Nielsen-Ninomiya Theorem with Bulk Topology: Duality in Floquet and Non-Hermitian Systems

Takumi Bessho and Masatoshi Sato

Phys. Rev. Lett. **127**, 196404 — Published 5 November 2021

DOI: 10.1103/PhysRevLett.127.196404

Nielsen-Ninomiya Theorem with Bulk Topology: Duality in Floquet and non-Hermitian Systems

Takumi Bessho[1,*] and Masatoshi Sato[1,†]

[1] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

(Dated: October 4, 2021)

The Nielsen-Ninomiya theorem is a fundamental theorem on the realization of chiral fermions in static lattice systems in high-energy and condensed matter physics. Here we extend the theorem in *dynamical systems*, which include the original Nielsen-Ninomiya theorem in the static limit. In contrast to the original theorem, which is a no-go theorem for bulk chiral fermions, the new theorem permits them due to bulk topology intrinsic to dynamical systems. The theorem is based on duality enabling a unified treatment of periodically driven systems and non-Hermitian ones. We also present the extended theorem for non-chiral gapless fermions protected by symmetry. Finally, as an application of our theorem and duality, we predict a new type of chiral magnetic effect — the non-Hermitian chiral magnetic skin effect.

The Nielsen-Ninomiya (NN) theorem is a fundamental constraint in realizing chiral fermions in lattice systems [1–3]. It initially was a no-go theorem for the lattice realization of the Standard Model in particle physics, but it also applies to condensed matter physics. For instance, the Nielsen-Ninomiya theorem requires that bulk Weyl points in Weyl semimetals always appear in a pair so that the total chiral charge of Weyl points vanishes [4–6]. The NN theorem severely restricts bulk low energy modes in topological materials [7–14].

However, recent studies have revealed that the NN theorem does not hold when considering topological states in dynamical systems [15–71]: Periodically driven systems may support unpaired chiral fermions both in one-[72–76] and three-dimensions[77, 78]. Furthermore, systems with non-Hermitian Hamiltonians also retain unpaired chiral fermions after the long-time dynamics [79]. These examples have suggested a reformulation of the NN theorem in dynamical systems.

In this Letter, we extend the NN theorem in dynamical systems. As a particular case of the static limit, the extended theorem includes the original one. A key of our extension is a duality between periodically driven systems and non-Hermitian ones. A one-cycle time evolution operator U_{F} generally describes a periodically driven system. By identifying iU_{F} as a non-Hermitian Hamiltonian H, we treat a periodically driven system and a non-Hermitian one in a unified manner. Another key is multiple gap structures intrinsic to non-Hermitian systems. The complex energy spectrum of non-Hermitian systems may introduce two different gap structures: point and line gaps [80, 81]. A non-Hermitian system can be gapped in the sense of point gap even if it supports gapless fermions in the sense of line gap. Because the point gap enables a novel bulk topological number, this means that bulk chiral (so gapless) fermions in dynamical systems may coexist with non-trivial bulk topology. This situation never happens in conventional static systems and makes it possible to reformulate the NN theorem.

The extended NN theorem provides an exact relation between the total chiral charge of chiral fermions and the bulk topological number. This theorem infers that if the bulk topological number is nonzero, so is the total chiral charge, and thus the system realizes unpaired chiral fermions. The extended theorem also applies to systems with symmetry. Symmetry protects non-chiral gapless fermions, giving them a topological charge other than chirality. In this case, the bulk topological number is equal to the total topological charge from our theorem.

As an application of our theorem, we consider a non-Hermitian version of the chiral magnetic effect (CME). The CME is an electric current generation along an applied magnetic field due to unpaired Weyl fermions in three dimensions [82]. While the chiral magnetic effect does not occur in static systems because of the NN theorem [9], the extended theorem allows it in dynamical systems. Periodically driven systems may exhibit the CME [77, 78], and thus our duality relation suggests that so do non-Hermitian systems. We demonstrate that a wave packet in a non-Hermitian Weyl semimetal moves in the direction of an applied magnetic field, manifesting the CME. Furthermore, the extended theorem implies a nonzero spectral winding number of non-Hermitian Weyl semimetals under a magnetic field. This result leads to predicting a new type of CME—the chiral magnetic skin effect.

We assume without loss of generality that the Fermi energy E_{F}, *i.e.* the reference energy of a gap, is zero unless otherwise mentioned. One can recover E_{F} by replacing the Hamiltonian $H(\boldsymbol{k})$ with $H(\boldsymbol{k}) - E_{\mathrm{F}}$ if necessary.

1D chiral fermions in dynamical systems.— Let us start with a simple 1D non-Hermitian system hosting a chiral mode. The Hamiltonian of the model is

$$H(k) = \sin k + i \cos k, \tag{1}$$

where k is the crystal momentum and $H(k)$ is periodic in k [79]. The energy $E(k)$ of the system is $H(k)$ itself, and the group velocity $v(k)$ is $v(k) = \mathrm{Re}(\partial E(k)/\partial k)$.

* takumi.bessho@yukawa.kyoto-u.ac.jp
† msato@yukawa.kyoto-u.ac.jp

At the Fermi energy $\mathrm{Re}E(k)=0$, there are two gapless modes with $k=0,\pi$: A right-moving mode ($v(k)>0$) with $k=0$ and a left-moving mode ($v(k)<0$) with $k=\pi$. While the right-moving mode has a positive $\mathrm{Im}E(k)$, the left-moving mode has a negative one; thus, the left-moving mode decays, and only the right-moving mode survives after the long-time dynamics. Therefore, the system realizes a chiral fermion, *i.e.* a right-moving chiral mode.

Another simple 1D model with a chiral mode is a periodically driven system evolved by the one-component unitary operator [77],

$$U_{\mathrm{F}}(k)=e^{-ik}. \tag{2}$$

The Floqet Hamiltonian $H_{\mathrm{F}}(k)$ defined by $e^{-iH_{\mathrm{F}}(k)\tau}=U_{\mathrm{F}}(k)$ with a driving period τ describes the stroboscopic time-evolution of the system, $|t+\tau\rangle=U_{\mathrm{F}}(t)|t\rangle=e^{-iH_{\mathrm{F}}(k)\tau}|t\rangle$. The eigenvalue of $H_{\mathrm{F}}(k)$, called the quasi-energy, is $\epsilon_{\mathrm{F}}(k)=k/\tau$ up to an integer multiple of $2\pi/\tau$. Because the group velocity $v_{\mathrm{F}}(k)=\partial\epsilon_{\mathrm{F}}(k)/\partial k$ is positive, the system has a right-moving chiral mode.

These chiral modes have a common topological origin. The equation

$$H(k)=iU_{\mathrm{F}}(k), \tag{3}$$

relates the above models, then the 1D (spectral) winding number

$$w_1=-\int_0^{2\pi}\frac{dk}{2\pi i}\mathrm{tr}[H^{-1}(k)\partial_k H(k)]. \tag{4}$$

gives $w_1=1$ for both models. (The trace is trivial in the above models.) For the non-Hermitian model in Eq.(1), the non-zero spectral winding number results in so-called the non-Hermitian skin effect [40]: For $w_1=1$, all bulk states localize to the right end [83, 84]. This effect suggests a right-moving chiral mode because a uni-directed movement of the mode forces all bulk states to move to the right end. For the periodically driven model in Eq.(2), on the other hand, the non-zero spectral winding number implies a non-zero average of the group velocity,

$$w_1=-\int_0^{2\pi}\frac{dk}{2\pi i}\partial_k\ln\det H(k)=\int_0^{2\pi}\frac{dk}{2\pi}v_{\mathrm{F}}(k)\tau, \tag{5}$$

which also indicates a right-moving chiral mode.

The above examples suggest a general relation between the spectral winding number and the chirality $\mathrm{sgn}\,v_{\mathrm{F}}(k)$ of gapless modes. For 1D non-Hermitian systems, the exact link is as follows [85]:

Theorem 1: Let $H(k)$ be a 1D non-Hermitian Hamiltonian and $E_p(k)$ be the complex eigen-energy of band p. Then, we have

$$w_1=\sum_{\mathrm{Im}E_p(k_{p\alpha})>0}\mathrm{sgn}\,v_{p\alpha}=-\sum_{\mathrm{Im}E_p(k_{p\alpha})<0}\mathrm{sgn}\,v_{p\alpha}, \tag{6}$$

where $k_{p\alpha}$ is the α-th Fermi point of band p defined by $\mathrm{Re}E_p(k_{p\alpha})=0$, and $v_{p\alpha}=\mathrm{Re}(\partial E_p(k)/\partial k)_{k=k_{p\alpha}}$ is the group velocity at $k_{p\alpha}$. The summation in Eq.(6) is over all p and α.

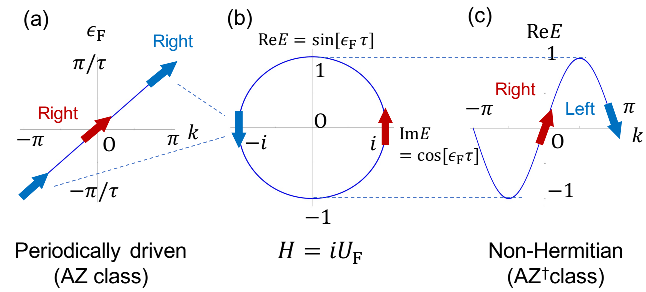

FIG. 1. Duality between a periodically driven system and a non-Hermitian one. We illustrate the 1D case here. w_1 is the winding number of the spectral in the complex energy plane in (b). Theorem 1' is evident in the relation between (a) and (b). The duality holds in any dimensions.

For a Hermitian Hamiltonian $H(k)$, the above theorem reproduces the NN theorem. The spectral winding number w_1 is zero for any Hermitian Hamiltonian, and by adding a small imaginary term $i\eta$ to $H(k)$, all the Fermi points can have a positive imaginary part of the energy. Thus, from Eq. (6), we have $\sum_{k_{p\alpha}}\mathrm{sgn}\,v_{p\alpha}=0$, which is the NN theorem in one-dimension [1].

Using the relation in Eq.(3), we can also derive a counterpart theorem for 1D periodically driven systems: Equation (3) maps the quasi-energy $\epsilon_p(k)$ of $U_{\mathrm{F}}(k)$ to the complex energy $E_p(k)$ of $H(k)$, $E_p(k)=\sin[\epsilon_p(k)\tau]+i\cos[\epsilon_p(k)\tau]$. Thus, a Fermi point defined by $\epsilon_p(k)=0$ (π/τ) gives a Fermi point of $E_p(k)$ with a positive (negative) $\mathrm{Im}E_p(k)$. Comparing the group velocities at the Fermi points, we obtain the theorem:

Theorem 1': Let $H_{\mathrm{F}}(k)$ be a 1D Floquet Hamiltonian and $\epsilon_p(k)$ be the quasi-energy of band p. Then, gapless modes of the quasi energy obey

$$w_1=\sum_{\epsilon_p(k_{p\alpha})=\mu}\mathrm{sgn}\,v_{p\alpha}, \tag{7}$$

where $k_{p\alpha}$ is the Fermi point of band p defined by $\epsilon(k_{p\alpha})=\mu$, and $v_{p\alpha}=(\partial\epsilon_p(k)/\partial k)_{k=k_{p\alpha}}$ is the group velocity at $k_{p\alpha}$ [86].

Here we have shifted the origin of the quasi-energy by $U_{\mathrm{F}}\to e^{i\mu\tau}U_{\mathrm{F}}$ and omitted the term corresponding to the last term in Eq.(6) since it is just a particular case of Eq.(7).

Non-Hermitian Weyl semimetals— Weyl fermions are 3D massless (or gapless) fermions with a definite chirality. They are realized as band crossing points (Weyl points) and behave like magnetic monopoles in the momentum space, of which the magnetic charge provides the chirality charge. They have finite lifetimes –the imaginary part of the energies– in the presence of non-Hermiticity. For Weyl fermions, we have the following theorem [87]:

Theorem 2: Let $H(\boldsymbol{k})$ be a 3D non-Hermitian Hamiltonian and $E_p(\boldsymbol{k})$ be the complex eigen-energy of band

p. Then, Weyl fermions in the complex energy spectrum obey

$$w_3 = \sum_{\mathrm{Im}E_p(S_{p\alpha})>0} \mathrm{Ch}_{p\alpha} = -\sum_{\mathrm{Im}E_p(S_{p\alpha})<0} \mathrm{Ch}_{p\alpha}. \quad (8)$$

Here w_3 is the 3D winding number,

$$w_3 = -\frac{1}{24\pi^2}\int_{\mathrm{BZ}} \mathrm{tr}[H^{-1}dH]^3, \quad (9)$$

$S_{p\alpha}$ is the α-th Fermi surface of band p defined by $S_{p\alpha} = \{\boldsymbol{k} \in \mathrm{BZ}|\mathrm{Re}E_p(\boldsymbol{k}) = 0\}$, and $\mathrm{Ch}_{p\alpha}$ is the Chern number on the Fermi surface $S_{p\alpha}$,

$$\mathrm{Ch}_{p\alpha} = \frac{1}{2\pi i}\int_{S_{p\alpha}} (\nabla \times \boldsymbol{A}(\boldsymbol{k}))\cdot \mathrm{d}\boldsymbol{S}, \quad (10)$$

where $\boldsymbol{A}(\boldsymbol{k}) = \langle\langle \psi_p(\boldsymbol{k})|\nabla\psi_p(\boldsymbol{k})\rangle$ with $H(\boldsymbol{k})|\psi_p(\boldsymbol{k})\rangle = E_p(\boldsymbol{k})|\psi_p(\boldsymbol{k})\rangle$, $H^\dagger(\boldsymbol{k})|\psi_p(\boldsymbol{k})\rangle\rangle = E_p^*(\boldsymbol{k})|\psi_p(\boldsymbol{k})\rangle\rangle$, and the orientation of $S_{p\alpha}$ is along the direction of the Fermi velocity $\mathrm{Re}(\partial E_p(\boldsymbol{k})/\partial \boldsymbol{k})_{\boldsymbol{k}\in S_{p\alpha}}$. $\mathrm{Ch}_{p\alpha}$ counts the total chirality of Weyl points inside $S_{p\alpha}$.

Theorem 2 reproduces the NN theorem again when $H(\boldsymbol{k})$ is Hermitian: By adding a tiny positive imaginary term to $H(k)$, we have $\sum_{p\alpha}\mathrm{Ch}_{p\alpha} = 0$, which is one of the variants of the NN theorem in three dimensions [88]. Indeed, this equation forbids an unpaired Weyl point in Hermitian systems: If an unpaired Weyl point were to exist, we would have a Fermi surface surrounding it by choosing the Fermi energy near the Weyl point. This configuration would give a nonzero $\sum_{p\alpha}\mathrm{Ch}_{p\alpha}$, which contradicts $\sum_{p\alpha}\mathrm{Ch}_{p\alpha} = 0$.

When w_3 is nonzero, Theorem 2 predicts chiral fermions. For instance, consider the following model,

$$H(\boldsymbol{k}) = (d_0 + \boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\sigma})\,\tau_1 + m(\boldsymbol{k})\tau_3 + i\gamma(\tau_3 - \tau_0), \quad (11)$$

with $d_i(\boldsymbol{k}) = \sin k_i$, $m(\boldsymbol{k}) = m_0 + \sum_{i=1}^{3}\cos k_i$. This model has a point gap at $E_\mathrm{F} = -i\gamma$ and hosts Weyl points in the complex energy plane as shown in Fig. 2(a), satisfying Theorem 2 [89].

Duality.— The relation (3), which enables a unified treatment of a periodically driven system and a non-Hermitian one, is not accidental. This duality relation holds in arbitrary dimensions. Evidently, one can immediately identify any one-cycle time evolution operator $U_\mathrm{F}(\boldsymbol{k})$ with a non-Hermitian Hamiltonian $H(\boldsymbol{k})$ by

$$H(\boldsymbol{k}) = iU_\mathrm{F}(\boldsymbol{k}). \quad (12)$$

However, the opposite is also true for a class of non-Hermitian systems. We say that a non-Hermitian Hamiltonian $H(\boldsymbol{k})$ has a point gap if $\det H(\boldsymbol{k}) \neq 0$. Then, one can regard any point-gapped Hamiltonian as a one-cycle time evolution operator because a point gapped $H(\boldsymbol{k})$ can smoothly deform into a unitary matrix without closing the point gap [80, 81].

The duality relation (12) brings out common properties of periodically driven systems and non-Hermitian ones: In terms of the Floquet Hamiltonian $H_\mathrm{F}(\boldsymbol{k}) = (i/\tau)\ln U_\mathrm{F}(\boldsymbol{k})$, the above relation reads $H(\boldsymbol{k}) = \sin[H_\mathrm{F}(\boldsymbol{k})\tau] + i\cos[H_\mathrm{F}(\boldsymbol{k})\tau]$. Thus, eigenstates of $H(\boldsymbol{k})$ are identical to those of $H_\mathrm{F}(\boldsymbol{k})$. Also, a gapless state in $H_\mathrm{F}(\boldsymbol{k}) \sim \boldsymbol{k}\cdot\boldsymbol{\Gamma}$ results in a gapless state in $H(\boldsymbol{k})$, and vice versa. ($\boldsymbol{\Gamma}$ are some matrices.) Furthermore, these systems share a topological number; the topological number is given by that of the Hermitian Hamiltonian [78, 80, 81],

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} 0 & H(\boldsymbol{k}) \\ H^\dagger(\boldsymbol{k}) & 0 \end{pmatrix}. \quad (13)$$

From Eq.(12), $\mathcal{H}(\boldsymbol{k})$ satisfies $\mathcal{H}^2(\boldsymbol{k}) = 1$ and thus has eigenvalues ± 1. Therefore, $\mathcal{H}(\boldsymbol{k})$ defines an insulator, giving a well-defined topological number.

Note that the above identification links a periodically driven system and a non-Hermitian one in different symmetry classes. To see this, consider time-reversal, particle-hole, and chiral symmetries for the Floquet Hamiltonian $H_\mathrm{F}(\boldsymbol{k})$, given by $TH_\mathrm{F}(\boldsymbol{k})T^{-1} = H_\mathrm{F}(-\boldsymbol{k})$, $CH_\mathrm{F}(\boldsymbol{k})C^{-1} = -H_\mathrm{F}(-\boldsymbol{k})$, and $\Gamma H_\mathrm{F}(\boldsymbol{k})\Gamma^{-1} = -H_\mathrm{F}(\boldsymbol{k})$, respectively. Here T and C are antiunitary operators with $T^2 = \pm 1$, $C^2 = \pm 1$, and Γ is a unitary operator with $\Gamma^2 = 1$. The presence or absence of these symmetries define Altland-Zirnbauer (AZ) ten symmetry classes [90]. The relation (12) maps these symmetries as follows: $TH^\dagger(\boldsymbol{k})T^{-1} = H(-\boldsymbol{k})$, $CH(\boldsymbol{k})C^{-1} = -H(-\boldsymbol{k})$, and $\Gamma H^\dagger(\boldsymbol{k})\Gamma^{-1} = -H(\boldsymbol{k})$. The latter symmetries define another ten symmetry classes, called AZ† classes [81], which are intrinsic to non-Hermitian systems.

Extended NN theorem. — Symmetry protects gapless fermions other than chiral fermions. We now present the extended NN theorem, including such non-chiral (Dirac) fermions.

First, consider non-Hermitian systems. Depending on symmetry classes, two different situations may happen: (i) gapless fermions in classes A, AI†, AII† appear as band crossing points at general positions in the complex energy plane, and (ii) those in other AZ† classes appear on the $\mathrm{Re}E = 0$ axis. To define the topological charge of gapless fermions, we use the Fermi surface at $\mathrm{Re}E = 0$ in the former case, and a small sphere encircling a gapless fermion in the latter [91]. We have the following theorem:

Theorem 3: Let $H(\boldsymbol{k})$ be a point-gapped non-Hermitian Hamiltonian in an AZ† class. Then, bulk gapless fermions of $H(\boldsymbol{k})$ obeys

$$n = \sum_{\mathrm{Im}E_\alpha>0} \nu_\alpha = -\sum_{\mathrm{Im}E_\alpha<0} \nu_\alpha, \quad (14)$$

As we mentioned above, the point gap topological number n is given by the conventional topological number of the topological insulator described by the Hermitian Hamiltonian in Eq.(13). The explicit form of n is summarized in Ref. [81]. In case (i) in the above, α labels the Fermi surfaces at $\mathrm{Re}E = 0$, ν_α is the topological charge of gapless fermions inside the α-th Fermi surface, and E_α is the complex energy of the Fermi surface. In case (ii), α labels gapless fermions, ν_α is the topological charge of the α-th gapless fermion defined on the small sphere, and E_α is the complex energy of the gapless fermion [92].

Using the duality relation (12), we also have an accompanying theorem for gapless fermions in periodically driven systems. We find that (i') gapless fermions in classes A, AI, AII appear as band crossing points with arbitrary energies in the quasi-energy spectra, and (ii') those in other AZ classes appear with $\epsilon = 0$ or π/τ. Then, the accompanying theorem is as follows.

Theorem 3': For gapless fermions in a periodically driven system in an AZ class, we have

$$n = \sum_{\epsilon_\alpha = \mu} \nu_\alpha^\mu, \qquad \text{in case (i')}, \quad (15)$$

$$n = \sum_{\epsilon_\alpha = 0} \nu_\alpha^0 = -(-1)^d \sum_{\epsilon_\alpha = \pi/\tau} \nu_\alpha^\pi, \qquad \text{in case (ii')}. \quad (16)$$

Here n is the topological number of $iU_{\mathrm{F}}(\boldsymbol{k})$ given by $\mathcal{H}(\boldsymbol{k})$ in Eq.(13), and d is the dimension of the system. In case (i'), α labels the Fermi surfaces defined by $\epsilon = \mu$, and ν_α^μ is the topological charge of gapless fermions inside the α-th Fermi surface. In case (ii), α labels gapless fermions at $\epsilon = 0, \pi$, and $\nu_\alpha^{0,\pi}$ is the topological charge of the gapless fermion with the quasi-energy $\epsilon_\alpha = 0, \pi$ [93] .

Note that Eq.(16) have a sign depending on d in the second equality: A gapless fermion at π/τ, $H_{\mathrm{F}}(\boldsymbol{k}) = \boldsymbol{k} \cdot \boldsymbol{\Gamma} + \pi/\tau$, in a periodically driven system corresponds to $H(\boldsymbol{k}) = -\boldsymbol{k} \cdot \boldsymbol{\Gamma} - i$, in a non-Hermitian system. Since these Hamiltonians have an opposite topological charge in odd dimensions, we have the additional sign $(-1)^d$. Equation (7) is the 1D case of Eq.(15) in class A (no symmetry). We have also confirmed Eq.(16) using a 2D periodically driven model with chiral symmetry (class AIII) [94].

Chiral magnetic effect.— Weyl fermions in a periodically driven system may exhibit the CME [77, 78]. As a counterpart of this effect, we investigate the non-Hermitian CME. Figure 2(b) shows the energy spectrum of the model in Eq. (11) under a magnetic field B_z in the z-direction. The magnetic field opens the Landau gap at the Weyl point at $k = (0, 0, 0)$ in Fig. 2(a), and a right-moving chiral mode with a positive $\mathrm{Im}(E - E_{\mathrm{F}})$ appears. The chiral mode has a longer lifetime and produces a current along the magnetic field, leading to the CME. We confirm the CME by examining the wave packet dynamics. Figures 2(c) and 2(d) show the wave packet dynamics without and with a magnetic field. While wave packets without a magnetic field tend to move along the spin direction because of the spin-momentum locking of Weyl fermions, we observe different uni-directed motions with a magnetic field consistent with the CME.

Using the extended NN theorem, we can predict a general effect intrinsic to the non-Hermitian CME: From Theorem 2, a system with nonzero w_3 hosts Weyl fermions with the total chiral charge of w_3. As in Fig.2 (b), a magnetic field B_z opens the Landau gap at Weyl fermions, leaving a 1D chiral mode for each Weyl point, with the Landau degeneracy $(eB_z/2\pi)L_x L_y$ [95], where e is the electric charge of the Weyl fermion and $L_{i=x,y}$

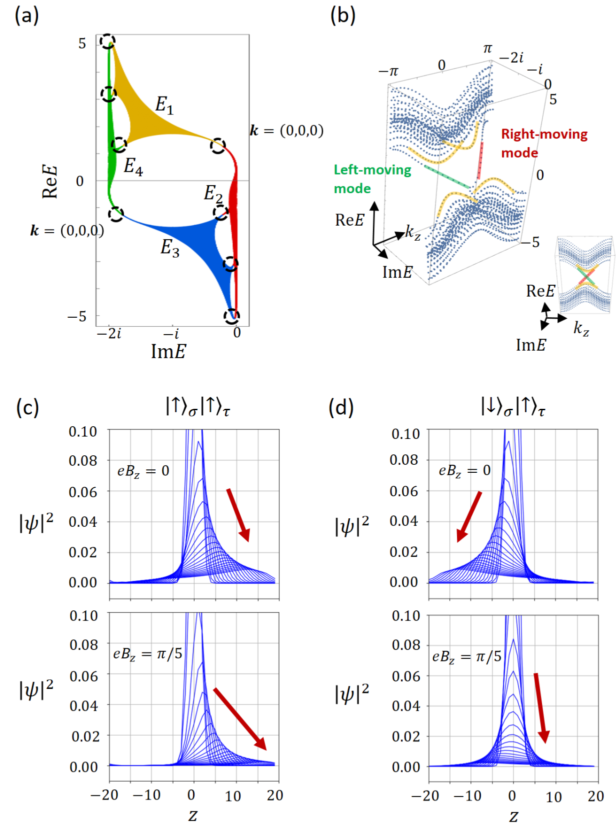

FIG. 2. (a, b) Energy spectrum of the non-Hermitian Weyl semimetal in Eq. (11) (a) without and (b) with a magnetic field B_z in the z direction. (a) Colors distinguish different bands, and dotted circles enclose Weyl points. (b) Right (Left) moving mode has positive (negative) $\mathrm{Im}(E - E_{\mathrm{F}})$ with $E_{\mathrm{F}} = -i$. The right and left moving modes originate from Weyl points with $\boldsymbol{k} = (0, 0, 0)$. The inset is the same figure viewed from a different angle. (c,d) Wave packet dynamics in the non-Hermitian Weyl semimetal of Eq. (11) (top) without and (bottom) with B_z. We draw snapshots of the probability densities $|\psi(z)|^2$ at each unit cycle, where the red arrows indicate the time evolution. We use the fourth-order Runge-Kutta method. The initial wave packets are $|\psi_0\rangle = \psi_0 \, |\sigma_z\rangle_\sigma \, |\tau_z\rangle_\tau$, where ψ_0 is a 3D Gaussian wave packet with the width $2\bar{\sigma}^2 = 5$ and $|\sigma_z\rangle_\sigma \, |\tau_z\rangle_\tau$ is specified in each figure. With $B_z = \pi/5$, all the wave packets tend to move in the $+\hat{z}$ direction. The parameters in Eq. (11) are $d_0 = \gamma = \gamma_0 = 1$ and $m_0 = -2$. The system size is (b) $L_x = L_y = L_z = 30$ and (c,d) $L_x = L_y = L_z = 40$ with the periodic boundary conditions.

is the system length in the i-direction. Therefore, the system supports 1D chiral modes with the total chiral charge $w_3(eB_z/2\pi)L_x L_y$. From Theorem 1, this result implies that the system also hosts the 1D spectral winding number w_1 given by

$$w_1 = \frac{eB_z}{2\pi} L_x L_y w_3. \quad (17)$$

Here w_1 is defined by Eq.(4), where $H(k)$ with $k = k_z$

is the Hamiltonian under the magnetic field B_z, and the trace includes the summation of k_x and k_y in the magnetic Brillouin zone. Note that $eB_zL_xL_y/2\pi$ is an integer under the periodic boundary conditions in x- and y-directions.

The relation (17) gives a profound implication. As mentioned above, a nonzero w_1 induces the non-Hermitian skin effect [83, 84], where extended bulk modes in the periodic boundary condition become localized boundary modes in the open boundary condition. Therefore, Eq. (17) predicts that the system with a nonzero w_3 inevitably shows the skin effect under a magnetic field. This prediction is consistent with the CME because bulk modes stack to a boundary in the direction parallel to the magnetic field due to uni-directed currents of the CME. We have confirmed the chiral magnetic skin effect in the model of Eq.(11) [96]. Photonic systems [97, 98] and cold atoms [99, 100] may provide the spin-selective (or sublattice selective) loss term in Eq.(11), and thus the experimental realization of the chiral magnetic skin effect is feasible.

We are grateful to Masaya Nakagawa, Ken Shiozaki, Nobuyuki Okuma, Kohei Kawabata, Masaya Kunimi, and Taigen Kawano for valuable discussions. This work was supported by JST CREST Grant No. JPMJCR19T2, KAKENHI Grant No. JP20H00131, and KAKENHI Grant No. JP21J11810 from the JSPS.

[1] H. B. Nielsen and M. Ninomiya, Nuclear Physics B **185**, 20 (1981).
[2] H. B. Nielsen and M. Ninomiya, Nuclear Physics B **193**, 173 (1981).
[3] L. H. Karsten, Physics Letters B **104**, 315 (1981).
[4] H. B. Nielsen and M. Ninomiya, Physics Letters B **130**, 389 (1983).
[5] S. Murakami, New Journal of Physics **9**, 356 (2007).
[6] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B **83**, 205101 (2011).
[7] A. A. Burkov, M. D. Hook, and L. Balents, Phys. Rev. B **84**, 235126 (2011).
[8] A. A. Zyuzin and A. A. Burkov, Phys. Rev. B **86**, 115133 (2012).
[9] M. M. Vazifeh and M. Franz, Phys. Rev. Lett. **111**, 027201 (2013).
[10] S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia, and M. Z. Hasan, Science **349**, 613 (2015).
[11] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. **90**, 015001 (2018).
[12] C.-K. Chiu and A. P. Schnyder, Phys. Rev. B **90**, 205136 (2014).
[13] S. Kobayashi, K. Shiozaki, Y. Tanaka, and M. Sato, Phys. Rev. B **90**, 024516 (2014).
[14] C. Fang, Y. Chen, H.-Y. Kee, and L. Fu, Phys. Rev. B **92**, 081201(R) (2015).
[15] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).
[16] T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B **84**, 235108 (2011).
[17] T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White, Nature communications **3**, 882 (2012).
[18] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Phys. Rev. X **3**, 031005 (2013).
[19] J. K. Asbóth and H. Obuse, Phys. Rev. B **88**, 121406(R) (2013).
[20] F. Nathan and M. S. Rudner, New Journal of Physics **17**, 125014 (2015).
[21] D. Carpentier, P. Delplace, M. Fruchart, and K. Gawedzki, Phys. Rev. Lett. **114**, 106806 (2015).
[22] R. Roy and F. Harper, Phys. Rev. B **96**, 155118 (2017).
[23] L. Zhou, C. Chen, and J. Gong, Phys. Rev. B **94**, 075443 (2016).
[24] T. Morimoto, H. C. Po, and A. Vishwanath, Phys. Rev. B **95**, 195155 (2017).
[25] M. Nakagawa, R.-J. Slager, S. Higashikawa, and T. Oka, Phys. Rev. B **101**, 075108 (2020).
[26] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996).
[27] C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. **86**, 787 (2001).
[28] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102**, 065703 (2009).
[29] M. Sato, K. Hasebe, K. Esaki, and M. Kohmoto, Progress of Theoretical Physics **127**, 937 (2012).
[30] Y. C. Hu and T. L. Hughes, Phys. Rev. B **84**, 153101 (2011).
[31] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, Phys. Rev. B **84**, 205128 (2011).
[32] H. Schomerus, Opt. Lett. **38**, 1912 (2013).
[33] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. **115**, 040402 (2015).
[34] T. E. Lee, Phys. Rev. Lett. **116**, 133903 (2016).
[35] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. **118**, 040401 (2017).
[36] L. Xiao, X. Zhan, Z. Bian, K. Wang, X. Zhang, X. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, *et al.*, Nature Physics **13**, 1117 (2017).
[37] M. Chernodub, Journal of Physics A: Mathematical and Theoretical **50**, 385001 (2017).
[38] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. **120**, 146402 (2018).
[39] H. Zhou and J. Y. Lee, Phys. Rev. B **99**, 235112 (2019).
[40] S. Yao and Z. Wang, Phys. Rev. Lett. **121**, 086803 (2018).
[41] S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. **121**, 136802 (2018).
[42] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).
[43] K. Kawabata, K. Shiozaki, and M. Ueda, Phys. Rev. B

98, 165148 (2018).
[44] H. Xu, D. Mason, L. Jiang, and J. Harris, Nature **537**, 80 (2016).
[45] T. Yoshida, R. Peters, and N. Kawakami, Phys. Rev. B **98**, 035141 (2018).
[46] A. A. Zyuzin and A. Y. Zyuzin, Phys. Rev. B **97**, 041203(R) (2018).
[47] S. Longhi, Phys. Rev. Lett. **122**, 237601 (2019).
[48] R. Okugawa and T. Yokoyama, Phys. Rev. B **99**, 041202(R) (2019).
[49] K. Kawabata, T. Bessho, and M. Sato, Phys. Rev. Lett. **123**, 066405 (2019).
[50] N. Okuma and M. Sato, Phys. Rev. Lett. **123**, 097701 (2019).
[51] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. **123**, 170401 (2019).
[52] Z.-Y. Ge, Y.-R. Zhang, T. Liu, S.-W. Li, H. Fan, and F. Nori, Phys. Rev. B **100**, 054105 (2019).
[53] T. Yoshida, R. Peters, N. Kawakami, and Y. Hatsugai, Phys. Rev. B **99**, 121101(R) (2019).
[54] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. **123**, 246801 (2019).
[55] K.-I. Imura and Y. Takane, Phys. Rev. B **100**, 165430 (2019).
[56] K. Kimura, T. Yoshida, and N. Kawakami, Phys. Rev. B **100**, 115124 (2019).
[57] W. B. Rui, M. M. Hirschmann, and A. P. Schnyder, Phys. Rev. B **100**, 245116 (2019).
[58] T. Matsushita, Y. Nagai, and S. Fujimoto, Phys. Rev. B **100**, 245205 (2019).
[59] K. Moors, A. A. Zyuzin, A. Y. Zyuzin, R. P. Tiwari, and T. L. Schmidt, Phys. Rev. B **99**, 041116(R) (2019).
[60] Z. Yang and J. Hu, Phys. Rev. B **99**, 081102(R) (2019).
[61] L. Jin and Z. Song, Phys. Rev. B **99**, 081103(R) (2019).
[62] L. Zhou, Phys. Rev. B **101**, 014306 (2020).
[63] T. Ohashi, S. Kobayashi, and Y. Kawaguchi, Phys. Rev. A **101**, 013625 (2020).
[64] S. Longhi, Phys. Rev. Lett. **124**, 066602 (2020).
[65] Z. Yang, C.-K. Chiu, C. Fang, and J. Hu, Phys. Rev. Lett. **124**, 186402 (2020).
[66] C. C. Wojcik, X.-Q. Sun, T. Bzdušek, and S. Fan, Phys. Rev. B **101**, 205417 (2020).
[67] Z. Yang, A. Schnyder, J. Hu, and C.-K. Chiu, arXiv:1912.02788.
[68] K. Kawabata and S. Ryu, arXiv:2005.00604.
[69] F. Terrier and F. K. Kunst, arXiv:2003.11042.
[70] M. N. Chernodub and A. Cortijo, Symmetry **12**, 761 (2020).
[71] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, Phys. Rev. Lett. **124**, 056802(R) (2020).
[72] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).
[73] P. Titum, E. Berg, M. S. Rudner, G. Refael, and N. H. Lindner, Phys. Rev. X **6**, 021013 (2016).
[74] J. C. Budich, Y. Hu, and P. Zoller, Phys. Rev. Lett. **118**, 105302 (2017).
[75] L. Privitera, A. Russomanno, R. Citro, and G. E. Santoro, Phys. Rev. Lett. **120**, 106601 (2018).
[76] M. M. Wauters, A. Russomanno, R. Citro, G. E. Santoro, and L. Privitera, Phys. Rev. Lett. **123**, 266601 (2019).
[77] X.-Q. Sun, M. Xiao, T. Bzdušek, S.-C. Zhang, and S. Fan, Phys. Rev. Lett. **121**, 196401 (2018).
[78] S. Higashikawa, M. Nakagawa, and M. Ueda, Phys. Rev. Lett. **123**, 066403 (2019).
[79] J. Y. Lee, J. Ahn, H. Zhou, and A. Vishwanath, Phys. Rev. Lett. **123**, 206404 (2019).
[80] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018).
[81] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X **9**, 041015 (2019).
[82] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[83] K. Zhang, Z. Yang, and C. Fang, arXiv:1910.01131.
[84] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Phys. Rev. Lett. **124**, 086801 (2020).
[85] See Sec. S2 in Supplemental Material, which includes Refs. [101–103].
[86] This result is known as the Brouwer degree in mathematics, and was shown in Ref.[77] in a different manner.
[87] See Sec. S2 in Supplemental Material.
[88] See Sec. S5 in Supplemental Material.
[89] See Sec. S6 in Supplemental Material.
[90] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[91] See Sec. S1 in Supplemental Material.
[92] For the precise definition of the topological charge ν_α and the proof of Theorem 3', see Sec. S2 in Supplemental Material.
[93] See Sec. S4 in Supplemental Material.
[94] See Sec. S7 in Supplemental Material.
[95] See Sec. S6 in Supplemental Material.
[96] See Sec. S6 in Supplemental Material.
[97] A. Cerjan, S. Huang, M. Wang, K. P. Chen, Y. Chong, and M. C. Rechtsman, Nature Photonics **13**, 623 (2019).
[98] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).
[99] J. Li, A. K. Harter, J. Liu, L. de Melo, Y. N. Joglekar, and L. Luo, Nature communications **10**, 1 (2019).
[100] Y. Takasu, T. Yagami, Y. Ashida, R. Hamazaki, Y. Kuno, and Y. Takahashi, Progress of Theoretical and Experimental Physics **2020**, 12A110 (2020).
[101] M. Sato, Phys. Rev. B **79**, 214526 (2009).
[102] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **81**, 134508 (2010).
[103] M. Sato, Y. Tanaka, K. Yada, and T. Yokoyama, Phys. Rev. B **83**, 224511 (2011).