Square Peg in a Circular Hole: Choosing the Right Ansatz for Isolated Black Holes in Generic Gravitational Theories

Yiqi Xie, Jun Zhang, Hector O. Silva, Claudia de Rham, Helvi Witek, and Nicolás Yunes

Phys. Rev. Lett. **126**, 241104 — Published 17 June 2021

DOI: 10.1103/PhysRevLett.126.241104

A Square Peg in a Circular Hole: Choosing the Right Ansatz for Isolated Black Holes in Generic Gravitational Theories

Yiqi Xie,[1,*] Jun Zhang,[2,1,†] Hector O. Silva,[3,1,‡] Claudia de Rham,[2,4,§] Helvi Witek,[1,¶] and Nicolás Yunes[1,**]

[1] *Illinois Center for Advanced Studies of the Universe & Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA*

[2] *Theoretical Physics, Blackett Laboratory, Imperial College, London, SW7 2AZ, United Kingdom*

[3] *Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, D-14476 Potsdam, Germany*

[4] *CERCA, Department of Physics, Case Western Reserve University, 10900 Euclid Ave, Cleveland, OH 44106, USA*

(Dated: April 12, 2021)

The metric of a spacetime can be greatly simplified if the spacetime is circular. We prove that in generic effective theories of gravity, the spacetime of a stationary, axisymmetric and asymptotically flat solution must be circular if the solution can be obtained perturbatively from a solution in the General Relativity limit. This result applies to a broad class of gravitational theories, that include arbitrary scalars and vectors in their light sector, so long as their non-standard kinetic terms and non-mininal couplings to gravity are treated perturbatively.

Introduction.—Despite the complexity and nonlinearity of the Einstein equations, rotating black holes in General Relativity (GR) are described by a remarkably simple analytical solution obtained by Kerr [1, 2]. A crucial step in finding the Kerr solution is that the ten unknown functions of four coordinate variables in the metric can be reduced to four unknown functions of only two variables. This simplification is only possible because stationary and axisymmetric vacuum solutions in GR belong to a specific class called *circular spacetimes* [3]. However, this is not necessarily the case in generic gravitational theories [4] and one should not expect *a priori* that black hole solutions in such theories will be circular. In particular, one should expect the validity of the circularity assumption to play a role as important as it did in GR to obtain rotating black hole solutions (either numerically or analytically) in such theories. In turn, knowledge of these solutions constitutes the stepping stone upon which many tests of strong-field gravity rely [5]. The use of an over-simplified ansatz based on the circularity condition can lead to spacetimes that are inconsistent with a given theory's field equations. This was recently observed, for instance, in the case of rotating black hole solutions with linearly time-dependent hair in cubic Galileon theories in which the circularity condition is not satisfied [6].

In this Letter, we investigate the circularity of stationary and axisymmetric solutions in generic gravitational theories, paving the way for finding rotating black hole solutions in GR and beyond. In order to remain generic on the gravitational theory, we work within the effective field theory (EFT) framework, in which UV modifications of GR manifest as higher dimensional operators in the low-energy EFT and can be treated perturbatively. The EFT framework works well for isolated astrophysical black holes, which have masses in the $\sim 5{-}10^{10} M_\odot$ range thanks to their low energy scale ($\lesssim 10^{-11}$eV), and is also supported by the agreement of GR predictions with gravitational wave detections [7] and other electromagnetic observations [8]. In particular, we focus on gravitational theories whose low-energy EFT represents extensions of GR involving additional (scalar) fields and other operators. These EFTs include $f(R)$ gravity or more general scalar-tensor theories [9, 10], and quadratic gravity [11], such as dynamical Chern–Simons gravity [12, 13] and Einstein–dilaton–Gauss–Bonnet gravity [14, 15], as well as gravitational EFTs without light scalar fields, like those studied in [16–18].

As the modifications of GR are small, black hole solutions in the EFTs can be obtained through a perturbative expansion around one (or more) coupling constants of such theories (see [19–33] for examples). We show here that the spacetime of stationary, axisymmetric, and asymptotically flat solutions is circular in these EFTs, hence also in the corresponding high-energy gravitational theories. In principle, there could be other branches of solutions which are not connected perturbatively to their GR counterparts (see [34, 35] for example), but these are not the focus of this Letter. We use geometric units ($c = 8\pi G = 1$) and employ mostly plus metric signature.

Circular spacetimes in GR.—Consider a stationary and axisymmetric spacetime associated with two Killing vectors ξ^μ and χ^μ that correspond to the two isometries respectively. Fig. 1 gives a schematic illustration of this geometry. Carter [36] showed that the two Killing vectors commute, which means one can choose adapted coordinates (t, r, θ, ϕ) on the spacetime such that $\xi = \partial_t$ and $\chi = \partial_\phi$. The isometries imply

$$\partial_t g_{\mu\nu} = 0 = \partial_\phi g_{\mu\nu} \,. \tag{1}$$

Moreover, there exist privileged 2-dimensional surfaces, called *surfaces of transitivity*, to which the Killing vectors are everywhere tangent, except on the rotation axis where χ^μ vanishes. In adapted coordinates, the surfaces of transitivity can be labelled by the values of (r, θ).

A circular spacetime is a subclass of stationary and axisymmetric spacetimes for which, in addition to (1), there exists a family of 2-surfaces known as meridional surfaces, that are everywhere orthogonal to the surfaces

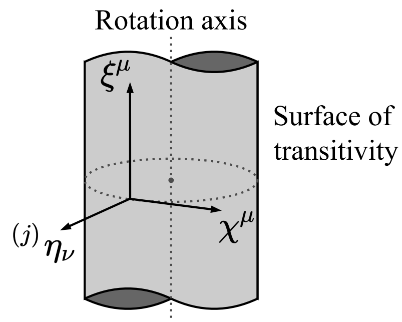

FIG. 1. Geometry of a stationary and axisymmetric spacetime. The Killing vector ξ^μ is associated with time translation and χ^μ is associated with rotations about the symmetry axis. Note that ξ^μ and χ^μ are not necessarily orthogonal. The surface of transitivity is generated by ξ^μ and χ^μ, and is degenerate on the rotation axis where χ^μ vanishes. The independent vectors $^{(j)}\eta_\nu$ $(j = 3, 4)$ are chosen to be orthogonal to the surface of transitivity. Here we only show one of the orthogonal vectors.

of transitivity. In this case, one can further choose the coordinates r and θ such that

$$g_{tr} = g_{t\theta} = g_{\phi r} = g_{\phi\theta} = 0\,. \tag{2}$$

Without loss of generality, the metric can then take the following ansatz

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = &-N^2 dt^2 + A^2\left(dr^2 + r^2 d\theta^2\right) \\ &+ B^2 r^2 \sin^2\theta \left(d\phi - \omega dt\right)^2\,, \end{aligned} \tag{3}$$

in *quasi-isotropic coordinates*, where N, A, B, and ω are functions of r and θ.

Papapetrou [3] (see also [37]) showed that a spacetime is circular if (i) $\xi_{[\mu}\chi_\nu\nabla_\rho\xi_{\sigma]}$ and $\xi_{[\mu}\chi_\nu\nabla_\rho\chi_{\sigma]}$ each vanish at least at one point of the spacetime, and (ii)

$$\xi^\mu R_\mu{}^{[\nu}\xi^\rho\chi^{\sigma]} = 0, \quad \chi^\mu R_\mu{}^{[\nu}\xi^\rho\chi^{\sigma]} = 0\,, \tag{4}$$

everywhere in spacetime, where the square brackets denote full antisymmetrization. For asymptotically flat spacetimes, which we focus on, Carter further showed that a rotation axis at which $\chi^\mu = 0$ exists [36], thus the first condition is satisfied.

Condition (4) is trivially satisfied if the Ricci tensor vanishes, which means that any stationary, axisymmetric and asymptotically flat vacuum solution in GR is circular, as well as those Ricci-flat solutions in modified gravity theories (e.g. [24, 38]). Condition (4) can also be recast as a requirement of the Ricci tensor being *invertible* [39, 40]. Let ${}_{(i)}\zeta^\mu$ $(i = 1, 2)$ be the two Killing vectors ξ^μ and χ^μ, and $^{(j)}\eta_\nu$ $(j = 3, 4)$ be two independent vectors everywhere orthogonal to ξ^μ and χ^μ. A tensor is said to be invertible in the isometry group, if the scalars obtained by contracting any combinations of the tensor's indices with any choice of ${}_{(i)}\zeta^\mu$ and $^{(j)}\eta_\nu$ vanish whenever the number of contracted ${}_{(i)}\zeta^\mu$ is odd. In particular, the Ricci tensor is invertible if

$$R_\mu{}^\nu\, {}_{(i)}\zeta^\mu\, {}^{(j)}\eta_\nu = 0\,, \quad i = 1, 2\,, \quad j = 3, 4\,. \tag{5}$$

Heuristically, condition (5) is equivalent to condition (4) because the latter is equivalent to requiring that ${}_{(i)}\zeta^\mu R_\mu{}^\nu$ be tangent to the surface of transitivity (i.e., proportional to any linear combination of ${}_{(i)}\zeta^\nu$), and thus, that any part tangent to the meridional surface (i.e., proportional to any linear combination of ${}_{(j)}\eta^\nu$) vanish. In the following, we shall omit the pre-sub/superscript of ζ^μ and η_ν, and bear in mind that each of them represents a vector in a two vector set.

Circularity in generic gravitational theories.—Let us consider a generic gravitational theory, potentially containing fields of arbitrary spin and coupling to gravity with Lagrangian,

$$\mathcal{L} = \frac{1}{2}R + \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_{\rm int}\left(\nabla_\rho,\, R_{\rho\sigma\alpha\beta},\, \varphi,\, \psi\right), \tag{6}$$

where the fields are classified as heavy fields ψ or light fields φ depending on whether their masses are above or below the curvature scale of the solution that we are interested in. Here, $\mathcal{L}_\varphi$ and $\mathcal{L}_\psi$ are the Lagrangians of φ and ψ, while $\mathcal{L}_{\rm int}$ captures all the interactions between the fields as well as any non-minimal couplings to gravity. In particular, we assume that non-standard kinetic terms of φ, if there is any in $\mathcal{L}_\varphi$, can be treated perturbatively. At the energy scale of the solution, we can integrate out the heavy fields with mass larger than the curvature of the solution we are interested in,

$$e^{i\int d^4x\sqrt{-g}\,\mathcal{L}_{\rm EFT}} = \int \mathcal{D}\psi\, e^{i\int d^4x\sqrt{-g}\,\mathcal{L}}, \tag{7}$$

and obtain a low-energy EFT with Lagrangian, (see Refs. [41–43] for explicit examples),

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}R + \mathcal{L}_0\left(\varphi,\, g_{\mu\nu}\right) + \alpha\,\mathcal{L}_{\rm M}\left(\nabla_\rho,\, R_{\rho\sigma\alpha\beta},\, \varphi\right), \tag{8}$$

where operators are sorted according to their dimensions. In particular, $\mathcal{L}_0$ are operators constructed by the light fields φ and their covariant derivatives with dimensions equal to or less than 4, while $\mathcal{L}_{\rm M}$ are higher dimension operators constructed by the Riemann tensor, the light fields and derivatives of both, and therefore are suppressed by a small parameter α. The heavy fields ψ in (6) have been integrated out and manifest themselves solely as higher curvature and derivative corrections in $\mathcal{L}_{\rm M}$. The curvature scale of isolated astrophysical black holes is expected to be smaller than 10^{-11} eV. Hence, in realistic situations, the heavy fields ψ include all massive particles of the Standard Model and beyond.

For now, we focus on the case in which the light fields, if any, are all scalar fields. We emphasize that φ denotes all light fields in the EFT, which we shall not distinguish with additional labels, and thus, inner products require an internal space metric, which we will also suppress [44]. This EFT reduces identically and smoothly to GR as $\alpha \to 0$, i.e. in this limit Eq. (8) reduces to the Einstein-Hilbert action minimally coupled to light scalar fields.

The modified Einstein equations in this theory are

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = T_{\mu\nu} + \alpha M_{\mu\nu}, \tag{9}$$

where $T_{\mu\nu} \equiv -2\delta\left(\sqrt{-g}\mathcal{L}_0\right)/\delta g^{\mu\nu}$ and $M_{\mu\nu} \equiv -2\delta\left(\sqrt{-g}\mathcal{L}_{\rm M}\right)/\delta g^{\mu\nu}$ are the energy-momentum tensors associated with $\mathcal{L}_0$ and $\mathcal{L}_{\rm M}$ respectively. In particular, terms in $T_{\mu\nu}$ are either proportional to $g_{\mu\nu}$ or proportional to $\partial_\mu\varphi\partial_\nu\varphi$ due to the dimension of the operators in $\mathcal{L}_0$. Given the smallness of α, a solution to Eq. (9) $\{g_{\mu\nu}, \varphi\}$ can be obtained order-by-order in α. For concreteness, we use $\{g^{(n)}_{\mu\nu}, \varphi^{(n)}\}$ to denote the solution to nth order in α, i.e. $g_{\mu\nu} = g^{(n)}_{\mu\nu} + \mathcal{O}(\alpha^{n+1})$, with $\mathcal{O}(\alpha^{n+1})$ accounting for all higher-order corrections. We also label a quantity with subscript or superscript (n), e.g. $T^{(n)}_{\mu\nu}$ or $g^{\mu\nu}_{(n)}$, if it is calculated up to nth order in α. The full solution is given by $\{g^{(n)}_{\mu\nu}, \varphi^{(n)}\}$ with n approaching infinity, for α sufficiently small.

In the following, we prove that the spacetime of a stationary, axisymmetric and asymptotically flat solution is necessarily circular, if the solution can be obtained order by order in α. Here we only consider solutions with stationary and axisymmetric scalar fields, which is not necessarily the case for the spacetime to be stationary and axisymmetric as we discuss later. We prove this statement in three steps.

First, we prove that the solution is circular at zeroth order in α, i.e. $g^{(0)}_{\mu\nu}$ is circular. At zeroth order, we get back to GR, and $g^{(0)}_{\mu\nu}$ is circular if $T^{(0)}_{\mu\nu}$ is invertible [39]. In order to show the invertibility, let us consider $T_\mu{}^\nu \zeta^\mu \eta_\nu$, where ζ^μ are the two Killing vectors and η_ν are the two independent vectors orthogonal to ζ^μ. Since the scalar fields are stationary and axisymmetric, the vanishing of their Lie derivatives along ζ^μ implies

$$\pounds_\zeta \varphi \equiv \zeta^\mu \partial_\mu \varphi = 0. \tag{10}$$

Thus, terms in $T_{\mu\nu}$ that are proportional to $\partial_\mu\varphi\partial_\nu\varphi$ vanish after contracting with ζ^μ. The rest of $T_{\mu\nu}$ is proportional to $g_{\mu\nu}$, and do not contribute to $T_\mu{}^\nu\zeta^\mu\eta_\nu$ given the orthogonality between ζ^μ and η_ν. Therefore,

$$T_\mu{}^\nu \zeta^\mu \eta_\nu = 0, \tag{11}$$

i.e. $T_{\mu\nu}$ is invertible. At zeroth order in α, Eq. (11) means $T^{(0)}_{\mu\nu}$ is invertible, and hence $g^{(0)}_{\mu\nu}$ is circular.

Next, we prove that if $g^{(0)}_{\mu\nu}$ is circular, then $g^{(1)}_{\mu\nu}$ is also circular. This can be proved if the Ricci tensor associated with $g^{(1)}_{\mu\nu}$ is invertible, or equivalently [45],

$$R_\mu{}^\nu \zeta^\mu \eta_\nu = 0 + \mathcal{O}(\alpha^2). \tag{12}$$

Contracting Eq. (9) with ζ^μ and η^ν, we find

$$R_\mu{}^\nu \zeta^\mu \eta_\nu = \alpha M_\mu{}^\nu \zeta^\mu \eta_\nu, \tag{13}$$

where the second term on the left hand side of Eq. (9) does not contribute due to the orthogonality between ζ^μ and η_ν, and the first term on the right hand side of Eq. (9) also vanishes because of the invertibility of $T_{\mu\nu}$.

On the other hand, since $g^{(0)}_{\mu\nu}$ is circular, the Riemann tensor associated with $g^{(0)}_{\mu\nu}$ is invertible (see the Supplemental Material [46] for a proof). Moreover, we show in the Supplemental Material [46] that any tensor constructed from stationary, axisymmetric, and invertible tensors and their covariant derivatives associated with $g^{(0)}_{\mu\nu}$ is also itself invertible. Together with the assumption that the scalar fields φ are stationary and axisymmetric, we conclude that $M_{\mu\nu}$ evaluated at zeroth order in α is invertible, and hence

$$M_\mu{}^\nu \zeta^\mu \eta_\nu = 0 + \mathcal{O}(\alpha). \tag{14}$$

Substituting Eq. (14) into Eq. (13), we find $R_\mu{}^\nu \zeta^\mu \eta_\nu$ vanishes to first order in α, and therefore, $g^{(1)}_{\mu\nu}$ is circular.

Finally, we assume the solution is circular to the nth order in α, and show that the solution to the $(n+1)$-th order is circular. The proof is similar to that in the second step. In this case, $M_\mu{}^\nu\zeta^\mu\eta_\nu$ can be evaluated to the n-th order in α with $g^{(n)}_{\mu\nu}$ and $\varphi^{(n)}$. The circularity of the n-th order solution implies that

$$M_\mu{}^\nu \zeta^\mu \eta_\nu = 0 + \mathcal{O}(\alpha^{n+1}). \tag{15}$$

Substituting this into Eq. (13), we find $R_\mu{}^\nu \zeta^\mu \eta_\nu$ vanishes to $(n+1)$-th order in α, and hence the solution to the $(n+1)$-th order is circular. By induction, we conclude that the solutions is circular to all orders in α.

Extension to generalized light fields.—Our proof can be further extended to theories with more general light fields, as long as the light fields and their leading-order stress-energy tensor $T_{\mu\nu}$ are invertible.

For light scalar fields, $\mathcal{L}_0$ may also include higher dimension operators that are arbitrary functions of φ, $\nabla_\mu\varphi\nabla^\mu\varphi$ and $\Box\varphi$. In this case, the resulting leading order stress-energy tensor is

$$T_{\mu\nu} = -\frac{\partial\mathcal{L}_0}{\partial(\nabla_\lambda\varphi\nabla^\lambda\varphi)}\nabla_\mu\varphi\nabla_\nu\varphi + \nabla_{(\mu}\left(\frac{\partial\mathcal{L}_0}{\partial(\Box\varphi)}\right)\nabla_{\nu)}\varphi + \frac{1}{2}g_{\mu\nu}\left[\mathcal{L}_0 - \nabla_\lambda\left(\frac{\partial\mathcal{L}_0}{\partial(\Box\varphi)}\nabla^\lambda\varphi\right)\right], \tag{16}$$

where terms proportional to $\nabla_\mu\varphi\nabla_\nu\varphi$ or $g_{\mu\nu}$ are invertible for the same reasons discussed above. Moreover, $\partial\mathcal{L}_0/\partial(\Box\varphi)$ inherits the symmetries of φ, so its Lie derivatives along ζ^μ vanish. Thus, the second term on the right hand side of (16), and hence the aggregated $T_{\mu\nu}$, is invertible, indicating stationary, axisymmetric and asymptotically flat vacuum solutions in such more general scalar-tensor theories are also circular.

In addition to the light scalars as described above, our proof can also be generalized to gravitational theories that include light vectors, as long as the non-standard kinetic terms and non-minimal couplings to gravity may be treated perturbatively. In particular, our proof can be extended to include light vectors with the following restrictions: (i) $\mathcal{L}_0$ is totally constructed from the vector fields V_μ and their exterior derivatives $F_{\mu\nu} = 2\nabla_{[\mu}V_{\nu]}$, and (ii) the vector fields V_μ, apart from being stationary and axisymmetric, are invertible. In this case, since the exterior derivative does not depend on the metric, the energy-momentum tensor associated with $\mathcal{L}_0$ is completely constructed from V_μ and $F_{\mu\nu}$, too. We show in the Supplemental Material [46] that $F_{\mu\nu}$ inherits the vector field's invertibility without assuming circularity. Therefore $T_{\mu\nu}$ is invertible, and any such vector-tensor theory admits circular ansatz for stationary and axisymmetric vacuum solutions. In addition, any Generalized Proca theory, as introduced in [47–51], would inherit the same properties so long as the higher-order Lagrangians introduced in these theories are treated perturbatively.

Discussions.—Our main result is a proof that the spacetime of stationary, axisymmetric, and asymptotically flat rotating black holes in a broad class of gravitational EFTs is circular. We emphasize that in addition to the light fields we have considered, the theory may also include any heavy field of arbitrary spin and coupling to gravity, as long as the mass of these fields is larger than the curvature scale of the black holes. Our result is of immediate importance to the ongoing effort of testing the strong-field regime of gravity through gravitational wave [52–56] and electromagnetic observations [57, 58] both in which black holes play a central role [5]. These tests generically require knowledge of a rotating black hole solution (within a certain EFT) from which observable consequences are then deduced and then ultimately confronted against observations. Here we proved that circularity is shared among a broad class of solutions, justifying the use of this ansatz when searching for analytical and numerical solutions.

What are the implications of our result to some specific theories? Consider, for instance, dynamical Chern-Simons gravity, in which a scalar field couples to the Pontryagin density [12, 13]. This theory must be treated as an EFT to admit an well-posed initial value problem [59] and, in fact, this theory is captured within the assumption of our proof. Rotating black hole solutions in this theory are known both numerically [60, 61] and analytically [20, 27, 62–64], in a perturbative expansion in the coupling strength α and black hole spin $a \ll 1$ to $\mathcal{O}(\alpha^2 a^5)$ [26, 30, 62] and in the extremal limit [65]. Our results indicate that the spacetime of rotating black holes in this theory is circular, justifying the use of the ansatz (3) in numerical calculations. The same applies to scalar Gauss–Bonnet gravity with shift-symmetric and dilatonic couplings where rotating black hole spacetimes are known both analytically [24, 25, 27–29] and numerically [66–69], including the final state of black holes that results at late times after highly dynamical black hole formation [70–72]. In fact, it applies to any EFT extension of GR, including any low-energy EFT of gravity that include massive fields of arbitrary spins.

Our results agree with those of [73], which suggested the non-existence of rotating non-circular black holes in dynamical Chern-Simons gravity and shift-symmetric scalar-Gauss-Bonnet gravity, by working perturbatively to $\mathcal{O}(\alpha^2 a^2)$. Our conclusions extend to all orders in these two parameters. Moreover, our results also apply to non-vacuum solutions in generic gravitational theories of the type discussed in this Letter, as long as the matter fields in the GR solution are stationary, axisymmetric and possess an invertible stress-energy tensor. That is, our conclusion holds for a gravitational theory minimally coupled to an ordinary matter source, such as a perfect fluid that satisfies the same symmetries as the metric (i.e. stationarity and axisymmetry).

We stress that our results only apply to solutions that reduce to a GR solution in the limit when the perturbative parameter α goes to zero. In general, this does not have to be the case, as other branches of solutions may be entropically favoured, as is the case in theories that exhibit spontaneous black hole scalarization [34, 35].

The requirement that the fields are stationary and axisymmetric (and invertible if of spin-1) is a sufficient but not a necessary condition for the solution to be circular, and it is not necessarily required by the isometries of the spacetime. There are cases in which the extra fields can be time- and angle-dependent, yet this dependence does not manifest itself in the gravitational equations. For example, there are hairy, nonlinear black hole solutions and solitonic solutions that arise in GR coupled to complex and massive (scalar) fields [74–77], where the metric is circular while the fields have time- or angle-dependent phases. Other examples are the stealth black holes of [78], in which the scalar field has a linear time dependence, although such black hole solutions usually suffer from a strong coupling problem [79–81].

Our results do imply that if a theory satisfies the conditions of our theorem, then *all* black hole solutions must have a circular spacetime, but the converse is not necessarily true. Imagine one were to find a black hole solution in a modified theory (in which our theorem does not apply) by requiring *a priori* that the spacetime be circular. The existence of this solution does not then mean that other non-circular solutions do not exist. For example, black hole solutions have been found in Einstein-Yang-Mills theories with [82] and without [83] a dilaton field, and in Einstein-æther theory in the slow-rotation approximation [84, 85] assuming *a priori* that the spacetime must be circular. In both cases, however, our theorem does not apply because either the Yang-Mills vector gauge field is non-invertible after gauge fixing or the æther field is non-invertible because of its timelike constraint. Thus, the existence of those solutions does not imply that other non-circular black hole solutions do not exist in these theories, which could be explored further.

Acknowledgements.—We thank Lydia Bieri, Daniela Doneva, David Garfinkle, Leonardo Gualtieri, Carlos A. R. Herdeiro and Jutta Kunz for discussions. Y.X, H.O.S and N.Y. acknowledge financial support through NSF grants No. PHY-1759615, PHY-1949838 and NASA ATP Grant No. 17-ATP17-0225, No. NNX16AB98G and No. 80NSSC17M0041. C.d.R and J.Z. acknowledge financial support provided by the European Union's Horizon 2020 Research Council grant 724659 MassiveCosmo ERC-2016-COG. H.W. acknowledges financial support provided by the NSF Grant No. OAC-2004879 and the Royal Society Research Grant No. RGF\R1\180073. C.d.R also acknowledges financial support provided by STFC grants ST/P000762/1 and ST/T000791/1, by the Royal Society through a Wolfson Research Merit Award, by the Simons Foundation award ID 555326 under the Simons Foundation's Origins of the Universe initiative, '*Cosmology Beyond Einstein's Theory*' and by the Simons Investigator award 690508.

* yiqixie2@illinois.edu
† jun.zhang@imperial.ac.uk
‡ hector.silva@aei.mpg.de
§ c.de-rham@imperial.ac.uk
¶ hwitek@illinois.edu
** nyunes@illinois.edu

[1] R. P. Kerr, Gravitational field of a spinning mass as an example of algebraically special metrics, Phys. Rev. Lett. **11**, 237 (1963).

[2] R. P. Kerr and A. Schild, Republication of: A new class of vacuum solutions of the Einstein field equations, General Relativity and Gravitation **41**, 2485 (2009).

[3] A. Papapetrou, Champs gravitationnels stationnaires a symetrie axiale, Ann. Inst. H. Poincare Phys. Theor. **4**, 83 (1966).

[4] E. Berti *et al.*, Testing General Relativity with Present and Future Astrophysical Observations, Class. Quant. Grav. **32**, 243001 (2015), arXiv:1501.07274 [gr-qc].

[5] K. Yagi and L. C. Stein, Black Hole Based Tests of General Relativity, Class. Quant. Grav. **33**, 054001 (2016), arXiv:1602.02413 [gr-qc].

[6] K. Van Aelst, E. Gourgoulhon, P. Grandclément, and C. Charmousis, Hairy rotating black holes in cubic Galileon theory, Class. Quant. Grav. **37**, 035007 (2020), arXiv:1910.08451 [gr-qc].

[7] B. P. Abbott *et al.* (LIGO Scientific, Virgo), Tests of General Relativity with the Binary Black Hole Signals from the LIGO-Virgo Catalog GWTC-1, Phys. Rev. D **100**, 104036 (2019), arXiv:1903.04467 [gr-qc].

[8] R. Abuter *et al.* (GRAVITY), Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole, Astron. Astrophys. **636**, L5 (2020), arXiv:2004.07187 [astro-ph.GA].

[9] T. P. Sotiriou and V. Faraoni, $f(R)$ Theories Of Gravity, Rev. Mod. Phys. **82**, 451 (2010), arXiv:0805.1726 [gr-qc].

[10] T. Kobayashi, Horndeski theory and beyond: a review, Rept. Prog. Phys. **82**, 086901 (2019), arXiv:1901.07183 [gr-qc].

[11] K. Yagi, L. C. Stein, and N. Yunes, Challenging the Presence of Scalar Charge and Dipolar Radiation in Binary Pulsars, Phys. Rev. D **93**, 024010 (2016), arXiv:1510.02152 [gr-qc].

[12] R. Jackiw and S. Y. Pi, Chern-Simons modification of general relativity, Phys. Rev. D **68**, 104012 (2003), arXiv:gr-qc/0308071.

[13] S. Alexander and N. Yunes, Chern-Simons Modified General Relativity, Phys. Rept. **480**, 1 (2009), arXiv:0907.2562 [hep-th].

[14] R. R. Metsaev and A. A. Tseytlin, Curvature Cubed Terms in String Theory Effective Actions, Phys. Lett. B **185**, 52 (1987).

[15] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Dilatonic black holes in higher curvature string gravity, Phys. Rev. D **54**, 5049 (1996), arXiv:hep-th/9511071.

[16] S. Endlich, V. Gorbenko, J. Huang, and L. Senatore, An effective formalism for testing extensions to General Relativity with gravitational waves, JHEP **09**, 122, arXiv:1704.01590 [gr-qc].

[17] N. Sennett, R. Brito, A. Buonanno, V. Gorbenko, and L. Senatore, Gravitational-Wave Constraints on an Effective Field-Theory Extension of General Relativity, Phys. Rev. D **102**, 044056 (2020), arXiv:1912.09917 [gr-qc].

[18] C. de Rham, J. Francfort, and J. Zhang, Black Hole Gravitational Waves in the Effective Field Theory of Gravity, Phys. Rev. D **102**, 024079 (2020), arXiv:2005.13923 [hep-th].

[19] B. A. Campbell, M. J. Duncan, N. Kaloper, and K. A. Olive, Gravitational dynamics with Lorentz Chern-Simons terms, Nucl. Phys. B **351**, 778 (1991).

[20] B. A. Campbell, M. J. Duncan, N. Kaloper, and K. A. Olive, Axion hair for Kerr black holes, Phys. Lett. B **251**, 34 (1990).

[21] B. A. Campbell, N. Kaloper, and K. A. Olive, Classical hair for Kerr-Newman black holes in string gravity, Phys. Lett. B **285**, 199 (1992).

[22] B. A. Campbell, N. Kaloper, R. Madden, and K. A. Olive, Physical properties of four-dimensional superstring gravity black hole solutions, Nucl. Phys. B **399**, 137 (1993), arXiv:hep-th/9301129.

[23] S. Mignemi and N. R. Stewart, Charged black holes in effective string theory, Phys. Rev. D **47**, 5259 (1993), arXiv:hep-th/9212146.

[24] N. Yunes and L. C. Stein, Non-Spinning Black Holes in Alternative Theories of Gravity, Phys. Rev. D **83**, 104002 (2011), arXiv:1101.2921 [gr-qc].

[25] P. Pani, C. F. B. Macedo, L. C. B. Crispino, and V. Cardoso, Slowly rotating black holes in alternative theories of gravity, Phys. Rev. D **84**, 087501 (2011), arXiv:1109.3996 [gr-qc].

[26] K. Yagi, N. Yunes, and T. Tanaka, Slowly Rotating Black Holes in Dynamical Chern-Simons Gravity: Deformation Quadratic in the Spin, Phys. Rev. D **86**, 044037 (2012), [Erratum: Phys.Rev.D 89, 049902 (2014)], arXiv:1206.6130 [gr-qc].

[27] D. Ayzenberg and N. Yunes, Slowly-Rotating Black Holes in Einstein-Dilaton-Gauss-Bonnet Gravity: Quadratic Order in Spin Solutions, Phys. Rev. D **90**, 044066

(2014), [Erratum: Phys.Rev.D 91, 069905 (2015)], arXiv:1405.2133 [gr-qc].

[28] A. Maselli, P. Pani, L. Gualtieri, and V. Ferrari, Rotating black holes in Einstein-Dilaton-Gauss-Bonnet gravity with finite coupling, Phys. Rev. D **92**, 083014 (2015), arXiv:1507.00680 [gr-qc].

[29] A. Maselli, H. O. Silva, M. Minamitsuji, and E. Berti, Slowly rotating black hole solutions in Horndeski gravity, Phys. Rev. D **92**, 104049 (2015), arXiv:1508.03044 [gr-qc].

[30] A. Maselli, P. Pani, R. Cotesta, L. Gualtieri, V. Ferrari, and L. Stella, Geodesic models of quasi-periodic-oscillations as probes of quadratic gravity, Astrophys. J. **843**, 25 (2017), arXiv:1703.01472 [astro-ph.HE].

[31] V. Cardoso, M. Kimura, A. Maselli, and L. Senatore, Black Holes in an Effective Field Theory Extension of General Relativity, Phys. Rev. Lett. **121**, 251105 (2018), arXiv:1808.08962 [gr-qc].

[32] F.-L. Julié and E. Berti, Post-Newtonian dynamics and black hole thermodynamics in Einstein-scalar-Gauss-Bonnet gravity, Phys. Rev. D **100**, 104061 (2019), arXiv:1909.05258 [gr-qc].

[33] P. A. Cano and A. Ruipérez, Leading higher-derivative corrections to Kerr geometry, JHEP **05**, 189, [Erratum: JHEP 03, 187 (2020)], arXiv:1901.01315 [gr-qc].

[34] D. D. Doneva and S. S. Yazadjiev, New Gauss-Bonnet Black Holes with Curvature-Induced Scalarization in Extended Scalar-Tensor Theories, Phys. Rev. Lett. **120**, 131103 (2018), arXiv:1711.01187 [gr-qc].

[35] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Spontaneous scalarization of black holes and compact stars from a Gauss-Bonnet coupling, Phys. Rev. Lett. **120**, 131104 (2018), arXiv:1711.02080 [gr-qc].

[36] B. Carter, The commutation property of a stationary, axisymmetric system, Commun. Math. Phys. **17**, 233 (1970).

[37] R. M. Wald, *General Relativity* (Chicago Univ. Pr., Chicago, USA, 1984).

[38] H. Motohashi and M. Minamitsuji, General Relativity solutions in modified gravity, Phys. Lett. B **781**, 728 (2018), arXiv:1804.01731 [gr-qc].

[39] B. Carter, Killing horizons and orthogonally transitive groups in space-time, J. Math. Phys. **10**, 70 (1969).

[40] B. Carter, Black holes equilibrium states, in *Les Houches Summer School of Theoretical Physics: Black Holes* (1973) pp. 57–214.

[41] I. G. Avramidi, The Covariant Technique for Calculation of One Loop Effective Action, Nucl. Phys. **B355**, 712 (1991), [Erratum: Nucl. Phys.B509,557(1998)].

[42] I. G. Avramidi, *Covariant methods for the calculation of the effective action in quantum field theory and investigation of higher derivative quantum gravity*, Ph.D. thesis, Moscow State U. (1986), arXiv:hep-th/9510140 [hep-th].

[43] C. de Rham and A. J. Tolley, Speed of gravity, Phys. Rev. D **101**, 063518 (2020), arXiv:1909.00881 [hep-th].

[44] T. Damour and G. Esposito-Farèse, Tensor multiscalar theories of gravitation, Class. Quant. Grav. **9**, 2093 (1992).

[45] To be precise, Eq. (12) does not immediately indicate $g^{(1)}_{\mu\nu}$ is circular. Instead, it indicates the metric corresponding to $\left(R_\mu{}^\nu\right)^{(1)}$, which could be different from $g^{(1)}_{\mu\nu}$ at $\mathcal{O}(\alpha^2)$, is circular. Nevertheless, it means there exist coordinates, in which certain components of the metric corresponding to $\left(R_\mu{}^\nu\right)^{(1)}$ vanish (cf. Eq. (2)). Those components remain zero after truncating all $\mathcal{O}(\alpha^2)$ terms, in which case the metric corresponding to $\left(R_\mu{}^\nu\right)^{(1)}$ reduces to $g^{(1)}_{\mu\nu}$. Thus, the Ricci tensor associated with $g^{(1)}_{\mu\nu}$ is invertible and $g^{(1)}_{\mu\nu}$ is circular.

[46] See Supplemental Material, which includes Refs. [39], for a detailed discussion on tensor invertibility and its preservation through common operations.

[47] G. Tasinato, Cosmic Acceleration from Abelian Symmetry Breaking, JHEP **04**, 067, arXiv:1402.6450 [hep-th].

[48] L. Heisenberg, Generalization of the Proca Action, JCAP **05**, 015, arXiv:1402.7026 [hep-th].

[49] E. Allys, P. Peter, and Y. Rodriguez, Generalized Proca action for an Abelian vector field, JCAP **02**, 004, arXiv:1511.03101 [hep-th].

[50] E. Allys, J. P. Beltran Almeida, P. Peter, and Y. Rodríguez, On the 4D generalized Proca action for an Abelian vector field, JCAP **09**, 026, arXiv:1605.08355 [hep-th].

[51] C. de Rham and V. Pozsgay, New class of Proca interactions, Phys. Rev. D **102**, 083508 (2020), arXiv:2003.13773 [hep-th].

[52] J. R. Gair, M. Vallisneri, S. L. Larson, and J. G. Baker, Testing General Relativity with Low-Frequency, Space-Based Gravitational-Wave Detectors, Living Rev. Rel. **16**, 7 (2013), arXiv:1212.5575 [gr-qc].

[53] N. Yunes and X. Siemens, Gravitational-Wave Tests of General Relativity with Ground-Based Detectors and Pulsar Timing-Arrays, Living Rev. Rel. **16**, 9 (2013), arXiv:1304.3473 [gr-qc].

[54] E. Berti, K. Yagi, and N. Yunes, Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (I) Inspiral-Merger, Gen. Rel. Grav. **50**, 46 (2018), arXiv:1801.03208 [gr-qc].

[55] E. Berti, K. Yagi, H. Yang, and N. Yunes, Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (II) Ringdown, Gen. Rel. Grav. **50**, 49 (2018), arXiv:1801.03587 [gr-qc].

[56] E. Barausse *et al.*, Prospects for Fundamental Physics with LISA, Gen. Rel. Grav. **52**, 81 (2020), arXiv:2001.09793 [gr-qc].

[57] H. Krawczynski, Difficulties of Quantitative Tests of the Kerr-Hypothesis with X-Ray Observations of Mass Accreting Black Holes, Gen. Rel. Grav. **50**, 100 (2018), arXiv:1806.10347 [astro-ph.HE].

[58] D. Psaltis, Testing General Relativity with the Event Horizon Telescope, Gen. Rel. Grav. **51**, 137 (2019), arXiv:1806.09740 [astro-ph.HE].

[59] T. Delsate, D. Hilditch, and H. Witek, Initial value formulation of dynamical Chern-Simons gravity, Phys. Rev. D **91**, 024027 (2015), arXiv:1407.6727 [gr-qc].

[60] L. C. Stein, Rapidly rotating black holes in dynamical Chern-Simons gravity: Decoupling limit solutions and breakdown, Phys. Rev. D **90**, 044061 (2014), arXiv:1407.2350 [gr-qc].

[61] T. Delsate, C. Herdeiro, and E. Radu, Non-perturbative spinning black holes in dynamical Chern–Simons gravity, Phys. Lett. B **787**, 8 (2018), arXiv:1806.06700 [gr-qc].

[62] N. Yunes and F. Pretorius, Dynamical Chern-Simons Modified Gravity. I. Spinning Black Holes in the Slow-Rotation Approximation, Phys. Rev. D **79**, 084043 (2009), arXiv:0902.4669 [gr-qc].

[63] K. Konno, T. Matsuyama, and S. Tanda, Rotating black

hole in extended Chern-Simons modified gravity, Prog. Theor. Phys. **122**, 561 (2009), arXiv:0902.4767 [gr-qc].
[64] K. Konno and R. Takahashi, Scalar field excited around a rapidly rotating black hole in Chern-Simons modified gravity, Phys. Rev. D **90**, 064011 (2014), arXiv:1406.0957 [gr-qc].
[65] R. McNees, L. C. Stein, and N. Yunes, Extremal black holes in dynamical Chern–Simons gravity, Class. Quant. Grav. **33**, 235013 (2016), arXiv:1512.05453 [gr-qc].
[66] B. Kleihaus, J. Kunz, and E. Radu, Rotating Black Holes in Dilatonic Einstein-Gauss-Bonnet Theory, Phys. Rev. Lett. **106**, 151104 (2011), arXiv:1101.2868 [gr-qc].
[67] P. Pani and V. Cardoso, Are black holes in alternative theories serious astrophysical candidates? The Case for Einstein-Dilaton-Gauss-Bonnet black holes, Phys. Rev. D **79**, 084031 (2009), arXiv:0902.1569 [gr-qc].
[68] J. F. M. Delgado, C. A. R. Herdeiro, and E. Radu, Spinning black holes in shift-symmetric Horndeski theory, JHEP **04**, 180, arXiv:2002.05012 [gr-qc].
[69] A. Sullivan, N. Yunes, and T. P. Sotiriou, Numerical Black Hole Solutions in Modified Gravity Theories: Axial Symmetry Case, (2020), arXiv:2009.10614 [gr-qc].
[70] R. Benkel, T. P. Sotiriou, and H. Witek, Black hole hair formation in shift-symmetric generalised scalar-tensor gravity, Class. Quant. Grav. **34**, 064001 (2017), arXiv:1610.09168 [gr-qc].
[71] J. L. Ripley and F. Pretorius, Scalarized Black Hole dynamics in Einstein dilaton Gauss-Bonnet Gravity, Phys. Rev. D **101**, 044015 (2020), arXiv:1911.11027 [gr-qc].
[72] J. L. Ripley and F. Pretorius, Gravitational collapse in Einstein dilaton-Gauss–Bonnet gravity, Class. Quant. Grav. **36**, 134001 (2019), arXiv:1903.07543 [gr-qc].
[73] K. Nakashi and M. Kimura, Towards rotating noncircular black holes in string-inspired gravity, Phys. Rev. D **102**, 084021 (2020), arXiv:2008.04003 [gr-qc].
[74] C. A. R. Herdeiro and E. Radu, Kerr black holes with scalar hair, Phys. Rev. Lett. **112**, 221101 (2014), arXiv:1403.2757 [gr-qc].
[75] C. A. R. Herdeiro and E. Radu, Asymptotically flat black holes with scalar hair: a review, Int. J. Mod. Phys. D **24**, 1542014 (2015), arXiv:1504.08209 [gr-qc].
[76] C. Herdeiro, I. Perapechka, E. Radu, and Y. Shnir, Skyrmions around Kerr black holes and spinning BHs with Skyrme hair, JHEP **10**, 119, arXiv:1808.05388 [gr-qc].
[77] C. A. R. Herdeiro and J. a. M. S. Oliveira, On the inexistence of solitons in Einstein–Maxwell-scalar models, Class. Quant. Grav. **36**, 105015 (2019), arXiv:1902.07721 [gr-qc].
[78] C. Charmousis, M. Crisostomi, R. Gregory, and N. Stergioulas, Rotating Black Holes in Higher Order Gravity, Phys. Rev. D **100**, 084020 (2019), arXiv:1903.05519 [hep-th].
[79] E. Babichev, C. Charmousis, G. Esposito-Farèse, and A. Lehébel, Hamiltonian unboundedness vs stability with an application to Horndeski theory, Phys. Rev. D **98**, 104050 (2018), arXiv:1803.11444 [gr-qc].
[80] C. de Rham and J. Zhang, Perturbations of stealth black holes in degenerate higher-order scalar-tensor theories, Phys. Rev. D **100**, 124023 (2019), arXiv:1907.00699 [hep-th].
[81] H. Ogawa, T. Kobayashi, and T. Suyama, Instability of hairy black holes in shift-symmetric Horndeski theories, Phys. Rev. D **93**, 064078 (2016), arXiv:1510.07400 [gr-qc].
[82] B. Kleihaus, J. Kunz, and F. Navarro-Lerida, Rotating dilaton black holes with hair, Phys. Rev. D **69**, 064028 (2004), arXiv:gr-qc/0306058.
[83] B. Kleihaus and J. Kunz, Rotating hairy black holes, Phys. Rev. Lett. **86**, 3704 (2001), arXiv:gr-qc/0012081.
[84] E. Barausse and T. P. Sotiriou, Black holes in Lorentz-violating gravity theories, Class. Quant. Grav. **30**, 244010 (2013), arXiv:1307.3359 [gr-qc].
[85] E. Barausse, T. P. Sotiriou, and I. Vega, Slowly rotating black holes in Einstein-æther theory, Phys. Rev. D **93**, 044044 (2016), arXiv:1512.05894 [gr-qc].