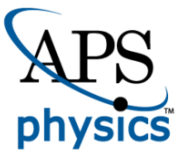

Eigenstate Entanglement: Crossover from the Ground State to Volume Laws

Qiang Miao and Thomas Barthel

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Eigenstate entanglement: Crossover from the ground state to volume laws

Qiang Miao and Thomas Barthel
Department of Physics, Duke University, Durham, North Carolina 27708, USA
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For the typical quantum many-body systems that obey the eigenstate thermalization hypothesis (ETH), we argue that the entanglement entropy of (almost) all energy eigenstates is described by a single crossover function. The ETH implies that the crossover functions can be deduced from subsystem entropies of thermal ensembles and have universal properties. These functions capture the full crossover from the groundstate entanglement regime at low energies and small subsystem size (area or log-area law) to the extensive volume-law regime at high energies or large subsystem size. For critical one-dimensional systems, a universal scaling function follows from conformal field theory (CFT) and can be adapted for nonlinear dispersions. We use it to also deduce the crossover scaling function for Fermi liquids in $d > 1$ dimensions. The analytical results are complemented by numerics for large non-interacting systems of fermions in $d \leq 3$ dimensions and have also been confirmed for bosonic systems and non-integrable spin chains.

Introduction. – A fundamental concept in modern physics and information theory is quantum entanglement. Specifically, the entanglement entropy quantifies quantum correlations and the utility of a given state for quantum information processing [1, 2]. It is also used to guide tensor network state simulations and to bound their computation costs [3–5]. Henceforth, consider the entanglement entropy S for the bipartition of a d-dimensional quantum many-body system into a compact subsystem $\mathcal{A}$ of volume ℓ^d and the rest $\mathcal{B}$, which is much larger or infinite. Ground states have been studied intensely [6–8] and one generally finds an area law, where S is proportional to the surface *area* of $\mathcal{A}$, or an area law with a logarithmic correction. In contrast, for random states and highly-excited states, one generally finds S to be proportional to the *volume* of $\mathcal{A}$. The transition from the groundstate scaling to the extensive scaling and, more generally, the distribution of S in excited states, have been largely unexplored.

The long-range physics of typical quantum many-body systems is captured by a field theory with local interactions. Then the groundstate entanglement entropy in gapped systems obeys an area law $S_{\rm gs} \propto \ell^{d-1}$ [5, 9–16]. Intuitively, only the vicinity of the boundary between $\mathcal{A}$ and $\mathcal{B}$ contributes to $S_{\rm gs}$ due to the finite correlation length induced by the energy gap. In critical systems, the correlation length diverges and the scaling of $S_{\rm gs}$ depends on the dimension d and particle statistics. Critical 1d systems are usually captured by CFT, giving $S_{\rm gs} \propto \ln \ell$ [9, 11, 17–21]. For critical fermionic systems, $S_{\rm gs}$ generally obeys a log-area law [22–25]

$$S_{\rm gs} \propto \ell^{d-1} \ln \ell. \tag{1}$$

For critical bosonic systems in $d > 1$ dimensions, $S_{\rm gs}$ still obeys the area law [9, 17, 24, 26, 27].

Our previous understanding of excited states is rather limited. Area, log-area laws and subleading corrections were found for states with few-particle excitations (vanishing excitation-energy density) [28–34] and for special rare excited states which are often ground states of other Hamiltonians [28, 34–36]. For broad classes of highly excited states, the entanglement volume law has been found in Refs. [35, 37–39]. Extensive scaling of the average eigenstate entanglement was shown in Refs. [40–44].

In this Letter, we address the long-standing question about the scaling of S in excited states and its transition from the groundstate scaling to an extensive scaling $S \propto \ell^d$ at higher energies. We argue and demonstrate that, generally, the eigenstate thermalization hypothesis (ETH) [45–54] implies the existence of crossover functions that capture the entanglement entropies of (almost) all eigenstates. Moreover, for system parameters and energies corresponding to the quantum critical regime of a system, the crossover function has universal scaling properties. For critical 1d systems, the result follows from CFT. We also derive the crossover scaling function for Fermi liquids. In addition, we discuss the scaling in gapped systems and the eigenstate entanglement distribution. The general arguments and derivations are confirmed numerically for systems in $d = 1, 2, 3$ dimensions.

ETH and excited-state entanglement. – According to the strong ETH, local expectation values of all energy eigenstates approach those of corresponding microcanonical ensembles with the same energy, where deviations decrease with increasing system size. Weak ETH allows for an exponentially small number of untypical energy eigenstates [48, 50, 55].

While strong ETH is difficult to establish in a general way, weak ETH [48, 56, 57] can be understood rather easily and, in contrast to strong ETH, also applies to integrable systems: Consider an observable $\hat{O}$ with finite spatial support and the microcanonical ensemble $D^{-1}\sum_n |E_n\rangle\langle E_n|$ for a small energy window $E - \Delta E \leq E_n \leq E$ containing D energy eigenstates $|E_n\rangle$. The weak ETH bounds the variance

$$\Delta O^2_{\rm eth} = D^{-1} \sum_n \left(\langle E_n|\hat{O}|E_n\rangle - \langle\hat{O}\rangle_{\rm mc}\right)^2 \leq \Delta O^2_{\rm mc} \tag{2}$$

for deviations between eigenstate and microcanoni-

cal expectation values. The inequality, follows from $\langle E_n|\hat{O}|E_n\rangle^2 \leq \langle E_n|\hat{O}^2|E_n\rangle$. For a translation-invariant system, $|E_n\rangle$ can be chosen as momentum eigenstates and we can replace $\hat{O}$ by the sum $\hat{O}' := \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} \hat{O}_i$ over all lattice translates $\hat{O}_i$ of $\hat{O}$ without changing matrix elements in Eq. (2). This yields

$$\Delta O_{\text{eth}}^2 \leq \Delta O_{\text{mc}}'^2 = \frac{1}{\mathcal{N}^2}\sum_{i,j}\left(\langle \hat{O}_i\hat{O}_j\rangle - \langle \hat{O}_i\rangle\langle \hat{O}_j\rangle\right)_{\text{mc}}. \quad (3)$$

Thus, if connected correlation functions decay exponentially or according to a sufficiently fast power law, ΔO_{eth} indeed vanishes in the thermodynamic limit, $\lim_{\mathcal{N}\to\infty}\Delta O_{\text{eth}} = 0$. Due to the equivalence of thermodynamic ensembles for large systems [58–61], we can also use other canonical ensembles. We will employ the grand-canonical ensemble (GCE) $\hat{\varrho}_{\text{gc}}$ with temperature and chemical potential chosen to match the energy and particle number of the energy eigenstates, i.e.,

$$\hat{\varrho}_{\text{gc}} = e^{-\beta(\hat{H}-\mu\hat{N})}/Z \quad \text{with} \quad \langle\hat{H}\rangle_{\text{gc}} = E,\ \langle\hat{N}\rangle_{\text{gc}} = N. \quad (4)$$

The coincidence of eigenstate expectation values with thermal expectation values for all observables $\hat{O}$ supported on $\mathcal{A}$, implies the coincidence of the corresponding subsystem density matrices, i.e.,

$$\langle E_n|\hat{O}|E_n\rangle \approx \langle\hat{O}\rangle_{\text{gc}} \quad \Rightarrow \quad \hat{\rho}_n := \text{Tr}_{\mathcal{B}}\,|E_n\rangle\langle E_n| \approx \text{Tr}_{\mathcal{B}}\,\hat{\varrho}_{\text{gc}}$$

for typical eigenstates $|E_n\rangle$. Hence, entanglement entropies $S_n(\ell) = -\,\text{Tr}\,\hat{\rho}_n \ln\hat{\rho}_n$ of typical eigenstates are very close to subsystem entropies of the GCE and become extensive for large subsystems,

$$S_n(\ell) \overset{\text{typical}}{\approx} S_{\text{gc}}(\ell,\beta) \xrightarrow{\ell\gg\xi} \ell^d\, s_{\text{th}}(\beta), \quad (5)$$

where $s_{\text{th}}(\beta)$ denotes the thermodynamic entropy density, and ξ is the thermal correlation length. Eq. (5) has important implications: (a) As long as (weak) ETH applies, the entanglement entropies of (almost) all eigenstates are captured by a single crossover function, determined by $S_{\text{gc}}(\ell,\beta(E_n))$. (b) This function follows the groundstate entanglement scaling for small ℓ and crosses over to an extensive scaling at large ℓ. (c) If the system parameters and energy (temperature) lie in a quantum critical regime of the considered model, general principles dictate that the entanglement entropies should follow a universal scaling function [62–64]. Point (b) is due to a *resolution limitation* effect: With observations on a subsystem $\mathcal{A}$ of linear size ℓ, one cannot resolve variations of momentum-space Green's functions below a scale $\sim 1/\ell$. Hence, one can coarse-grain accordingly and, for ℓ below a crossover length ℓ_c, the coarse-grained Green's functions of excited states approach that of the ground state. One recovers the extensive scaling predicted by thermodynamics for $\ell \gtrsim \ell_c$, and ℓ_c increases with decreasing energy (β^{-1}). More detail is provided in the Supplemental Material (SM).

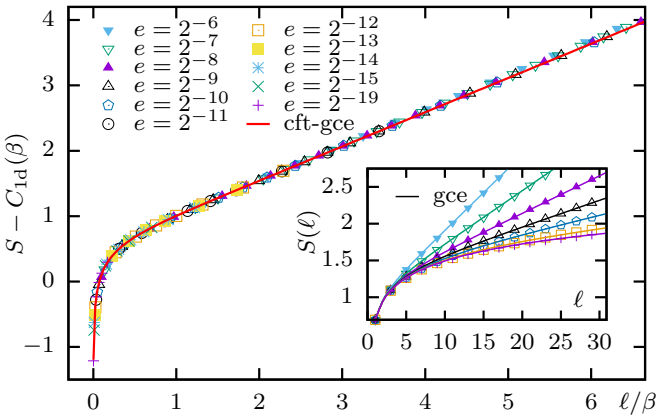

FIG. 1. Entanglement entropies of randomly sampled eigenstates for a fermionic tight-binding chain with half filling ($\mu = 0$, $v = 2$), system size $L \approx 4\times 10^6$, excitation energy densities $e := (E - E_{\text{gs}})/|E_{\text{gs}}|$, and window size $\Delta E = 1$. The main panel confirms the data collapse to the crossover scaling function (7). The inset asserts the validity of the ETH by comparison to GCE subsystem entropies (lines).

Crossover in critical 1d systems. – Let us now investigate crossover functions for specific classes of systems. The long-range physics of critical 1d systems with linear dispersion at low energies, interacting or noninteracting, is described by 1+1d CFT. The GCE subsystem entropy can be computed using the replica trick and analytic continuation [11, 65]. One obtains

$$S_{\text{gc}}^{\text{cft}}(\ell,\beta) = \frac{c}{3}\ln\left[\frac{\beta v}{\pi a}\sinh\left(\frac{\pi\ell}{\beta v}\right)\right] + c' \quad (6)$$

with the central charge c, group velocity v, ultraviolet cutoff $1/a$, and a nonuniversal constant c'. For small subsystem size ℓ or temperature β^{-1}, one recovers the log-area law $\frac{c}{3}\ln(\ell/a)$ [Eq. (1)] as motivated above with the resolution argument. The crossover to extensive scaling $S \sim \frac{c}{3}\ell/\ell_c$ occurs at $\ell_c = \beta v/\pi$ and the universal scaling function is simply the leading term in

$$S_{\text{gc}}^{\text{cft}}(\ell,\beta) = \frac{c}{3}\ln(\sinh\ell/\ell_c) + \mathcal{O}(\ell^0). \quad (7)$$

It applies whenever CFT does, including critical fermionic, bosonic and spin systems.

To confirm this numerically, we sample energy eigenstates $|E_n\rangle$ from small windows of width ΔE around energies E for fermionic tight-binding chains $\hat{H} = -\sum_i(\hat{c}_i^\dagger\hat{c}_{i+1} + h.c.) = \sum_k \varepsilon_k\hat{n}_k$ at half filling [66]. These obey weak ETH and are captured by a CFT with $c = 1$. Figure 1 shows the results. The variances of the sampled $S_n(\ell)$ are much smaller than the symbol sizes. The inset asserts perfect agreement with the corresponding GCE subsystem entropies. The main plot shows the data collapse to the universal scaling function (7) after subtrac-

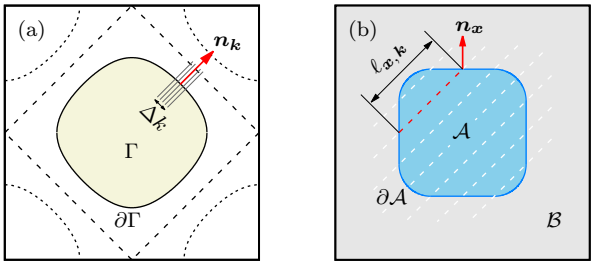

FIG. 2. (a) Fermi surfaces $\partial\Gamma$ for the 2d tight-binding model (11) with fillings $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. (b) Real-space bipartition and chord lengths $\ell_{\boldsymbol{x},\boldsymbol{k}}$ for direction $\boldsymbol{n}_{\boldsymbol{k}}$.

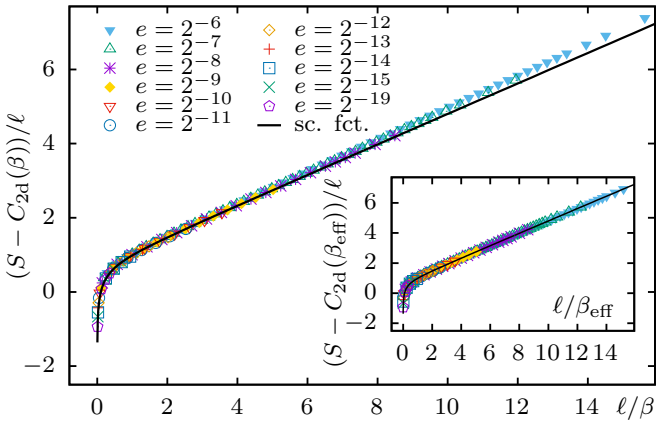

FIG. 3. Eigenstate entanglement entropies for the critical 2d model (11) with size 4096×4096 and $\mu \approx -1.44$ collapse onto a scaling function after subtraction of a subleading area-law term $C_{2d}(\beta)$. The latter corresponds to the factor $\beta/\pi a$ in the logarithm of Eq. (9) with $\beta = \beta(E)$.

tion of $C_{1d}(\beta) := \frac{c}{3}\ln(\beta v/\pi a)$ and proper rescaling of ℓ with $\beta = \beta(E)$.

Critical fermions in $d > 1$**.** – For higher-dimensional critical systems, it is more complex to extract crossover functions. Let us first discuss translation-invariant systems of non-interacting fermions with a $(d-1)$-dimensional Fermi surface and consider interactions later on. Employing the Widom conjecture [67], Gioev and Klich found the coefficient in the log-area law (1) for the groundstate entanglement as an integral over the Fermi surface $\partial\Gamma$ and the boundary $\partial\mathcal{A}$ of the subsystem [23, 68]. To leading order,

$$S_{\text{gs}}(\ell) = \frac{\ln \ell}{12} \int_{\partial\mathcal{A}} \int_{\partial\Gamma} \frac{\mathrm{d}A_x \mathrm{d}A_k}{(2\pi)^{d-1}} \, |\boldsymbol{n}_{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{k}}| \tag{8}$$

with the normal vectors $\boldsymbol{n}_{\boldsymbol{x}}$ and $\boldsymbol{n}_{\boldsymbol{k}}$ on $\partial\mathcal{A}$ and on $\partial\Gamma$ as indicated in Fig. 2. Eq. (8) can be interpreted as an integral over entanglement contributions of lines perpendicular to the Fermi surface [69, 70].

For the case of finite temperatures (energies), we can use the latter intuition and will involve the chord length as indicated in Fig. 2b to correctly capture all temperature scales: First consider the trivial example of uncoupled critical 1d chains oriented in a fixed direction $\boldsymbol{n}_{\boldsymbol{k}}$ with density $\rho_\perp$ in the perpendicular direction. To get the entropy $S_{\text{gc}}(\ell, \beta)$ of a subsystem $\mathcal{A}$, we simply need to add the contributions from all chains. For convex $\mathcal{A}$ and the continuum limit, this gives $S_{\mathcal{A}} = \frac{\rho_\perp}{2} \int_{\partial\mathcal{A}} \mathrm{d}A_x \, |\boldsymbol{n}_{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{k}}| \, S_{1d}(\ell_{\boldsymbol{x},\boldsymbol{k}})$, where $\rho_\perp |\boldsymbol{n}_{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{k}}|$ is the density of chains piercing $\partial\mathcal{A}$ at point $\boldsymbol{x}$ and $\ell_{\boldsymbol{x},\boldsymbol{k}}$ is the chord length across $\mathcal{A}$ in direction $\boldsymbol{n}_{\boldsymbol{k}}$. We want to reduce the case of true d-dimensional systems with couplings in all directions to that of uncoupled chains as follows. Consider a patch $\mathcal{P}$ of size Δk^{d-1} around $\boldsymbol{k} \in \partial\Gamma$ on the Fermi surface and group wave vectors around that point into lines in direction $\boldsymbol{n}_{\boldsymbol{k}}$ as indicated in Fig. 2a. We know that, at sufficiently low temperatures, modes far away from the Fermi surface are irrelevant for the long-range physics and can be disregarded. The dispersion is linear in direction $\boldsymbol{n}_{\boldsymbol{k}}$, corresponding to chiral fermions, but the dispersion is flat in the perpendicular directions. Let us parametrize these directions by $k_\parallel$ and $\boldsymbol{k}_\perp$. Due to the flat dispersion with respect to $\boldsymbol{k}_\perp$, any unitary transformation (basis change) among the single-particle states corresponding to $\boldsymbol{k}_\perp \in \mathcal{P}$ does not generate any coupling of these modes. In particular, we can use this to transform to single-particle states that are spatially localized around points with spacings $\Delta y = 2\pi/\Delta k$ in the $(d-1)$-dimensional perpendicular plane. A subsequent inverse Fourier transform with respect to $k_\parallel$ then yields uncoupled chains in direction $\boldsymbol{n}_{\boldsymbol{k}}$ and density $1/\Delta y$ in the directions perpendicular to $\boldsymbol{n}_{\boldsymbol{k}}$. This is the situation we considered initially with $\rho_\perp = 1/\Delta y^{d-1}$, and we hence know how patch $\mathcal{P}$ contributes to the subsystem entropy. For $S_{1d}(\ell_{\boldsymbol{x},\boldsymbol{k}})$, we can plug in the finite-temperature CFT result (6), where we substitute ℓ by the chord length $\ell_{\boldsymbol{x},\boldsymbol{k}}$ and use $c = 1/2$ because of chirality. Finally, we need to integrate over the entire Fermi surface to take into account the contributions from all patches, resulting to leading order in the subsystem entropy

$$S_{\text{gc}}(\ell, \beta) = \frac{1}{12} \int_{\partial\mathcal{A}} \int_{\partial\Gamma} \frac{\mathrm{d}A_x \mathrm{d}A_k}{(2\pi)^{d-1}} \, |\boldsymbol{n}_{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{k}}| \, \ln \left[\frac{\beta v_{\boldsymbol{k}}}{\pi a} \sinh\left(\frac{\pi \ell_{\boldsymbol{x},\boldsymbol{k}}}{\beta v_{\boldsymbol{k}}} \right) \right], \tag{9}$$

where $v_{\boldsymbol{k}}$ is the Fermi velocity at point $\boldsymbol{k} \in \partial\Gamma$.

This formula for the subsystem entropy is remarkable: For zero temperature, we recover the groundstate result (8). For large subsystems, the logarithm approaches $\pi \ell_{\boldsymbol{x},\boldsymbol{k}}/\beta v_{\boldsymbol{k}}$ such that the integral over $\partial\mathcal{A}$ gives the subsystem volume $\operatorname{vol}\mathcal{A}$ and we are left with

$$\frac{1}{6} \operatorname{vol}\mathcal{A} \int_{\partial\Gamma} \frac{\mathrm{d}A_k}{(2\pi)^{d-1}} \frac{\pi}{\beta v_{\boldsymbol{k}}} = \frac{\pi^2 g(\mu)}{3\beta} \operatorname{vol}\mathcal{A}. \tag{10}$$

This is the well-known extensive thermodynamic entropy due to the Sommerfeld expansion [71] with the density of states $g(\mu)$ at the Fermi energy. Further, if we move the β factor in the logarithm of Eq. (9) to a subleading area law term and replace $\ell_{\boldsymbol{x},\boldsymbol{k}} \equiv \ell \cdot \lambda_{\boldsymbol{x},\boldsymbol{k}}$, where $\operatorname{vol}\mathcal{A} =$

ℓ^d as before, the leading order term of S/ℓ^{d-1} is just a function of ℓ/β. In this way, we obtain the desired scaling function for the crossover from groundstate to extensive subsystem entropies.

It is non-trivial to make the arguments leading to Eq. (9) totally rigorous. The limit $\ell, \beta \to \infty$ has been captured using the theory of semiclassical trace formulas [72–75]. Eq. (9) should be treated as a conjecture. Our numerical tests in Fig. 3 show however that it is very precise for all ℓ and β. The figure shows the data collapse of sampled eigenstate entanglement to the scaling function for fermions on a square lattice,

$$\hat{H} - \mu\hat{N} = -\sum_{\langle i,j\rangle} (\hat{c}_i^\dagger \hat{c}_j + h.c.) - \mu \sum_i \hat{c}_i^\dagger \hat{c}_i \qquad (11)$$

with $\mu \approx -1.44$ (quarter filling at zero temperature). Deviations at large ℓ/β are due to the finite band width of the model, i.e., due a nonlinear dispersion at higher energies, differing from the assumptions of CFT. The inset shows that this can be fixed by replacing β in Eq. (9) by β_{eff}. Specifically, we define an effective temperature by matching the exact thermodynamic entropy density and the large-ℓ limit (10) of Eq. (9), i.e., $s_{\text{th}}(\beta) =: \pi^2 g(\mu)/3\beta_{\text{eff}}(\beta)$ such that $\beta_{\text{eff}}(\beta) \to \beta$ at low temperatures. For the model (11) in 2d at half filling, $v_{\boldsymbol{k}}$ is zero at some points on the Fermi surface. Hence, CFT and Eq. (9) are not applicable, but the described rescaling procedure works nevertheless. Eq. (9) works without problems in 3d. Both cases are discussed in the SM.

The crossover scaling function (9) also applies to *interacting* metals described by Fermi liquid theory, because the quasi-particle lifetime diverges when approaching the Fermi energy. The scaling function is universal in the sense that the only remnant of microscopic details is the dependence on the Fermi surface shape and $v_{\boldsymbol{k}}$ as pointed out for other quantities in Ref. [64].

Crossover in gapped systems. – If one adds a mass term to a critical theory, the resulting gap $\sim m$ represents an additional energy scale. At zero temperature, the entanglement entropy should then be determined by a function of $m\ell$ [26, 76–80]. So far, very few works investigated excited-state entanglement entropies in gapped systems [38] and, to our knowledge, the crossover from area law to extensive scaling and its universal properties have not been addressed.

As argued before, a crossover function should exist and now depend on subsystem size ℓ, mass m, and β. However, we expect that it can be expressed in terms of a scaling function that only depends on the parameters $m\ell$ and $m\beta$, characterizing the full crossover behavior. For fermionic systems, in particular, it follows a log-area law as in Eq. (1) for $m\ell \ll \min(1, m\beta)$ and a volume law for $m\ell \gg \min(1, m\beta)$. Moreover, at low temperatures, $m\beta \gg 1$, one has the typical behavior of thermally activated excitations, and the growth of the entropy density

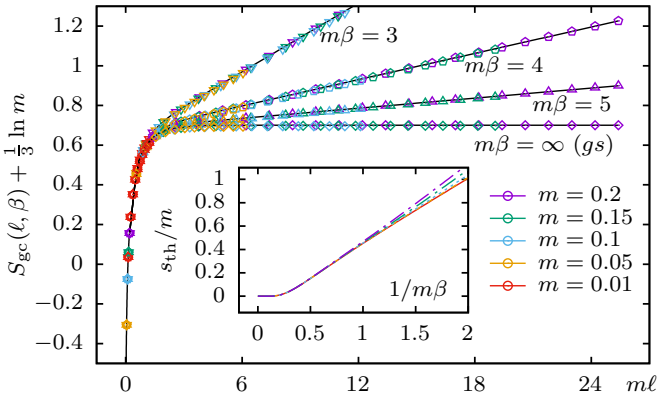

FIG. 4. Subsystem entropies $S_{\text{gc}}(\ell, \beta)$ in the staggered tight-binding chain (12) with half filling ($\mu = 0$), mass m, and $L \approx 10^6$. We only use odd ℓ because of odd-even effects.

s_{th} in Eq. (5) as a function of temperature changes from exponential to linear around $m\beta \sim 1$.

Figure 4 shows the data collapse of GCE subsystem entropies to the two-parameter scaling function for the staggered tight-binding chain

$$\hat{H} = -\sum_i \left(t\hat{c}_{2i-1}^\dagger \hat{c}_{2i} + t'\hat{c}_{2i}^\dagger \hat{c}_{2i+1} + h.c. \right) \qquad (12)$$

at half filling. The dispersion relation is $\varepsilon_k = \pm\sqrt{t^2 + t'^2 + 2tt'\cos k}$ with mass $m = |t - t'|$ and $v = \sqrt{tt'}$. It takes the relativistic form $\pm\sqrt{m^2 + (vp)^2}$ for small $p = k \pm \pi$. The inset displays scaled entropy densities s_{th} which show the thermally activated exponential behavior at low temperatures $m\beta \gg 1$ and linear scaling for higher temperature.

Entanglement distribution. – Recently, interesting results were derived for the entanglement entropy *averaged over all* energy eigenstates [40–44]. In particular, bounds on the average entanglement for chaotic local Hamiltonians and quadratic fermionic systems were given. As long as one only considers the case of small subsystems, we can get much more – the *entire distribution* of eigenstate entanglement. We have shown that the eigenstate entanglement entropies are given by certain crossover functions. To obtain the entanglement distribution, we only need to multiply these with the many-body density of states. For many purposes, the latter can be approximated by a Gaussian that describes well the bulk of the spectrum [81].

From this, properties of the average entanglement follow rather easily. For example, Ref. [40] addresses for non-interacting translation-invariant fermions how the average entanglement between a subsystem of fixed size ℓ^d converges to the maximum $\langle S\rangle = \ell^d$ in the thermodynamic limit. This can also be explained as follows. With the same arguments as in the derivation for the weak ETH above, we can bound the deviation of expectation values $\langle E_n|\hat{O}|E_n\rangle$ of a local observable

from the infinite temperature value $\langle\hat{O}\rangle_\infty = \mathrm{Tr}(\hat{O})/D$, averaged over *all* energy eigenstates, by $\Delta O^2_{\mathrm{avg}} := D^{-1}\sum_{n=1}^{D}\left(\langle E_n|\hat{O}|E_n\rangle - \langle\hat{O}\rangle_\infty\right)^2 \leq \Delta O^2_\infty$. And ΔO_∞ decays with $\mathcal{O}(\mathcal{N}^{-1/2})$ in the thermodynamic limit. Hence, almost all energy eigenstates look locally like the infinite temperature ensemble and their entanglement entropies are maximal.

Discussion. – In conclusion, ETH can be employed to understand the full crossover of eigenstate entanglement entropies from the groundstate scaling at small subsystem sizes and low energies to the extensive scaling at large sizes and higher energies. With increasing system size, the entanglement entropies of all (strong ETH) or almost all (weak ETH) energy eigenstates converge onto a single crossover function, which can be determined from corresponding subsystem entropy of corresponding thermodynamic ensembles – also in integrable systems. Importantly, for the quantum critical regime, one obtains *universal* scaling functions that capture large classes of microscopic models [62–64]. For critical 1d systems and fermions in $d > 1$ dimensions, analytic forms for the scaling functions were given and numerically confirmed. We substantiate the scaling properties further for bosonic systems in Ref. [82] and for interacting integrable and non-integrable spin chains in Ref. [83]. These results generalize immediately to Rényi entanglement entropies and can, for example, be used to derive upper bounds on computation costs in tensor network simulations [84, 85].

Scaling functions for thermal subsystem entropies are so far largely unexplored. The connection to eigenstate entanglement makes them very interesting and it will be an exciting endeavor to derive crossover functions for specific lattice models and field theories, similar to efforts on groundstate entanglement. Furthermore, the accuracy of the $d > 1$ scaling function (9) for fermions suggests an extension of the famous Widom formula for groundstate entanglement to finite temperatures.

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[1] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Concentrating partial entanglement by local operations*, Phys. Rev. A **53**, 2046 (1996).

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[3] F. Verstraete and J. I. Cirac, *Renormalization algorithms for quantum-many body systems in two and higher dimensions*, arXiv:cond-mat/0407066 (2004).

[4] G. Vidal, *Class of quantum many-body states that can be efficiently simulated*, Phys. Rev. Lett. **101**, 110501 (2008).

[5] M. B. Hastings, *Entropy and entanglement in quantum ground states*, Phys. Rev. B **76**, 035114 (2007).

[6] J. Eisert, M. Cramer, and M. B. Plenio, *Colloquium: Area laws for the entanglement entropy*, Rev. Mod. Phys. **82**, 277 (2010).

[7] J. I. Latorre and A. Riera, *A short review on entanglement in quantum spin systems*, J. Phys. A: Math. Theor. **42**, 504002 (2009).

[8] N. Laflorencie, *Quantum entanglement in condensed matter systems*, Phys. Rep. **646**, 1 (2016).

[9] C. Callan and F. Wilczek, *On geometric entropy*, Phys. Lett. B **333**, 55 (1994).

[10] J. I. Latorre, E. Rico, and G. Vidal, *Ground state entanglement in quantum spin chains*, Quant. Inf. Comput. **4**, 48 (2004).

[11] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, J. Stat. Mech. P06002 (2004).

[12] M. B. Plenio, J. Eisert, J. Dreißig, and M. Cramer, *Entropy, entanglement, and area: analytical results for harmonic lattice systems*, Phys. Rev. Lett. **94**, 060503 (2005).

[13] M. Cramer, J. Eisert, M. B. Plenio, and J. Dreißig, *Entanglement-area law for general bosonic harmonic lattice systems*, Phys. Rev. A **73**, 012309 (2006).

[14] F. G. S. L. Brandão and M. Horodecki, *An area law for entanglement from exponential decay of correlations*, Nat. Phys. **9**, 721 (2013).

[15] J. Cho, *Realistic area-law bound on entanglement from exponentially decaying correlations*, Phys. Rev. X **8**, 031009 (2018).

[16] T. Kuwahara and K. Saito, *Area law of noncritical ground states in 1D long-range interacting systems*, Nat. Commun. **11**, 4478 (2020).

[17] M. Srednicki, *Entropy and area*, Phys. Rev. Lett. **71**, 666 (1993).

[18] C. Holzhey, F. Larsen, and F. Wilczek, *Geometric and renormalized entropy in conformal field theory*, Nucl. Phys. B **424**, 443 (1994).

[19] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, *Entanglement in quantum critical phenomena*, Phys. Rev. Lett. **90**, 227902 (2003).

[20] B. Q. Jin and V. E. Korepin, *Quantum spin chain, Toeplitz determinants and Fisher-Hartwig conjecture*, J. Stat. Phys. **116**, 79 (2004).

[21] H.-Q. Zhou, T. Barthel, J. O. Fjærestad, and U. Schollwöck, *Entanglement and boundary critical phenomena*, Phys. Rev. A **74**, 050305(R) (2006).

[22] M. M. Wolf, *Violation of the entropic area law for fermions*, Phys. Rev. Lett. **96**, 010404 (2006).

[23] D. Gioev and I. Klich, *Entanglement entropy of fermions in any dimension and the Widom conjecture*, Phys. Rev. Lett. **96**, 100503 (2006).

[24] T. Barthel, M.-C. Chung, and U. Schollwöck, *Entanglement scaling in critical two-dimensional fermionic and bosonic systems*, Phys. Rev. A **74**, 022329 (2006).

[25] W. Li, L. Ding, R. Yu, T. Roscilde, and S. Haas, *Scaling behavior of entanglement in two- and three-dimensional free-fermion systems*, Phys. Rev. B **74**, 073103 (2006).

[26] H. Casini and M. Huerta, *Entanglement entropy in free quantum field theory*, J. Phys. A: Math. Theor. **42**, 504007 (2009).

[27] H.-H. Lai, K. Yang, and N. E. Bonesteel, *Violation of the entanglement area law in bosonic systems with Bose*

surfaces: Possible application to Bose metals, Phys. Rev. Lett. **111**, 210402 (2013).
[28] S. Das and S. Shankaranarayanan, *How robust is the entanglement entropy-area relation?*, Phys. Rev. D **73**, 121701(R) (2006).
[29] S. Das, S. Shankaranarayanan, and S. Sur, *Power-law corrections to entanglement entropy of horizons*, Phys. Rev. D **77**, 064013 (2008).
[30] L. Masanes, *Area law for the entropy of low-energy states*, Phys. Rev. A **80**, 052104 (2009).
[31] F. C. Alcaraz, M. I. Berganza, and G. Sierra, *Entanglement of low-energy excitations in conformal field theory*, Phys. Rev. Lett. **106**, 201601 (2011).
[32] M. I. Berganza, F. C. Alcaraz, and G. Sierra, *Entanglement of excited states in critical spin chains*, J. Stat. Mech. P01016 (2012).
[33] J. Mölter, T. Barthel, U. Schollwöck, and V. Alba, *Bound states and entanglement in the excited states of quantum spin chains*, J. Stat. Mech. P10029 (2014).
[34] S. Moudgalya, N. Regnault, and B. A. Bernevig, *Entanglement of exact excited states of Affleck-Kennedy-Lieb-Tasaki models: Exact results, many-body scars, and violation of the strong eigenstate thermalization hypothesis*, Phys. Rev. B **98**, 235156 (2018).
[35] V. Alba, M. Fagotti, and P. Calabrese, *Entanglement entropy of excited states*, J. Stat. Mech. P10020 (2009).
[36] O. Vafek, N. Regnault, and B. A. Bernevig, *Entanglement of exact excited eigenstates of the Hubbard model in arbitrary dimension*, SciPost Phys. **3**, 043 (2017).
[37] F. Ares, J. G. Esteve, F. Falceto, and E. Sánchez-Burillo, *Excited state entanglement in homogeneous fermionic chains*, J. Phys. A: Math. Theor. **47**, 245301 (2014).
[38] M. Storms and R. R. P. Singh, *Entanglement in ground and excited states of gapped free-fermion systems and their relationship with Fermi surface and thermodynamic equilibrium properties*, Phys. Rev. E **89**, 012125 (2014).
[39] J. P. Keating, N. Linden, and H. J. Wells, *Spectra and eigenstates of spin chain Hamiltonians*, Commun. Math. Phys. **338**, 81 (2015).
[40] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, *Entanglement entropy of eigenstates of quadratic fermionic Hamiltonians*, Phys. Rev. Lett. **119**, 020601 (2017).
[41] L. Vidmar and M. Rigol, *Entanglement entropy of eigenstates of quantum chaotic Hamiltonians*, Phys. Rev. Lett. **119**, 220603 (2017).
[42] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, *Volume law and quantum criticality in the entanglement entropy of excited eigenstates of the quantum Ising model*, Phys. Rev. Lett. **121**, 220602 (2018).
[43] T.-C. Lu and T. Grover, *Renyi entropy of chaotic eigenstates*, Phys. Rev. E **99**, 032111 (2019).
[44] Y. Huang, *Universal eigenstate entanglement of chaotic local Hamiltonians*, Nucl. Phys. B **938**, 594 (2019).
[45] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, Phys. Rev. A **43**, 2046 (1991).
[46] M. Srednicki, *Chaos and quantum thermalization*, Phys. Rev. E **50**, 888 (1994).
[47] M. Rigol, V. Dunjko, and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, Nature **452**, 854 (2008).
[48] G. Biroli, C. Kollath, and A. M. Läuchli, *Effect of rare fluctuations on the thermalization of isolated quantum systems*, Phys. Rev. Lett. **105**, 250401 (2010).
[49] W. Beugeling, R. Moessner, and M. Haque, *Finite-size scaling of eigenstate thermalization*, Phys. Rev. E **89**, 042112 (2014).
[50] H. Kim, T. N. Ikeda, and D. A. Huse, *Testing whether all eigenstates obey the eigenstate thermalization hypothesis*, Phys. Rev. E **90**, 052105 (2014).
[51] V. Alba, *Eigenstate thermalization hypothesis and integrability in quantum spin chains*, Phys. Rev. B **91**, 155123 (2015).
[52] H.-H. Lai and K. Yang, *Entanglement entropy scaling laws and eigenstate typicality in free fermion systems*, Phys. Rev. B **91**, 081110(R) (2015).
[53] A. Dymarsky, N. Lashkari, and H. Liu, *Subsystem eigenstate thermalization hypothesis*, Phys. Rev. E **97**, 012140 (2018).
[54] J. M. Deutsch, *Eigenstate thermalization hypothesis*, Rep. Prog. Phys. **81**, 082001 (2018).
[55] T. Yoshizawa, E. Iyoda, and T. Sagawa, *Numerical large deviation analysis of the eigenstate thermalization hypothesis*, Phys. Rev. Lett. **120**, 200604 (2018).
[56] T. Mori, *Weak eigenstate thermalization with large deviation bound*, arXiv:1609.09776 (2016).
[57] E. Iyoda, K. Kaneko, and T. Sagawa, *Fluctuation theorem for many-body pure quantum states*, Phys. Rev. Lett. **119**, 100601 (2017).
[58] J. L. Lebowitz, J. K. Percus, and L. Verlet, *Ensemble dependence of fluctuations with application to machine computations*, Phys. Rev. **153**, 250 (1967).
[59] D. Ruelle, *Statistical Mechanics: Rigorous Results* (W. A. Benjamin, Inc., Reading, Massachusetts, 1969).
[60] M. P. Müller, E. Adlam, L. Masanes, and N. Wiebe, *Thermalization and canonical typicality in translation-invariant quantum lattice systems*, Commun. Math. Phys **340**, 499 (2015).
[61] H. Tasaki, *On the local equivalence between the canonical and the microcanonical ensembles for quantum spin systems*, J. Stat. Phys. **172**, 905 (2018).
[62] R. Shankar, *Renormalization-group approach to interacting fermions*, Rev. Mod. Phys. **66**, 129 (1994).
[63] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, 2011).
[64] T. Senthil, *Critical Fermi surfaces and non-Fermi liquid metals*, Phys. Rev. B **78**, 035103 (2008).
[65] V. E. Korepin, *Universality of entropy scaling in one dimensional gapless models*, Phys. Rev. Lett. **92**, 096402 (2004).
[66] See the the Supplemental Material for details.
[67] H. Widom, *On a class of integral operators with discontinuous symbol*, Oper. Th.: Adv. Appl. **4**, 477 (1982).
[68] H. Leschke, A. V. Sobolev, and W. Spitzer, *Scaling of Rényi entanglement entropies of the free Fermi-gas ground state: a rigorous proof*, Phys. Rev. Lett. **112**, 160403 (2014).
[69] B. Swingle, *Entanglement entropy and the Fermi surface*, Phys. Rev. Lett. **105**, 050502 (2010).
[70] B. Swingle, *Conformal field theory approach to Fermi liquids and other highly entangled states*, Phys. Rev. B **86**, 035116 (2012).
[71] F. Schwabl, *Statistical Mechanics*, *Advanced Texts in Physics* (Springer, Heidelberg, 2006).
[72] H. Widom, *On a class of integral operators on a half-space with discontinuous symbol*, J. Funct. Anal. **88**, 166 (1990).
[73] A. Sobolev, *Pseudo-differential operators with discontinuous symbols: Widoms conjecture*, Mem. Am. Math. Soc.

1043, (2013).

[74] H. Leschke, A. V. Sobolev, and W. Spitzer, *Large-scale behaviour of local and entanglement entropy of the free Fermi gas at any temperature*, J. Phys. A: Math. Theor. **49**, 30LT04 (2016).

[75] H. Leschke, A. V. Sobolev, and W. Spitzer, *Trace formulas for Wiener-Hopf operators with applications to entropies of free fermionic equilibrium states*, J. Funct. Anal. **273**, 1049 (2017).

[76] H. Casini, C. D. Fosco, and M. Huerta, *Entanglement and alpha entropies for a massive Dirac field in two dimensions*, J. Stat. Mech. P07007 (2005).

[77] H. Casini and M. Huerta, *Entanglement and alpha entropies for a massive scalar field in two dimensions*, J. Stat. Mech. P12012 (2005).

[78] J. Cardy, O. A. Castro-Alvaredo, and B. Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, J. Stat. Phys. **130**, 129 (2008).

[79] O. A. Castro-Alvaredo and B. Doyon, *Bi-partite entanglement entropy in integrable models with backscattering*, J. Phys. A: Math. Theor. **41**, 275203 (2008).

[80] B. Doyon, *Bipartite entanglement entropy in massive two-dimensional quantum field theory*, Phys. Rev. Lett. **102**, 031602 (2009).

[81] The approach to a Gaussian in the bulk of the spectrum follows from the central limit theorem for non-interacting systems and can be derived more generally with random matrix theory [86, 87]. At the spectral edges, the density of states is exponentially suppressed and follows, e.g., power laws [5] or stretched exponentials [88–90].

[82] T. Barthel and Q. Miao, *Scaling functions for eigenstate entanglement crossovers in harmonic lattices*, arXiv:1912.10045 (2019).

[83] Q. Miao and T. Barthel, *Eigenstate entanglement scaling for critical interacting spin chains*, arXiv:2010.07265 (2020).

[84] F. Verstraete and J. I. Cirac, *Matrix product states represent ground states faithfully*, Phys. Rev. B **73**, 094423 (2006).

[85] T. Barthel, *Typical one-dimensional quantum systems at finite temperatures can be simulated efficiently on classical computers*, arXiv:1708.09349 (2017).

[86] J. French and S. Wong, *Validity of random matrix theories for many-particle systems*, Phys. Lett. B **33**, 449 (1970).

[87] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, *Random-matrix physics: spectrum and strength fluctuations*, Rev. Mod. Phys. **53**, 385 (1981).

[88] H. A. Bethe, *An attempt to calculate the number of energy levels of a heavy nucleus*, Phys. Rev. **50**, 332 (1936).

[89] J. L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nucl. Phys. B **270**, 186 (1986).

[90] E. Verlinde, *On the holographic principle in a radiation dominated universe*, arXiv:hep-th/0008140 (2000).

[91] A. L. Fetter and J. D. Walecka, *Quantum theory of many-particle systems* (McGraw-Hill, San Francisco, 1971).

[92] E. T. Jaynes, *Information theory and statistical mechanics*, Phys. Rev. **106**, 620 (1957).