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The Lieb-Robinson light cone for power-law interactions

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The Lieb-Robinson theorem states that information propagates with a finite velocity in quantum systems on a lattice with nearest-neighbor interactions. What are the speed limits on information propagation in quantum systems with power-law interactions, which decay as $1/r^\alpha$ at distance r? Here, we present a definitive answer to this question for all exponents $\alpha > 2d$ and all spatial dimensions d. Schematically, information takes time at least $r^{\min\{1,\alpha-2d\}}$ to propagate a distance r. As recent state transfer protocols saturate this bound, our work closes a decades-long hunt for optimal Lieb-Robinson bounds on quantum information dynamics with power-law interactions.

Over a century ago, Einstein realized that there is a speed limit to information propagation. If no physical object or signal can travel faster than light, then the speed of light itself must constrain the dynamics of quantum information and entanglement. In ordinary quantum systems, however, *emergent* speed limits can arise that place more stringent restrictions on information propagation than does the speed of light. For example, in quantum spin systems with nearest-neighbor interactions on a lattice, Lieb and Robinson proved in 1972 that there is a *finite velocity* of information propagation [1].

Of course, most non-relativistic physical systems realized in experiments include long-range interactions such as the Coulomb interaction, the dipole-dipole interaction, or the van-der-Waals interaction. Each of these decays with distance as a power law $1/r^\alpha$ for some exponent α. What is the fundamental speed limit on the propagation of quantum information in these systems?

Despite the importance of this question in designing and constraining the operation of future quantum technologies [2–6], bounding information propagation in systems with power-law interactions has been a notoriously challenging mathematical physics problem. In 2005, Hastings and Koma [7] showed that it takes a time $t \gtrsim \log r$ to send information a distance r, for all $\alpha > d$, where d is the dimension of the lattice. By analogy to Einstein's relativity, we say that there is at least a "logarithmic light cone" for such power-law interactions. However, it was suspected that this bound was far from tight, and ten years later it was shown that $t \gtrsim r^\gamma$, for an exponent $0 < \gamma < 1$ when $\alpha > 2d$ [8–10]. In 2019, Chen and Lucas [11] proved the existence of a linear light cone ($t \gtrsim r$) for all $\alpha > 3$ in $d = 1$; Kuwahara and Saito [12] later generalized this result to higher dimensions, finding a linear light cone for all $\alpha > 2d+1$. These recent results prove that power-law interactions are, for all practical purposes, entirely local for sufficiently large α.

A natural question is then how small α must be in order to *break* a linear light cone. Fast state-transfer and entanglement-generation protocols developed in the past year [12–15] have ultimately demonstrated that the time t required to send information a distance r obeys $t \lesssim r^{\min(\alpha-2d,1)}$ for any $\alpha > 2d$ and $t \lesssim r^{o(1)}$ for $\alpha < d$, where $o(1)$ is an arbitrarily small constant. Combining all best known results in the literature leads to the diagram shown in Fig. 1, which compares known information-transfer protocols to corresponding Lieb-Robinson bounds.

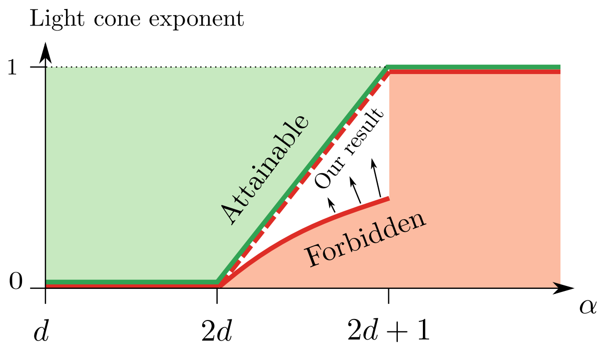

FIG. 1. The gap in the Lieb-Robinson literature in $d > 1$ dimensions. The red solid lines represent the exponent γ of the Lieb-Robinson light cone $t \gtrsim r^\gamma$ in literature. The green solid lines correspond to the light cone exponents of best-known information-propagating protocols. Accordingly, the green region corresponds to attainable light cone exponents, whereas the red region is forbidden by the known bounds. Our result (red dashed line) closes the gap in our understanding of the Lieb-Robinson light cone.

In this Letter, we complete this extensive literature on Lieb-Robinson bounds for power-law interactions [7–22], by proving that quantum information is contained within the Lieb-Robinson light cone $t \gtrsim r^{\min(\alpha-2d-\varepsilon,1)}$, for any $\varepsilon > 0$. This result closes the remaining gap, up to subalgebraic corrections, between bounds and protocols in Fig. 1, and concludes the fifteen-year quest to understand the fundamental speed limit on quantum information in the presence of power-law interactions. We sketch the proof of the result in the main text and refer readers to the Supplemental Material (SM) [23] for a rigorous treatment.

Main result.—We consider a d-dimensional regular lattice Λ, a finite-level system at every site of the lattice, and a two-body power-law Hamiltonian $H(t)$ with an exponent α supported on the lattice. Specifically, we assume $H(t) = \sum_{i,j\in\Lambda} h_{ij}(t)$ is a sum of two-body terms h_{ij} supported on sites i, j such that $\|h_{ij}(t)\| \leq 1/\mathrm{dist}(i,j)^\alpha$ for all $i \neq j$, where $\|\cdot\|$ is the operator norm and $\mathrm{dist}(i,j)$ is the distance between i, j. In the following discussion, we assume Λ is a hypercubic lattice of qubits for simplicity.

We use $\mathcal{L}$ to denote the Liouvillian corresponding to the Heisenberg evolution under Hamiltonian H, i.e. $\mathcal{L}\,|O) \equiv |i[H,O])$ for any operator O, and use $e^{\mathcal{L}t}\,|O) \equiv |O(t))$ to denote the time-evolved version of the operator O. We also use $\mathbb{P}_r^{(i)}\,|O)$ to denote an operator constructed from O by decomposing O into a sum of Pauli strings and removing strings that are supported entirely within a ball of radius r from i. Colloquially speaking, $\mathbb{P}_r^{(i)}\,|O)$ is the component of O that has non-trivial support on sites a distance at least r from site i. If i is the origin of the lattice, we drop the superscript i and simply write $\mathbb{P}_r$ for brevity.

Given a unit-norm operator O initially supported at the origin, our main result is a bound on how much O spreads to a distance r and beyond under the evolution $e^{\mathcal{L}t}$:

Theorem 1. For any $\alpha \in (2d, 2d+1)$ and an arbitrarily small $\varepsilon > 0$, there exist constants $c, C \geq 0$ such that

$$\left\|\mathbb{P}_r e^{\mathcal{L}t}\,|O)\right\| \leq C\left(\frac{t}{r^{\alpha-2d-\varepsilon}}\right)^{\frac{\alpha-d}{\alpha-2d}-\frac{\varepsilon}{2}} \tag{1}$$

holds for all $1 \leq t \leq c r^{\alpha-2d-\varepsilon}$.

Because $\left\|\mathbb{P}_r e^{\mathcal{L}t}\,|O)\right\|$ can be both upper- and lower-bounded by linear functions of $\sup_A \left\|\left[A, e^{\mathcal{L}t}O\right]\right\|$, where A is a unit-norm operator supported at least a distance r from O, Eq. (1) is equivalent to a bound on the unequal-time commutators commonly used in the Lieb-Robinson literature.

For $\alpha \in (2d, 2d+1)$, by setting the left-hand side of Eq. (1) to a constant, Theorem 1 implies the light cone $t \gtrsim r^{\alpha-2d-\varepsilon}$ for some ε that can be made arbitrarily small. Note that our definition does not require $\|h_{ij}\|$ to decay exactly as $1/\mathrm{dist}(i,j)^\alpha$; it may actually decay faster than $1/\mathrm{dist}(i,j)^\alpha$ and still satisfy the condition of a power-law interaction with an exponent α. Therefore, for $\alpha \geq 2d+1$ and power-law Hamiltonians $H = \sum_{ij} h_{ij}$ satisfying $\|h_{ij}\| \leq 1/\mathrm{dist}(i,j)^\alpha < 1/\mathrm{dist}(i,j)^{2d+1-\varepsilon}$, Theorem 1 implies a linear light cone $t \gtrsim r^{1-2\varepsilon}$.

Sketch of proof.— We sketch the proof of Theorem 1, denoting time and distance in the theorem by T and R to distinguish with time t and distance r in the intermediate steps of the proof. For simplicity, we assume here that the lattice diameter is $\mathcal{O}(R)$. Similar to recent works [8, 11, 12], we group the interactions of the Hamiltonian by their ranges, prove a bound for short-range interactions, and recursively add longer-range interactions to the Hamiltonian. The key difference is in how we group the interactions.

Specifically, motivated by the recent optimal protocol [15] and the expected bound in Theorem 1 (see the SM [23] for more details), we choose $\ell_k \equiv L^k$ for $k = 1, \ldots, n$, where L, n are to-be-determined functions of T, R, and α. We use H_k to denote those terms of H with range at most ℓ_k and use $\mathcal{L}_k \equiv i[H_k, \cdot]$ to denote the corresponding Liouvillian. We start with the standard Lieb-Robinson bound for H_1 [1]:

$$\left\|\mathbb{P}_r e^{\mathcal{L}_1 t}\,|O)\right\| \lesssim e^{\frac{v_1 t - r}{\ell_1}}, \tag{2}$$

where $v_1 \propto \ell_1 = L$ is the rescaled Lieb-Robinson velocity, and prove a bound for H_2 by adding $V_2 \equiv H_2 - H_1$, i.e., interactions of range between ℓ_1 and ℓ_2, to the Hamiltonian H_1.

For that, we employ a technique introduced in Ref. [8] and move into the interaction picture of H_1 so that we can decompose the evolution $e^{\mathcal{L}_2 t} = e^{\mathcal{L}_{2,I}t}e^{\mathcal{L}_1 t}$ into two consecutive evolutions, where $e^{\mathcal{L}_{2,I}t}$ is the evolution under $V_{2,I} \equiv e^{\mathcal{L}_1 t}V_2$. Loosely speaking, the light cone induced by H_2 will be a "sum" of the light cones induced by H_1 and $V_{2,I}$ individually (see the SM [23] for a proof.) With the light cone of H_1 given by Eq. (2), our task is to find the light cone of $V_{2,I}$.

For this purpose, we consider the structure of $V_{2,I}$ and show that, with a suitable rescaling of the lattice, the interactions in $V_{2,I}$ decay exponentially with distance. We then obtain the light cone of $V_{2,I}$ using the standard Lieb-Robinson bound on the rescaled lattice. Specifically, we divide the lattice into non-overlapping hypercubes of length ℓ_2 (see Fig. 2). Given x, y as the centers of two hypercubes, we define $\mathrm{dist}(x,y)/\ell_2$ to be the rescaled distance between the hypercubes. We shall estimate the strength of the interaction between hypercubes under the Hamiltonian $V_{2,I}$.

We first consider the case $t = 0$ so that $V_{2,I} = V_2$. Because each interaction in V_2 has range at most ℓ_2, no interaction h_{ij} is supported on two distinct hypercubes unless they are nearest neighbors. Therefore, only nearest-neighboring hypercubes may interact under $V_{2,I} = V_2$.

The case $t > 0$ is slightly more complicated. The support of an interaction h_{ij} in V_2 may expand under $e^{\mathcal{L}_1 t}$, and, hence, non-nearest-neighboring hypercubes may interact with each other. However, due to Eq. (2), the support of $e^{\mathcal{L}_1 t}h_{ij}$ would largely remain inside the balls of radius $v_1 t$ around i, j. The interactions between hypercubes are exponentially suppressed with distance by Eq. (2). Therefore, the system of hypercubes would interact via a nearly finite-range interaction (see Fig. 2).

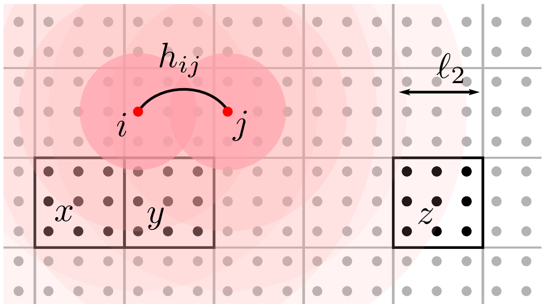

FIG. 2. We study the structure of $V_{2,I}$ by dividing the lattice into hypercubes of length ℓ_2 (labeled by x, y, and z for example). In the interaction picture, how much each $e^{\mathcal{L}_1 t} h_{ij}$ contributes to the pair-wise "effective interaction" between two hypercubes depends on how strongly the support of $e^{\mathcal{L}_1 t} h_{ij}$ (represented by the shaded area) overlaps with the hypercubes. Because of the bound in Eq. (2), the evolved operator $e^{\mathcal{L}_1 t} h_{ij}$ is largely confined to the light cones induced by $\mathcal{L}_1$ around i and j (the smallest disks around i and j). The component of $e^{\mathcal{L}_1 t} h_{ij}$ supported outside this light cone is exponentially suppressed with distance (represented by lighter shade). Consequently, the effective interaction between the hypercubes x and z is exponentially smaller than the one between x and y.

To apply the standard Lieb-Robinson bound for this system of hypercubes, we estimate the maximum effective interaction between any pair of nearest-neighboring hypercubes centered on x, y. In particular, assuming $v_1 t \leq \ell_2$, the primary contributions to such an interaction come from $\propto \ell_2^d \times \ell_2^d = \ell_2^{2d}$ interaction terms $e^{\mathcal{L}_1 t} h_{ij}$ whose light cones under H_1 may overlap with the hypercubes. Because each interaction h_{ij} has norm at most $1/\ell_1^\alpha$ by our assumption, the total contribution to the interactions between the cubes x, y is $\mathcal{O}(\ell_2^{2d}/\ell_1^\alpha)$. Applying the standard finite-range Lieb-Robinson bound on the system of hypercubes, where the maximum energy per interaction is $\mathcal{O}(\ell_2^{2d}/\ell_1^\alpha)$ and the distance is rescaled by a factor ℓ_2, we obtain the bound for the evolution under $V_{2,I}$:

$$\left\|\mathbb{P}_r e^{\mathcal{L}_I t} |O)\right\| \lesssim \exp\left(\mathcal{O}\left(\frac{\ell_2^{2d}}{\ell_1^\alpha}\right) t - \frac{r}{\ell_2}\right) \equiv e^{\frac{\Delta v t - r}{\ell_2}}, \quad (3)$$

where $\Delta v = \mathcal{O}(\ell_2^{2d+1}/\ell_1^\alpha)$.

After getting the light cone for the evolution under $V_{2,I}$, we now combine it with the evolution under H_1 to obtain the light cone of H_2. Intuitively, the evolutions under H_1 and $V_{2,I}$ for time t may each grow the support radius of an operator by $v_1 t$ and $\Delta v t$ respectively. Therefore, one would expect an operator evolved under H_1 and $V_{2,I}$ consecutively, each for time t, may have the support radius at most $(v_1 + \Delta v)t$. In the SM [23], we show that

$$\left\|\mathbb{P}_r e^{\mathcal{L}_2 t} |O)\right\| = \left\|\mathbb{P}_r e^{\mathcal{L}_{2,I} t} e^{\mathcal{L}_1 t} |O)\right\| \lesssim e^{\frac{v_2 t - r}{\ell_2}}, \quad (4)$$

where

$$v_2 \propto \log(r) v_1 + \Delta v = \log(r) v_1 + \frac{\ell_2^{2d+1}}{\ell_1^\alpha}. \quad (5)$$

The additional factor of $\log(r)$ (compared to our intuition) comes from the enhancement to the operator spreading due to the increased support size after the first evolution $e^{\mathcal{L}_1 t}$.

Up to this point, we have used the bound Eq. (2) for H_1 to prove a bound for H_2 [Eq. (4)], which has the same form. Repeating this process, we arrive at similar bounds for H_k $(k = 3, 4, \ldots, n)$:

$$\left\|\mathbb{P}_r e^{\mathcal{L}_k t} |O)\right\| \lesssim e^{\frac{v_k t - r}{\ell_k}}, \quad (6)$$

where the velocity v_k is defined iteratively:

$$v_k \propto \log(r) v_{k-1} + \frac{\ell_k^{2d+1}}{\ell_{k-1}^\alpha}. \quad (7)$$

Increasing k makes the bound in Eq. (6) applicable for longer and longer interactions. However, doing so also increases ℓ_k, resulting in weaker and weaker bounds. In particular, if $\ell_k > R$, Eq. (6) becomes trivial at the final time T and distance R, even when $T \leq R/v_k$. Therefore, we stop the iteration at $k = n$ such that ℓ_n is slightly smaller than R. Specifically, we choose n such that $\ell_n = L^n = R/\chi(T, R)$, where $\chi(T, R) > 1$ is a function of T, R. For $v_n T \leq R/2$ and at the final time T and distance R, the right-hand side of Eq. (6) becomes

$$e^{\frac{v_n T - R}{\ell_n}} \lesssim e^{-\frac{R}{2\ell_n}} \lesssim e^{-\frac{1}{2}\chi(T,R)} \lesssim \frac{1}{\chi(T, R)^\omega}, \quad (8)$$

where we upper-bound an exponentially decaying function of $\chi(T, R)$ by a power-law decaying function of $\chi(T, R)$ with an exponent $\omega > 0$. Choosing $\chi(T, R) = (R^{\alpha-2d}/T)^\zeta$, where $\zeta > 0$ is an arbitrarily small constant, and $\omega = \frac{\alpha - d}{\zeta(\alpha - 2d)}$, we obtain the desired bound

$$\left\|\mathbb{P}_R e^{\mathcal{L}_n T} |O)\right\| \lesssim \left(\frac{T}{R^{\alpha-2d}}\right)^{\frac{\alpha-d}{\alpha-2d}}. \quad (9)$$

Note that Eq. (9) only holds for $T \leq R/2v_n$. To maximize the range of validity of Eq. (9), we aim to choose L such that v_n is as small as possible. Without the second term in Eq. (7), we would expect v_k to increase by a factor of $\log r$ between iterations. Meanwhile, given $\ell_k = L^k$, the second term in Eq. (7) also increases by a factor $L^{2d+1-\alpha}$ in every iteration. Choosing $L^{2d+1-\alpha} \propto \log R$ so that the two terms in Eq. (7) have roughly equal contributions to v_k, we expect

$$v_n \propto (\log R)^n \propto L^{n(2d+1-\alpha)} = \left(\frac{R}{\chi(T, R)}\right)^{2d+1-\alpha} \quad (10)$$

up to a small logarithmic correction in R. Substituting the earlier choice of $\chi(T, R)$, we have

$$v_n T \propto R \left(\frac{T}{R^{\alpha-2d}}\right)^{1+o(1)} \leq R, \quad (11)$$

where $o(1)$ represents an arbitrarily small constant, for all $T \leq R^{\alpha-2d}$. Therefore, the bound in Eq. (9) holds as long as $T \lesssim R^{\alpha-2d}$.

The bound in Eq. (9) applies to the Hamiltonian H_n constructed from H by taking interactions of range at most ℓ_n, which is slightly smaller than R for all $T \lesssim R^{\alpha-2d}$. To add interactions of range larger than ℓ_n to the bound, we use the identity [24]:

$$e^{\mathcal{L}t} = e^{\mathcal{L}_n t} + \sum_{i,j:\mathrm{dist}(i,j)>\ell_n} \int_0^t ds\; e^{\mathcal{L}(t-s)} \mathcal{L}_{h_{ij}} e^{\mathcal{L}_n s}, \quad (12)$$

where $\mathcal{L}_{h_{ij}} = i[h_{ij}, \cdot]$ is the Liouvillian corresponding to the interaction h_{ij}. We will argue that the contribution from the second term of the right-hand side to the bound on $\left\|\mathbb{P}_R e^{\mathcal{L}T} |O)\right\|$ is small.

Note that $\mathcal{L}_{h_{ij}} e^{\mathcal{L}_n s} |O)$ vanishes if $e^{\mathcal{L}_n s} |O)$ has no support on the sites i, j. Suppose site i is closer to the origin than site j. Then, most contributions to the right-hand side of Eq. (12) come from terms h_{ij} where i lies within the light cone of $e^{\mathcal{L}_n s} |O)$. Let $\mathcal{V}$ be the volume inside this light cone at time T. Using the triangle inequality on Eq. (12) specifying to the final time T and distance R, we would arrive at

$$\left\|\mathbb{P}_R e^{\mathcal{L}T} |O)\right\| \lesssim \left\|\mathbb{P}_R e^{\mathcal{L}_n T} |O)\right\| + \frac{\mathcal{V}T}{\ell_n^{\alpha-d}}, \quad (13)$$

where $\mathcal{V}$ is the result of the sum over i inside the light cone, summing over j where $\mathrm{dist}(i,j) > \ell_n$ gives a factor proportional to $1/\ell_n^{\alpha-d}$, and the integral over time in Eq. (12) gives the factor T.

Suppose we can apply the desired light cone $t \gtrsim r^{\alpha-2d}$. Then we can estimate the volume inside the light cone $\mathcal{V} \lesssim T^{d/(\alpha-2d)}$. Substituting it into the above bound together with the value of ℓ_n, we would arrive at

$$\left\|\mathbb{P}_R e^{\mathcal{L}T} |O)\right\| \lesssim \left(\frac{T}{R^{\alpha-2d}}\right)^{\frac{\alpha-d}{\alpha-2d}}, \quad (14)$$

which gives about the same light cone as in Theorem 1.

However, we are proving Theorem 1 and so cannot yet apply the light cone $t \gtrsim r^{\alpha-2d}$. Instead, we use the light cone from Ref. [8], which is weaker than Theorem 1, to estimate $\mathcal{V}$. Substituting this value of $\mathcal{V}$ into Eq. (13), we obtain a *tighter* light cone than that of Ref. [8]. Iteratively using the resulting light cone to estimate $\mathcal{V}$ (see the SM [23] for a more detailed derivation), we obtain tighter and tighter bounds. These bounds converge to a stable point that is exactly Eq. (14). Therefore, we obtain Theorem 1.

We note the above iterative procedures actually result in a bound that depends on $\log r_*$, where r_* is the lattice diameter. We show in the SM [23] how to remove this mild dependence on the lattice diameter and again obtain Theorem 1 without any r_*-dependence.

Discussion.—Theorem 1 implies a light cone that can be made arbitrarily close to $t \gtrsim r^{\alpha-2d}$ for all $\alpha \in (2d, 2d+1)$. In addition, as discussed earlier, Theorem 1 also implies $t \gtrsim r^{1-o(1)}$ for $\alpha \geq 2d+1$, providing an alternative proof of the linear light cone in Refs. [11, 12] for two-body Hamiltonians. Together with Refs. [7, 11, 12], we have the final Lieb-Robinson light cone for power-law interactions:

$$t \gtrsim \begin{cases} \log r & \text{if } d < \alpha \leq 2d \\ r^{\alpha-2d-o(1)} & \text{if } 2d < \alpha \leq 2d+1 \\ r & \text{if } \alpha > 2d+1 \end{cases}, \quad (15)$$

which we can saturate, up to subalgebraic corrections, using the protocol for state transfer and entanglement generation in Ref. [15]. While it is unlikely that these subalgebraic corrections have significant physical implications, removing them might be interesting open problem mathematically. In the SM [23], we provide a table which briefly summarizes the Lieb-Robinson bounds and the saturating protocols for all $\alpha \geq 0$.

Additionally, at any fixed time, our bound decays with distance as $1/r^{\alpha-d-o(1)}$. Because the total strength of the interactions between the origin and all sites that are at distance at least r from the origin already scales as $1/r^{\alpha-d}$, this so-called "tail" of our bound is also optimal.

Our result tightens the constraints on various quantum information tasks in power-law systems, including the growth of connected correlation functions, the generation of topological order, and the digital simulation of local observables. Intuitively, as a local operator evolves, it is mostly constrained to lie within a light cone defined by a Lieb-Robinson bound, with total leakage outside this light cone constrained by the tail of this bound. To simulate the dynamics of such observables, it is sufficient to simulate only the dynamics inside the light cone [10, 14, 25], resulting in a more efficient simulation than simulating the entire lattice. Similarly, the connected correlator between initially local observables remains small during the dynamics if their corresponding light cones have little overlap [14, 18, 26]. Topologically ordered states—those that cannot be distinguished by local observables—would also remain topologically ordered until local observables have enough time to substantially grow their supports [14, 26]. Crucially, then, Theorem 1, which has a provably optimal light cone and tail, provides the best-known asymptotic constraints for the dynamics of these quantities. The mathematical details of precisely how they are bounded and the improvements that our new bound provides are detailed in the SM. Additionally, our result may also provide a tighter constraint on the capacity of quantum communication channels based on power-law interacting spins [27].

While we assume that the Hamiltonian is two-body throughout the paper, we expect the result extends to general many-body interactions. Specifically, we conjecture that Theorem 1 holds for all Hamiltonians $H = \sum_{X\subset\Lambda} h_X$, where the sum is over all subsets of the lattice and $\sum_{X\ni i,j} \|h_X\| \leq 1/\mathrm{dist}(i,j)^\alpha$ for all $i \neq j$.

Additionally, while the Lieb-Robinson bounds assume arbitrarily time-dependent Hamiltonians, physical systems typically come with additional constraints, such as time-independence and more general conservation laws. It is conceivable that these constraints may result in tighter light cones for the dynamics of such systems. In fact, no super-linear protocols based on static Hamiltonians are known to saturate the Lieb-Robinson bounds, supporting the existence of tighter Lieb-Robinson-like bounds for time-independent systems.

Lastly, while Theorem 1 demonstrates the optimality of the single-particle state transfer protocol of [15], other information-theoretic tasks are constrained by tighter light cones. Our techniques may help extend recent progress [14, 28–30] in constraining the remaining light cone hierarchy that has been demonstrated with power-law interactions.

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