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Quantum geometry and flat band Bose-Einstein condensation

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We study the properties of a weakly interacting Bose-Einstein condensate (BEC) in a flat band lattice system by using multiband Bogoliubov theory, and discover fundamental connections to the underlying quantum geometry. In a flat band, the speed of sound and the quantum depletion of the condensate are dictated by the quantum geometry, and a finite quantum distance between the condensed and other states guarantees stability of the BEC. Our results reveal that a suitable quantum geometry allows one to reach the strong quantum correlation regime even with weak interactions.

Introduction — Geometric and topological properties of Bloch wave functions in periodic lattice systems [1, 2] - i.e. the quantum geometry – are important to describe a range of physical phenomena. Tremendous progress in the understanding of the physical relevance of concepts such as the quantum metric [3–5], Berry curvature, Chern number, and other topological invariants has been made [6–12], and experimental techniques to probe quantum geometry have been developed [13, 14]. Quantum geometric phenomena are especially striking in systems that feature dispersionless (flat) Bloch bands, where the kinetic energy is quenched and quantum states are strongly localized [15]. Due to a vanishing kinetic energy, the transport properties of a flat band are determined by the overlap between Bloch states, that is, by the quantum geometry [16]. Indeed, previous studies have shown that the superfluid density of flat band systems is determined by the Chern number, quantum metric or Berry curvature [17–19] despite the fact that effective mass of the electrons in a flat band is infinite. Recently it has been proposed [20–23] that the observed superconductivity in twisted bilayer graphene [24, 25] stems from quantum geometric properties of quasi-flat Bloch bands.

Geometric properties of quantum states are widely studied in fermionic systems but less is known about their role in bosonic systems where particles can undergo Bose-Einstein condensation (BEC). While bosonic flat band geometries have been studied experimentally [26–34], and quantum geometry is experimentally accessible in bosonic systems [13], understanding how the quantum geometry affects the physical properties of a BEC is still lacking. In this letter and in our more detailed joint work of Ref. [35], by using multiband Bogoliubov formalism, we theoretically unravel fundamental connections between a weakly-interacting BEC taking place in a multiband lattice system and the quantum geometric properties of the underlying Bloch states. Our focus is on the systems where the condensation takes place within a flat band. We show how the quantum geometry crucially determines the stability and excitation properties of a flat band BEC.

A fundamental question on flat band BEC relates to the stability of the condensate: can the bosons coherently condense to a single flat Bloch band when all the other flat band states have the same energy? As a first guess, one could think that the interaction effects renormalize the energy dispersion so that the lowest excitation band is not flat anymore, ensuring the stability of a BEC. We, however, show that one can realize a stable BEC even in the limit of *vanishing interaction strength* U. Intriguingly, a non-zero quantum distance $D(\mathbf{q})$ (defined below with $\mathbf{q}$ being the quasi-momentum and $0 \leq D(\mathbf{q}) \leq 1$) between the flat band states prevents the scenario where all the particles escape the condensate even if such excitations do not in the limit of $U \to 0$ cost any extra energy. This mechanism guarantees a stable flat band BEC. Because some of the non-condensed Bloch states can overlap with the condensed state (i.e. $D(\mathbf{q}) < 1$), in the limit of $U \to 0$ there can exists finite quantum depletion, i.e. finite density of non-condensed bosons $n_{\rm ex}$. This is in stark contrast to conventional dispersive-band BEC where $\lim_{U\to 0} n_{\rm ex} = 0$ [36, 37]. We also find that the quantum geometric origin of a stable BEC is manifested by the speed of sound c_s which turns out to be determined by the quantum metric [1, 2] at the condensed state, i.e. the second derivative of the quantum distance.

Importantly, we show that $\lim_{U\to 0} n_{\rm ex}$ is determined by the quantum geometry only and not by the total density $n_{\rm tot}$. Therefore, by decreasing the condensation density, one can increase the relative depletion of the condensate, $n_{\rm ex}/n_{\rm tot}$, even in the $U \to 0$ limit. In this way, the importance of quantum fluctuations and correlations can be significantly enhanced. We demonstrate this in our joint work [35] where we calculate the density-density correlation function to show that the quantum geometry can provide access to a regime dominated by interaction effects even with infinitesimally small U. This is highly relevant in systems where interactions are inherently small such as photon and polariton condensates.

In this letter, we consider a two-dimensional kagome lattice geometry that supports a flat band. In our joint work of Ref. [35], we provide the details of the calculations for a generic flat band system and furthermore show the results for density-density correlations and superfluid density. These two works together thus establish fundamental connections between quantum geometry and various physical properties of weakly interacting flat band condensates.

Kagome flat band model — We consider a Bose-Hubbard Hamiltonian $H = H_0 + H_{int}$ in kagome lattice [see Fig. 1(a)] whose one-particle Hamiltonian in momentum-space reads $H_0 = \sum_{\mathbf{k}} \big(c^\dagger_{\mathbf{k}\alpha}\mathcal{H}_{\alpha\beta}(\mathbf{k})c_{\mathbf{k}\beta} - \mu c^\dagger_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}\big)$ with summations over repeated indices assumed. Here, $c_{\mathbf{k}\alpha}$ annihilates a boson of momentum $\mathbf{k}$ in the αth sublattice and μ is the chemical potential. For kagome lattice there exists three sublattices and the hopping matrix $\mathcal{H}(\mathbf{k})$ is

$$\mathcal{H}(\mathbf{k}) = 2t\begin{bmatrix} 0 & \cos(k_1/2) & \cos(k_2/2) \\ \cos(k_1/2) & 0 & \cos(k_3/2) \\ \cos(k_2/2) & \cos(k_3/2) & 0 \end{bmatrix}, \qquad (1)$$

where $k_i = \mathbf{k}\cdot\mathbf{a}_i$ for $i = \{1, 2\}$ and $k_3 = k_1 - k_2$. Here $\mathbf{a}_i$ are the basis vectors [Fig. 1(a)] and $t > 0$ is the nearest-neighbour hopping. One can diagonalize $\mathcal{H}(\mathbf{k})$ as $\mathcal{H}(\mathbf{k})|u_n(\mathbf{k})\rangle = \epsilon_n(\mathbf{k})|u_n(\mathbf{k})\rangle$, where $\epsilon_n(\mathbf{k})$ ($|u_{n\mathbf{k}}\rangle$) are the eigenenergies (Bloch states) and n is the band index so that $\epsilon_1(\mathbf{k}) \le \epsilon_2(\mathbf{k}) \le \epsilon_3(\mathbf{k})$. The lowest Bloch band is strictly flat, i.e. $\epsilon_1(\mathbf{k}) = -2t$, see Fig. 1(b).

The interaction Hamiltonian is $H_{int} = \frac{U}{2N}\sum_{\alpha\mathbf{k},\mathbf{k}',\mathbf{q}} c^\dagger_{\mathbf{k}\alpha}c_{\mathbf{k}-\mathbf{q}\alpha}c^\dagger_{\mathbf{k}'\alpha}c_{\mathbf{k}'+\mathbf{q}\alpha}$, where N is the number of unit cells and $U > 0$ describes the repulsive on-site interaction. Because the lowest band is flat, it is the interaction term that determines the momentum $\mathbf{k}_c$ and Bloch state $|\phi_0\rangle \equiv |u_1(\mathbf{k}_c)\rangle$ in which the BEC takes place [38]. Via a mean-field analysis [37, 38] it is shown that for kagome lattice the condensation takes place in one of the Dirac points, e.g. in $\mathbf{k}_c = [4\pi/3, 0]$ [black dot in Fig. 1(b)] with $|\phi_0\rangle = [-1, -1, 1]^T$. For this Bloch state the particle density is distributed uniformly among all three sublattices so that the repulsive Hubbard interaction is minimized [38].

To analyze the stability and excitation properties of BEC, we utilize the multiband Bogoliubov approximation (details are provided in Ref. [35]) where the bosonic operators for the condensate are treated as complex numbers, i.e. we write $c_{\mathbf{k}_c\alpha} = \sqrt{Nn_0}\langle\alpha|\phi_0\rangle$, where n_0 is the number of condensed bosons per unit cell and $\langle\alpha|\phi_0\rangle$ is the projection of $|\phi_0\rangle$ to the αth sublattice. In the Bogoliubov theory, one considers only the interaction terms that are quadratic in fluctuations $c_{\mathbf{k}\alpha}$ and $c^\dagger_{\mathbf{k}\alpha}$ with $\mathbf{k} \neq \mathbf{k}_c$. The total Hamiltonian is then $H = E_c + H_B$, where E_c is a constant giving the ground energy of the condensate, and the Bogoliubov Hamiltonian H_B describes the fluctuations of the condensate:

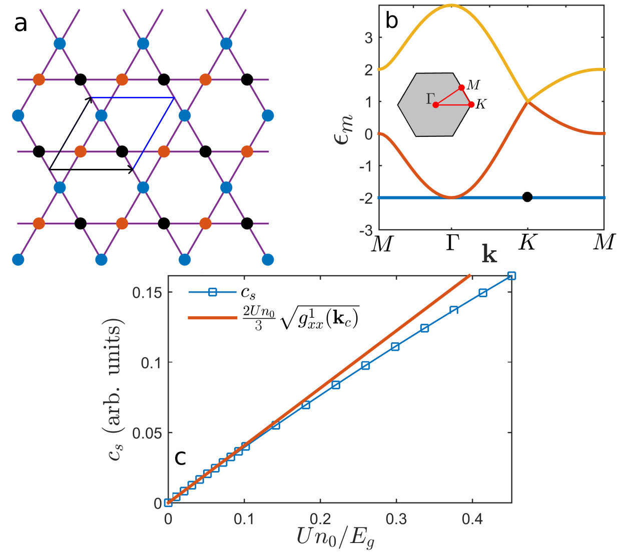

FIG. 1. (a) Kagome lattice geometry. The unit cell is shown as a blue parallelogram and black arros are the basis vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Purple lines depict NN hopping terms of strength t. (b) Bloch bands of the kagome lattice with $t = 1$ along the path connecting the high-symmetry points shown in the inset. The lowest band is strictly flat. The black dot marks the Dirac point $\mathbf{k} = [4\pi/3, 0]$ in which BEC can take place. (c) Speed of sound c_s for the kagome flat band BEC as a function of U. Total density was chosen to be $n_{\rm tot} = 3$, i.e. one particle per lattice site. We also show the weak-coupling result of Eq. (5) as a solid line. The energy scale $E_g = 3t$ is the energy gap from the flat band to the dispersive bands at $\mathbf{k}_c$.

$$H_B = \frac{1}{2}\sum_{\mathbf{k}}{}' \Psi^\dagger_{\mathbf{k}}\mathcal{H}_B(\mathbf{k})\Psi_{\mathbf{k}}, \qquad (2)$$

where $\mathcal{H}_B(\mathbf{k})$ is a 6×6 matrix given by

$$\begin{aligned}
\mathcal{H}_B(\mathbf{k}) &= \begin{bmatrix} \mathcal{H}(\mathbf{k}) - \mu_{\rm eff} & \Delta \\ \Delta^* & \mathcal{H}^*(2\mathbf{k}_c - \mathbf{k}) - \mu_{\rm eff}, \end{bmatrix}, \\
\Psi_{\mathbf{k}} &= [c_{\mathbf{k}1}, c_{\mathbf{k}2}, c_{\mathbf{k}3}, c^\dagger_{2\mathbf{k}_c-\mathbf{k}1}, c^\dagger_{2\mathbf{k}_c-\mathbf{k}2}, c^\dagger_{2\mathbf{k}_c-\mathbf{k}3}]^T, \\
[\Delta]_{\alpha\beta} &= \delta_{\alpha,\beta}Un_0/3, \\
[\mu_{\rm eff}]_{\alpha\beta} &= (\epsilon_0 - \frac{Un_0}{3})\delta_{\alpha,\beta}. \qquad (3)
\end{aligned}$$

The primed sum in Eq. (2) includes the momenta for non-condensed states only, i.e. $\mathbf{k} \neq \mathbf{k}_c$ and $2\mathbf{k}_c - \mathbf{k} \neq \mathbf{k}_c$.

The excitation energies of the BEC can be accessed by diagonalizing $L(\mathbf{k}) \equiv \sigma_z\mathcal{H}_B(\mathbf{k})$, where σ_z is the Pauli matrix acting in the particle-hole space [39]. We then obtain Bogoliubov bands of the energies $E_3(\mathbf{k}) \ge E_2(\mathbf{k}) \ge E_1(\mathbf{k}) \ge 0 \ge -E_1(2\mathbf{k}_c-\mathbf{k}) \ge -E_2(2\mathbf{k}_c-\mathbf{k}) \ge -E_3(2\mathbf{k}_c-\mathbf{k})$. Positive (negative) energies describe quasi-particle (-hole) excitations and the corresponding quasi-particle (-hole) states are labelled as $|\psi^+_m(\mathbf{k})\rangle$ ($|\psi^-_m(\mathbf{k})\rangle$). The lowest quasi-particle energy band becomes gapless at $\mathbf{k}_c$, i.e. $E_1(\mathbf{k} \to \mathbf{k}_c) = 0$, which corresponds to the Goldstone mode emerging from the spontaneous gauge $U(1)$ symmetry breaking of the complex phase of the BEC wavefunction [36, 37].

Speed of sound of kagome flat band BEC— As the speed of sound c_s for a BEC is given by the slope of the gapless Goldstone mode $E_1(\mathbf{k})$ at $\mathbf{k}_c$, we write $\mathbf{k} = \mathbf{k}_c + \mathbf{q}$, where $\mathbf{q} << 1$. We then unitary transform $L(\mathbf{k})$ to the Bloch band basis and discard the dispersive bands of freedom to obtain the 2×2 matrix $L_p(\mathbf{k})$ projected to the flat band space:

$$L_p(\mathbf{k}) = \frac{Un_0}{3} \begin{bmatrix} 1 & \alpha(\mathbf{q}) \\ -\alpha^*(\mathbf{q}) & -1 \end{bmatrix} \tag{4}$$

for $\mathbf{q} \to 0$. Here, $\alpha(\mathbf{q}) \equiv \langle u_1(\mathbf{k}_c + \mathbf{q}) | u_1(\mathbf{k}_c - \mathbf{q}) \rangle$. Diagonalizing (4), we find the Goldstone mode as $E_1(\mathbf{k}_c + \mathbf{q}) = \frac{Un_0}{3} D(\mathbf{q})$, where $D(\mathbf{q}) = \sqrt{1 - |\alpha(\mathbf{q})|^2}$ is the Hilbert-Schmidt quantum distance [40] which for fermionic flat band systems was recently shown to dictate the spread of the Landau levels [41]. By definition, $0 \le D(\mathbf{q}) \le 1$. We can immediately see that non-zero $D(\mathbf{q})$ is required to have finite speed of sound c_s.

By Taylor expanding the Bloch states up to second order in $\mathbf{q}$, one finds for c_s

$$c_s = \frac{2Un_0}{3} \sqrt{g^1(\mathbf{k}_c)}, \tag{5}$$

where the quantity inside the square root is called *quantum metric* and defined as [1]

$$g^n_{\mu\nu}(\mathbf{k}) = \mathrm{Re}\Big[\langle \partial_\mu u_n(\mathbf{k}) | \Big(1 - |u_n(\mathbf{k})\rangle\langle u_n(\mathbf{k})| \Big) | \partial_\nu u_n(\mathbf{k}) \rangle \Big]. \tag{6}$$

with the notation $\partial_\mu = \frac{\partial}{\partial k_\mu}$. In case of kagome lattice we have $g^1_{xx}(\mathbf{k}_c) = g^1_{yy}(\mathbf{k}_c) \equiv g^1(\mathbf{k}_c)$ and $g^1_{xy}(\mathbf{k}_c) = g^1_{yx}(\mathbf{k}_c) = 0$. For anisotropic systems (for derivation see [35]), $c_s(\theta_\mathbf{q}) = \frac{2Un_0}{M} \sqrt{\hat{\mathbf{e}}^T_\mathbf{q} g^1(\mathbf{k}_c) \hat{\mathbf{e}}_\mathbf{q}}$, where $\hat{\mathbf{e}}_\mathbf{q} = \mathbf{q}/|\mathbf{q}|$, $\tan\theta_\mathbf{q} = q_y/q_x$, $[g^1]_{\mu\nu} = g^1_{\mu\nu}$, and M the number of orbitals.

A remarkable consequence of Eq. (5) is that a *finite quantum metric of the condensed state* guarantees finite c_s – and thus possibility for superfluidity – even if the condensation takes place within a strictly flat band. Conversely, by measuring the speed of sound of a flat band condensate, one can extract the quantum metric at the condensation point $\mathbf{k}_c$. This should be compared to fermionic systems, where flat band superfluidity is guaranteed by finite Chern numbers or *integrals* of the quantum metric over the first Brillouin zone (BZ) [17, 18, 42]. Moreover, in Ref. [43] it was shown that for a fermionic two-body problem, the effective mass m^C_{eff} of the Cooper pairs within a flat band is inversely proportional to the the quantum metric integrated over the whole BZ. Via the usual dependence of $c_s \propto 1/\sqrt{m^C_{\text{eff}}}$, one could anticipate a similar relationship between c_s and quantum geometry. However, the result presented here is different: only the quantum metric of the condensed Bloch state is needed, not an integral over the whole BZ. Furthermore, in Ref. [44] the speed of sound was analyzed for spin-orbit coupled Fermi gases: the Goldstone mode was shown to depend on the momentum-space integrals in which the quantum metric is convoluted with other non-geometric terms. Thus, the significance of quantum geometry was obscured due to the presence of more prominent non-geometric contributions. In contrast to this, we have shown that the quantum geometry plays a dominant role for determining the speed of sound in a flat band BEC.

In Fig. 1(c) we plot c_s for the kagome flat band condensate as a function of U by numerically extracting the speed of sound from the full Bogoliubov Hamiltonian (2). Moreover, we also plot the weak-coupling result of Eq. (5). The agreement at small U is excellent.

Note that we find linear Goldstone modes for flat band condensates. In contrast, in Refs. [45, 46] the sound mode for spin-orbit (SO) coupled BEC is quadratic in the direction of dispersionless one-dimensional flat band. This is due to the inter-sublattice interaction term, induced by the SO coupling, and thus does not contradict our results as we only consider intra-sublattice interaction.

Excitation density — An important question related to the stability of flat band BEC is how the excitation density n_{ex} behaves, in particular when $U \to 0$. For the usual dispersive band BEC, one has $\lim_{U\to 0} n_{ex} = 0$ [36]. However, for a strictly flat band, the $U \to 0$ limit of Eq. (4) implies that the Goldstone modes becomes flat. One could then conclude that the condensate becomes unstable as exciting particles out of the condensate does not cost energy. We now show that this is not the case as the quantum distance ensures the stability of a flat band BEC in the non-interacting limit.

The expression for n_{ex} reads [35]:

$$n_{ex} = \frac{1}{N} \sum_{\mathbf{k}m}{}' \langle c^\dagger_{\mathbf{k}m} c_{\mathbf{k}m} \rangle = \frac{1}{2N} \sum_{\mathbf{k}m}{}' [-1 + \langle \psi^-_m(\mathbf{k}) | \psi^-_m(\mathbf{k}) \rangle]$$
$$\equiv \frac{1}{N} \sum_{\mathbf{k}}{}' n_{ex}(\mathbf{k}), \tag{7}$$

where $c^\dagger_{\mathbf{k}m}$ creates a boson in the Bloch band m with momentum $\mathbf{k}$. We again consider the projected $L_p(\mathbf{k})$ of Eq. (4) and neglect the higher bands as we are considering the $U \to 0$ limit. By diagonalizing Eq. (4), one obtains

$$\lim_{U\to 0} \langle c^\dagger_{\mathbf{k}1} c_{\mathbf{k}1} \rangle = \frac{1 - D(\mathbf{q})}{2D(\mathbf{q})}, \tag{8}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}_c$. Equation (8) provides a remarkable link between the density of non-condensed bosons, n_{ex}, and the quantum distance $D(\mathbf{q})$. We see that $n_{ex}(\mathbf{k})$ diverges for $D(\mathbf{q}) = 0$, implying the breakdown of the Bogoliubov theory. This is intuitively easy to understand as $D(\mathbf{q}) = 0$ indicates the perfect overlap between the condensed state $|\phi_0\rangle$ and other flat band condensates, i.e.

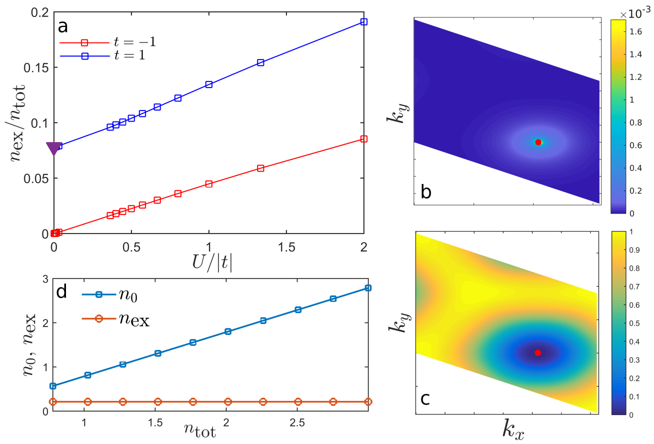

FIG. 2. (a) Excitation fraction $n_{\rm ex}/n_{\rm tot}$ at $n_{\rm tot} = 3$ as a function of U for the flat band BEC ($t = 1$) and dispersive band BEC ($t = -1$). Purple triangle depicts the analytical result of Eq. (8) integrated over the first BZ. (b) Momentum dependence of $n_{\rm ex}(\mathbf{k})$ at $Un_0/E_g = 5.13\times10^{-4}$. (c) Quantum distance $D(\mathbf{q})$ as a function of $\mathbf{k} = \mathbf{k}_c + \mathbf{q}$. In (b) and (c) the red dot depicts the momentum $\mathbf{k}_c = [4\pi/3, 0]$ of the flat band BEC. (d) Densities n_0 and $n_{\rm ex}$ as a function of $n_{\rm tot}$ for the flat band condensation at $U = |t|/1800$. Excitation density $n_{\rm ex}$ remains constant, as it is determined by the quantum distance.

$\langle u_1(\mathbf{k}_c+\mathbf{q})|\phi_0\rangle = 1$ [35]. On the other hand, finite $D(\mathbf{q})$ sets the limit for the excitation density, allowing a stable flat band BEC at arbitrarily small interaction values. The Eqs. 5 and 8 are valid for any flat band with real Bloch functions $|u_1\rangle$; the relation to quantum geometry is similar also for arbitrary wavefunctions although the formula are slightly more complicated [35].

In Fig. 2(a) we present $n_{\rm ex}/n_{\rm tot}$, where $n_{\rm tot}$ is the total density, for the kagome lattice as a function of U. In addition to the flat band BEC, we also provide the result for dispersive band BEC. Condensation to one of the dispersive bands of the kagome lattice can be achieved by changing the sign of the NN hopping term, i.e. $t < 0$. This choice flips the Bloch band structure such that the dispersive band is the lowest band for which the condensation takes place at $\mathbf{k}_c = 0$. From Fig. 2(a) we see that $\lim_{U\to0} n_{\rm ex} = 0$ for the dispersive band BEC, as expected. However, for the flat band BEC, the non-interacting asymptote of $n_{\rm ex}$ is given by Eq. (8) integrated over the first BZ. This clearly illustrates that the quantum distance determines the excitation density and protects the stability of flat band BEC in the weak-coupling limit.

In Figs. 2(b)-(c) we show $n_{\rm ex}(\mathbf{k})$ for small U and $D(\mathbf{q} = \mathbf{k} - \mathbf{k}_c)$, respectively, as a function of momentum $\mathbf{k}$ across the first BZ. We see that indeed the quantum distance is imprinted to the momentum distribution of excitation density. Importantly, $n_{ex}(\mathbf{k})$ is the Fourier transform of the following first-order spatial coherence function: $\tilde{g}^{(1)}(j) \equiv \frac{1}{N}\sum_{i\alpha}\langle \delta c^\dagger_{i+j\alpha}\delta c_{i\alpha}\rangle$, where $\delta c_{i\alpha}$ annihilates a non-condensed boson in the ith unit cell and αth sublattice. Thus, first order coherence is fundamentally determined by quantum geometry, and measuring it provides a direct access to the quantum distance.

Surprisingly, the number of atoms excited out of the condensate for a vanishing interaction strength, $\lim_{U\to0} n_{\rm ex}$ given by Eq. (8), does not depend on the total density $n_{\rm tot}$ but is solely determined by the quantum geometry of the flat band. This implies that by decreasing $n_{\rm tot}$, the excitation fraction $n_{\rm ex}/n_{\rm tot}$ of the flat band BEC and the role of the interactions can be made large even at $U \to 0$. We demonstrate this in Fig. 2(d) by presenting n_0 and $n_{\rm ex}$ as a function of $n_{\rm tot}$ for small U. We see that $n_{\rm ex}$ remains constant, consistent with Eq. (8), whereas n_0 decreases with decreasing $n_{\rm tot}$, implying that *at the low density regime, the condensate depletion and interaction effects can be made significant, even at the non-interacting limit of* $U \to 0$. The validity of the Bogoliubov theory for large $n_{\rm ex}/n_{\rm tot}$ ratios is addressed in Ref. [35].

Discussion—By using Bogoliubov theory, we have studied fundamental connections between the excitations of a BEC and quantum geometry of the Bloch states. The properties of the flat band BEC are dictated by the underlying quantum geometry and are strikingly different from the dispersive band case. The speed of sound c_s is proportional to the quantum metric of the condensed state, and the excitation density $n_{\rm ex}$ does not vanish with interactions as in case of a dispersive band BEC. In contrast, it obtains a finite value given by the quantum distance between the Bloch states. These results have a common origin; the quantum metric is the small momentum limit of the quantum distance, meaning that long-wavelength physical quantities such as c_s and low energy excitations depend on the quantum metric, while those that involve higher momenta, e.g. $n_{\rm ex}$, are governed by the quantum distance. While the quantum distance has been previously connected to Landau level spreading in non-interacting flat band models [41], our results are among the first to unravel the deep connections between the quantum distance and relevant physical quantities in an *interacting many-body quantum system.*

Our predictions should be readily observable. The linear dependence of the speed of sound in a flat band BEC on the interaction strength is in stark contrast to the usual quadratic dependence of a dispersive band BEC and can be detected by tuning the interaction for example in experimental ultracold gas settings [47, 48]. Furthermore, as the excitation fraction is the Fourier transform of the first order coherence, measurement of the latter gives access to the quantum geometry effects. In addition to ultracold systems [26, 49], flat band condensates can be also created in polaritonic platforms [30–34] which therefore could be used to study quatum geometric effects discussed here.

Enhancing interaction effects has been a key motivation for studying flat band systems. The present work, alongside the accompanying study of Ref. [35], shows that indeed this promise is realized in the context of BEC. Even more importantly, we show that these effects are controlled by the non-trivial quantum geometry. Therefore, bosons in a flat band provide a highly promising platform to explore beyond mean-field physics and effects of the quantum geometry, as well as to realize strong correlations even in the weak interaction limit. This is particularly important for photon and polariton systems where effective interactions in general are small. The results presented here are thus relevant for efforts of realizing strongly correlated photons, important for both fundamental research and opto-electronic components. In the future, it would be interesting to explore how quantum geometry affects the spatial and temporal dependence of the first and second order correlation functions, the physics of the strong interaction limit [50], and driven-dissipative BECs.

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