béard, D. A. Bonn, W. N. Hardy, R. Liang, N. Doiron-Leyraud, Louis Taillefer, and Cyril Proust. Change of carrier density at the pseudogap critical point of a cuprate superconductor. *Nature*, 531:210 EP –, 02 2016.

[5] G. Grissonnanche, S. Thériault, A. Gourgout, M. E. Boulanger, E. Lefrançois, A. Ataei, F. Laliberté, M. Dion, J. S. Zhou, S. Pyon, T. Takayama, H. Takagi, N. Doiron-Leyraud, and L. Taillefer. Chiral phonons in the pseudogap phase of cuprates. *Nature Physics*, 16(11):1108–1111, 2020.

[6] Y. Kasahara, K. Sugii, T. Ohnishi, M. Shimozawa, M. Yamashita, N. Kurita, H. Tanaka, J. Nasu, Y. Motome, T. Shibauchi, and Y. Matsuda. Unusual thermal hall effect in a kitaev spin liquid candidate $\alpha-\mathrm{rucl}_3$. *Phys. Rev. Lett.*, 120:217205, May 2018.

[7] Xiaokang Li, Benoît Fauqué, Zengwei Zhu, and Kamran Behnia. Phonon thermal Hall effect in strontium titanate. *Physical review letters*, 124(10):105901, 2020.

[8] Ryogo Kubo. Statistical-Mechanical Theory of Irreversible Processes. I. General Theory and Simple Applications to Magnetic and Conduction Problems. *Journal of the Physical Society of Japan*, 12(6):570–586, 1957.

[9] JM Luttinger. Theory of thermal transport coefficients. *Physical Review*, 135(6A):A1505, 1964.

[10] NR Cooper, BI Halperin, and IM Ruzin. Thermoelectric response of an interacting two-dimensional electron gas in a quantizing magnetic field. *Physical Review B*, 55(4):2344, 1997.

[11] Tao Qin, Qian Niu, and Junren Shi. Energy magnetization and the thermal Hall effect. *Physical review letters*, 107(23):236601, 2011.

[12] Cong Xiao and Qian Niu. Unified bulk semiclassical theory for intrinsic thermal transport and magnetization currents. *Phys. Rev. B*, 101:235430, Jun 2020.

[13] J.E. Avron and R. Seiler. Quantization of the Hall conductance for general, multiparticle Schrödinger Hamiltonians. *Physical review letters*, 54(4):259, 1985.

[14] P Streda and L Smrcka. Thermodynamic derivation of the Hall current and the thermopower in quantising magnetic field. *Journal of Physics C: Solid State Physics*, 16(24):L895, 1983.

[15] Herbert Oji and P Streda. Theory of electronic thermal transport: Magnetoquantum corrections to the thermal transport coefficients. *Physical Review B*, 31(11):7291, 1985.

[16] Assa Auerbach. Equilibrium formulae for transverse magnetotransport of strongly correlated metals. *Physical Review B*, 99(11):115115, 2019.

[17] On OBC, the wavevector $\boldsymbol{q}$ is continuous and independent on V (in contrast to the finite torus), and $(\lim_{\boldsymbol{q}\to 0}, \lim_{V\to\infty})$ commute. The complex frequency $\omega + i\varepsilon \to 0$, can be taken (after $V \to \infty$) along the imaginary axis. .

[18] On OBC one can define *uniform* charge and thermal polarizations, and the charge and thermal magnetizations needed for Eq. (2). Continuous and uniform thermal gradients can only be implemented on OBC. For the mesoscopic regime, boundary conditions matter. Our formulas can be relevant for small mesoscopic samples where the dephasing length scales is of the order of the sample size. Our finite volume formulas can implement the effects of attached leads, by keeping ε larger than the leads level spacing.

[19] For models with no C4 symmetry, the Hall coefficients, which are antisymmetric in magnetic field B, can be defined by antisymmetrizing the Kubo formulas with respect to $j^x_{\mathrm{O}} \to j^y_{\mathrm{O}'}$, in accordance with Onsager's relations.

[20] For long range interactions, the spatial decomposition may not be unique.

[21] In the literature [10], one frequently encounters specialized expressions of the magnetization and thermal magnetization $M_{\mathrm{OO}'}$ for continuum particle Hamiltonians, $M^{\mathrm{P}}_{\mathrm{CT}} = -\frac{e}{2}\sum_i^{N_p} \hat{\boldsymbol{z}} \cdot \boldsymbol{x}_i \times \mathbf{v}_i$, and $M^{\mathrm{P}}_{\mathrm{TT}} = -\frac{1}{2}\sum_i^{N_p} \hat{\boldsymbol{z}} \cdot \boldsymbol{x}_i \times \{\mathbf{v}_i, h_i\}$, where $\mathbf{v} = (\boldsymbol{p}_i - \frac{e}{c}\boldsymbol{A}_i)/m$. Here we consider more general lattice Hamiltonians.

[22] Barry Bradlyn and N Read. Low-energy effective theory in the bulk for transport in a topological phase. *Physical Review B*, 91(12):125303, 2015.

[23] We describe normal diagonalized hamiltonians. Anomalous bosonic terms e.g. $a^\dagger a^\dagger$ may be present in the energy-diagonalized thermal polarizations. These terms contribute off-shell matrix elements which can be ignored.

[24] The following argument holds also for sharp and well separated edges in the bulk-gapped QH phases. .

[25] John M Ziman. *Electrons and phonons: the theory of transport phenomena in solids*. Oxford university press, 2001.

[26] Andreas M. Läuchli, Julien Sudan, and Erik S. Sørensen. Ground-state energy and spin gap of spin-$\frac{1}{2}$ Kagomé-Heisenberg antiferromagnetic clusters: Large-scale exact diagonalization results. *Phys. Rev. B*, 83:212401, Jun 2011.

[27] Ulrich Schollwock. The density-matrix renormalization group in the age of matrix product states. *Annals of Physics*, 326(1):96 – 192, 2011. January 2011 Special Issue.

[28] See Supplementary Material.

[29] Abhisek Samanta, Daniel P. Arovas, and Assa Auerbach. Hall Coefficient of Semimetals. *Phys. Rev. Lett.*, 126:076603, Feb 2021.

[30] JT Chalker and PD Coddington. Percolation, quantum tunnelling and the integer hall effect. *Journal of Physics C: Solid State Physics*, 21(14):2665, 1988.