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Gaussian versus non-Gaussian filtering of phase-insensitive nonclassicality

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Measures of quantum properties are essential to understanding the fundamental differences between quantum and classical systems as well as quantifying resources for quantum technologies. Here two broad classes of bosonic phase-space functions, which are filtered versions of the Glauber-Sudarshan P function, are compared with regard to their ability to uncover nonclassical effects of light through their negativities. Gaussian filtering of the P function yields the family of s-parametrized quasiprobabilities, while more powerful regularized nonclassicality quasiprobabilities are obtained by non-Gaussian filtering. A method is proposed to directly sample such phase-space functions for the restricted case of phase-independent quantum states from balanced homodyne measurements. This overcomes difficulties of previous approaches that manually append uniformly distributed optical phases to the measured quadrature data. We experimentally demonstrate this technique for heralded single- and two-photon states using balanced homodyne detection with varying efficiency. The s-parametrized quasiprobabilities, which can be directly sampled, are non-negative for detection efficiencies below 0.5. By contrast, we show that significant negativities of non-Gaussian filtered quasiprobabilities uncover nonclassical effects for arbitrarily low efficiencies.

Introduction.—The distinction between quantum and classical properties of a physical system has played a fundamental role in the development of quantum theory since its inception [1]. Tools to probe the boundary between the classical and quantum domains have developed over more than a century since the foundations of quantum theory were set out. These techniques have become increasingly important in quantum information science, where the ability to quantify nonclassical, i.e. quantum, properties of a physical system determines how well the system can perform a particular technological task [2]. Phase-space distributions have emerged as canonical representations of quantum systems that can be utilized to distinguish their nonclassical properties [3, 4].

There are a number of distinguishing characteristics displayed by quantum states of light. For single-mode fields, such non-classical traits include non-Gaussian Wigner representations and singular Glauber-Sudarshan P-function. These reflect different characteristics of quantum states of light, which have different utility in quantum information science and technology. In 1963, it was discovered that all states of a single electromagnetic field mode can be represented in the form [5, 6]

$$\hat{\rho} = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|, \tag{1}$$

by means of the Glauber-Sudarshan P function $P(\alpha)$, which contains complete information about a state $\hat{\rho}$. Coherent states $|\alpha\rangle$ are known to be the only pure states with a non-negative P function [7] and can be considered as the analogue of a classical radiation field of amplitudes α. The P function of a statistical mixture of coherent states is thus the same as the classical phase-space distribution of the corresponding statistical distribution over classical amplitudes. Accordingly, it is reasonable to define a state as classical if it has a representation as in Eq. (1) with a classical probability distribution $P(\alpha) = P_{\mathrm{cl}}(\alpha)$ [8]. A remarkable aspect of quantum physics is that it allows states that cannot be represented by a completely positive P function and thus are called nonclassical states of the field [9]. Prominent examples of nonclassical states with negative P function values include single-photon states and squeezed states. It would be an easy task to experimentally certify nonclassicality if the P function of all states existed as a regular function, but the opposite is the case. In fact, this function can even contain infinite derivatives of the Dirac δ distribution [10]. Accordingly, the P function is in general not accessible experimentally. An established approach to uncover nonclassicality in phase space is by convolving the P function with a Gaussian function to transform it into a regular function, so-called s-parametrized quasiprobabilities, whose negativities unambiguously represent nonclassicality of the state [11]. A drawback of this method is that many states, such as the important class of squeezed Gaussian states, are not identified as nonclassical, since their Gaussian-regularized P function is nonnegative. In addition, such approaches to identifying nonclassicality of states place strict requirements on measurement efficiency. For this reason, non-Gaussian filter functions with specific properties have been introduced [12]. Filtering the P function with non-Gaussian functions allows a complete nonclassicality test, as the strength of filtering can be arbitrarily reduced while preserving the regularity of the resulting nonclassicality quasiprobability distribution, and benefits from reduced sensitivity to detector efficiency.

In the present Letter, we study the benefits of non-

Gaussian compared to Gaussian filtered P functions to uncover nonclassical effects in the realistic scenario of low quantum efficiency detection on the basis of experimental quadrature data of lossy single- and two-photon states. Experimental access to regular phase-space representations is provided by balanced homodyne detection (BHD) of light [13–15], which measures the quadrature statistics of the electric field strength of the radiation field. In Refs. [16, 17] direct sampling formulas were introduced to easily obtain quasiprobabilities from phase-sensitive BHD data. We go beyond this by developing direct sampling formulas of non-Gaussian regularized P functions from phase-insensitive quadrature measurements via BHD. This method is then applied to BHD of heralded single- and two-photon states for different detection efficiencies.

Regular phase-space functions.— Singularities of the Glauber-Sudarshan P function dictate alternative strategies to experimentally access nonclassicalities of quantum states. The P function is the Fourier transform

$$P(\alpha) = \frac{1}{\pi^2} \int d^2\beta \, e^{\alpha\beta^* - \alpha^*\beta} \Phi(\beta), \tag{2}$$

of the characteristic function $\Phi(\beta)$. The latter can grow maximally as $e^{|\beta|^2/2}$; see Ref. [10] for details. Accordingly its Fourier transform $P(\alpha)$ does in this case not exist as a regular function.

To obtain regular phase-space functions the characteristic function can be multiplied by a filter function $\Omega(\beta)$ which decays stronger than the Gaussian $e^{-|\beta|^2/2}$, resulting in a new phase-space function

$$P_\Omega(\alpha) = \frac{1}{\pi^2} \int d^2\beta \, e^{\alpha\beta^* - \alpha^*\beta} \Omega(\beta)\Phi(\beta). \tag{3}$$

If the Fourier transform of the filter function is nonnegative, the regularization procedure does not introduce negativities in $P_\Omega(\alpha)$. Accordingly, a negativity of the latter unambiguously certifies nonclassicality of the state. Non-Gaussian filter functions which fulfill these requirements were introduced in Ref. [12] as autocorrelations

$$\Omega_w^{(q)}(\beta) = \int d^2\gamma \, \chi_w^{(q)*}(\gamma) \, \chi_w^{(q)}(\beta + \gamma) \tag{4}$$

of functions of the form

$$\chi_w^{(q)}(\beta) = \frac{1}{w} 2^{1/q} \sqrt{\frac{q}{2\pi\Gamma(2/q)}} \exp\left[-\left(\frac{|\beta|}{w} \right)^q \right], \tag{5}$$

where $2 < q < \infty$, w is a positive value, and $\Gamma(\cdot)$ is the gamma function. The parameter w controls the width of the filter and thus the degree of filtering, which affects how smooth the resulting quasi-probability distribution becomes. For $w \to \infty$ one obtains the original P function. In the limit $q \to \infty$ the filter function in Eq. (4) has the analytical form [18]

$$\Omega(\beta) = \frac{2}{\pi} \left[\arccos\left(\frac{|\beta|}{2w} \right) - \frac{|\beta|}{2w} \sqrt{1 - \frac{|\beta|^2}{4w^2}} \right] \mathrm{rect}\left(\frac{|\beta|}{4w} \right), \tag{6}$$

where $\mathrm{rect}(x)$ is one for $x \leq 1/2$ and zero otherwise. In the opposite limiting case $q = 2$, the function in Eq. (4) is essentially the autocorrelation of two Gaussians and, therefore, it reduces to a Gaussian function. Rescaling the parameter w according to $s = 1 - 1/w^2$ with $s \leq 1$, yields the filter function

$$\Omega(\beta) = \exp\left(-\frac{1-s}{2} |\beta|^2 \right). \tag{7}$$

Inserting this Gaussian filter in Eq. (3), the established class of s-parametrized quasiprobabilities $P(\alpha; s)$ is retrieved [11]. To guarantee regularity of these quasiprobabilities s must be chosen less-equal to 0. In other words, only s-parametrized quasiprobabilities at least as smooth as the Wigner function, $W(\alpha) = P(\alpha; 0)$, are regular for all quantum states [19]. It is known that for $s \leq -1$ the corresponding quasiprobability is always nonnegative [20], i.e., nonclassicality cannot be identified by negativities in this range. Consequently, this family of phase-space functions is useful only for $-1 < s \leq 0$ to identify nonclassicality through negative values. Accordingly, only a subset of all states is uncovered to be nonclassical by these quasiprobabilities. Note that the important class of the squeezed vacuum states is not included in this set. A further disadvantage of the s-parametrized quasiprobability is the inability to certify nonclassicality in the presence of high constant losses or low detection efficiency η. In particular, all states detected with $\eta \leq 0.5$ have nonnegative s-parametrized quasiprobabilities for $s \leq 0$.

By contrast, filters with $q > 2$ in Eq. (4) provide a full nonclassicality test, since the corresponding filtered function P_Ω is always regular for arbitrarily large parameter w. We want to point out here that an arbitrarily small quantum efficiency η can be compensated by a rescaled larger filter parameter $w/\sqrt{\eta}$. That is why these filters are referred to as nonclassicality filters and the functions P_Ω are denoted as nonclassicality quasiprobabilities [12]. It is a central purpose of this work to demonstrate the power of these nonclassicality quasiprobabilities to certify nonclassical effects in the presence of high losses ($\eta < 0.5$), where the s-parametrized quasiprobabilities fail to display any nonclassicality.

Direct sampling of quasiprobabilities.— The phase-dependent statistics $p(x;\varphi) = \langle x;\varphi|\hat{\rho}|x;\varphi\rangle$ of the quadrature x, which is proportional to the electric field strength of the electromagnetic radiation at the optical phase φ, contains full information about a quantum state $\hat{\rho}$. Accordingly there exists a unique mapping from this probability distribution to the quasiprobabilities in Eq. (3).

In particular, both quantities are connected by the relation [16]

$$P_\Omega(\alpha) = \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\varphi \, \frac{p(x;\varphi)}{2\pi} \, f_\Omega(x,\varphi;\alpha), \quad (8)$$

via the pattern function

$$f_\Omega(x,\varphi;\alpha) = \frac{2}{\pi} \int_0^\infty db \, b \, e^{b^2/2} \Omega(b) \cos[\xi(x,\varphi,\alpha) b], \quad (9)$$

where

$$\xi(x,\varphi,\alpha) = x + 2|\alpha| \sin(\arg(\alpha) + \varphi - \pi/2). \quad (10)$$

The quadrature distribution $p(x;\varphi)$ can be measured experimentally using BHD [13–15]. BHD results in a set of N statistically-independent quadrature-phase pairs $\{(x_j,\varphi_j)\}_{j=1,\dots,N}$, where the phases φ_j must be scanned uniformly in the range $[0, 2\pi)$, ensuring that the quadrature measurements are properly averaged over the optical phases. The values of x_j depend upon the state and the number of measurement outcomes improves the estimate of nonclassicality by reducing errors from statistical fluctuations; see Ref. [17] for further details. Equation (8) allows one to formulate the convenient direct sampling formula

$$P_\Omega(\alpha) \approx \frac{1}{N} \sum_{j=1}^{N} f_\Omega(x_j, \varphi_j; \alpha), \quad (11)$$

to estimate the regular phase-space distribution. The statistical uncertainty of the estimate in Eq. (11) is straightforwardly calculated to be

$$\sigma\left\{P_\Omega(\alpha)\right\} = \frac{1}{\sqrt{N(N-1)}} \sqrt{\sum_{j=1}^{N} \left[f_\Omega(x_j,\varphi_j;\alpha) - P_\Omega(\alpha)\right]^2}. \quad (12)$$

Negativities of P_Ω certify nonclassicality, therefore, we evaluate the signed statistical significance

$$\Sigma = \min_\alpha \left[\frac{P_\Omega(\alpha)}{\sigma\left\{P_\Omega(\alpha)\right\}}\right] \quad (13)$$

to ensure reliable results. The direct sampling of nonclassicality quasiprobabilities was successfully demonstrated for a squeezed vacuum state [16].

In some situations it is known *a priori* that the state of the field is phase invariant, i.e.,

$$p(x;\varphi) = p(x). \quad (14)$$

A prominent example are the Fock states. Generally, the set of states belonging to the category of phase-insensitive states is formed by statistical mixtures of Fock states. In this case it is not necessary to record the optical phase φ_j associated with the measured quadrature x_j. Rather, only a set of N quadratures $\{x_j\}_{j=1,\dots,N}$ is recorded. One can still use Eq. (11) if a phase φ_j is randomly attributed—according to a uniform distribution—to each quadrature sample x_j. However, this can lead to ambiguity problems, particularly when N is small. Thus, we derive a pattern function, for the case of phase-insensitive quantum states that only depends on the quadrature x. Inserting Eq. (14) into Eq. (8), one obtains

$$P_\Omega(\alpha) = \int_{-\infty}^{\infty} dx \, p(x) \, \overline{f}_\Omega(x;\alpha), \quad (15)$$

with a new pattern function which reads as

$$\overline{f}_\Omega(x;\alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, f_\Omega(x,\varphi;\alpha). \quad (16)$$

On the basis of Eq. (15) it is straightforward to show that the phase-space distribution can be estimated by a direct sampling formula of the form

$$P_\Omega(\alpha) \approx \frac{1}{N} \sum_{j=1}^{N} \overline{f}_\Omega(x_j;\alpha), \quad (17)$$

which no longer contains phase values compared to Eq. (11). The statistical error of the estimate in Eq. (17) is given by

$$\sigma\left\{P_\Omega(\alpha)\right\} = \frac{1}{\sqrt{N(N-1)}} \sqrt{\sum_{j=1}^{N} \left[\overline{f}_\Omega(x_j;\alpha) - P_\Omega(\alpha)\right]^2}. \quad (18)$$

Now, we further evaluate the integral in Eq. (16). Using Eq. (9), it holds

$$\overline{f}_\Omega(x;\alpha) = \frac{2}{\pi} \int_0^\infty db \, b \, e^{b^2/2} \Omega(b) \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, \cos[\xi(x,\varphi,\alpha) b]. \quad (19)$$

Recalling the definition in Eq. (10), and performing the phase integration, yields

$$\overline{f}_\Omega(x;\alpha) = \frac{2}{\pi} \int_0^\infty db \, b \, e^{b^2/2} \Omega(b) \, J_0(2|\alpha| b) \cos(xb) \quad (20)$$

with $J_0(\cdot)$ being the Bessel function of the first kind. The expression for the phase-insensitive pattern function in Eq. (20) together with the direct sampling formulas in Eqs. (17) and (18) are central results of the present work.

Note that the nonclassicality quasiprobability of a phase-independent state, namely a single-photon-added thermal state, has been reconstructed from experimental data [21]. However, this was performed in an indirect manner by first sampling the characteristic function of the P function and on this basis the nonclassicality quasiprobability was determined. This required optimization of the filter function for the state that was

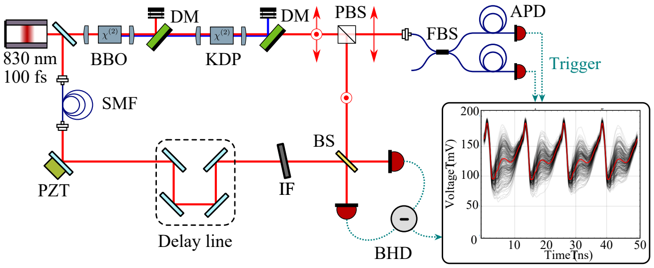

FIG. 1. Experimental scheme. PBS: Polarizing beam-splitter. APD: Avalanche Photodiode. PZT: Piezo-electric actuator. IF: Interference filter. BHD: Balanced homodyne detection. BS: Beam-splitter. FBS: Fiber beam-splitter. Inset: 250 acquisitions of four pulses from the BHD with the residual mean voltage.

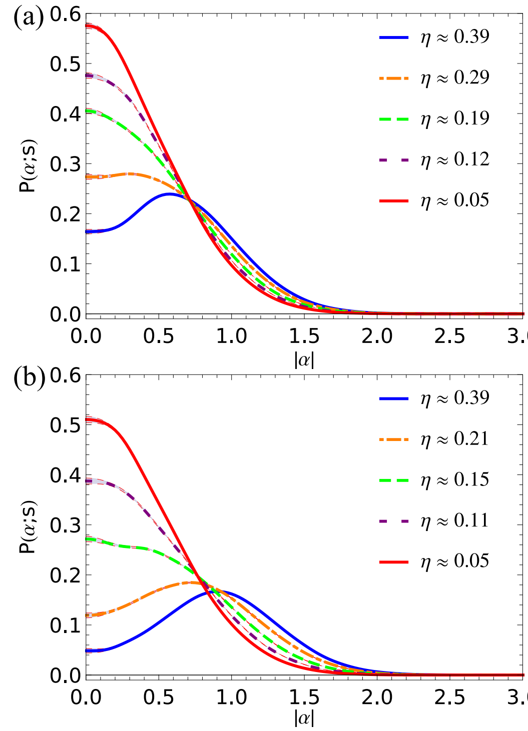

FIG. 2. (Color online) The directly sampled s-parametrized quasiprobabilities as a function of $|\alpha|$ for $s = -0.04$ and five different quantum efficiencies. (a) Single photon; (b) Two photons. The thin dashed line correspond to an error of one standard deviation, which is barely noticeable.

studied. The resulting filter function did not have an analytical form and the error calculation in this scenario becomes cumbersome already for the single-mode case. In the present manuscript, the analytical representation is a significant improvement in the methodology to certify non-classicality with the best statistical significance for the considered states.

As already stated in the preceding section, in the case of the Gaussian filter in Eq. (7) the parameter s must be smaller than zero in order to apply the direct sampling formula, since otherwise the pattern functions in Eqs. (9) and (20) do not exist as regular expressions. Beneficially, such a confinement does not exist in the case of the nonclassicality filters in Eq. (4) for $q > 2$. In Ref. [18] it was shown that among the possible values of the parameter q the filter in Eq. (6), which corresponds to $q \to \infty$, requires the minimal amount N of quadrature data points to significantly certify nonclassicality on the basis of the negativities of P_Ω. For this reason, we will apply this particular nonclassicality filter in the following considerations.

Experimental setup.— Single- and two-photon states were generated by degenerate, co-linear, type-II spontaneous parametric down conversion (SPDC) in potassium di-hydrogen phosphate (KDP) [22–24]. The SPDC process was pumped by 3.5 nm bandwidth pulses centered at 415 nm wavelength produced by second-harmonic generation from a Ti:Sapphire laser in beta-barium borate (BBO), as depicted in Fig. 1. The idler beam was coupled into a single-mode fiber and sent to spatially multiplexed single-photon counting modules (SPCMs). A single (double) click detection event in the multiplexed SPCMs heralds a single- (two-) photon state in the signal path. The signal beam was sent to a BHD for quadrature measurements.

The BHD [25] utilized a local oscillator (LO) pulse train derived from the Ti:Sapphire laser system mode matched to the signal photons. The maximum overall efficiency of the system was determined to be $\eta \approx 0.4$ (see supplementary information). The efficiency of the detection was set to approximately 0.4, 0.3, 0.2, 0.1 and 0.05, achieved by tuning the mode overlap between the signal photons and the LO, and data for heralded single- and two-photon states collected at each setting. We sampled $8 \cdot 10^5$ and $6 \cdot 10^5$ quadrature values for single- and two-photon states, respectively. The heralding rate was 500 kHz in the former case and 500 Hz in the latter. Further details about the experimental setup can be found in the supplementary information [26].

Comparing Gaussian with non-Gaussian filtering.—In this section, we apply the phase-independent direct sampling formula in Eq. (17) together with Eq. (18) to the measured quadrature data sets. First we sampled the s-parametrized quasiprobabilities, which is only possible for $s < 0$. Figure 2 (a) and (b) show these phase-space functions for the single- and two-photon state, respectively, for an s-parameter close to 0. As expected for the low efficiencies $\eta < 0.5$ under consideration, no negativities appear. This once more shows the inability of the s-parametrized quasiprobabilities, i.e., Gaussian filtered P functions, smoother than the Wigner function to visualize the quantum effects in the presence of high losses.

For comparison we utilize the non-Gaussian filter in Eq. (6) for various values of the filter parameter w

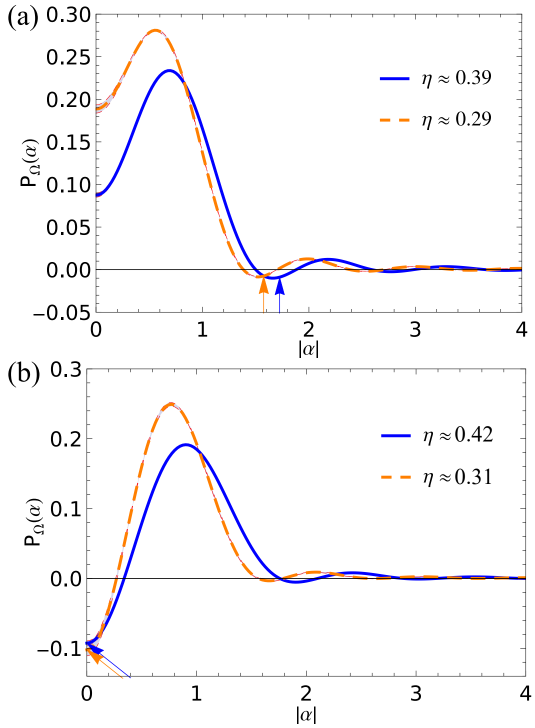

FIG. 3. (Color online) Nonclassicality quasiprobability as a function of $|\alpha|$, for the filter parameters $w_{\rm opt}$ which maximizes the statistical significance of the negativities; see Tab. I in [26]. They are shown for the two largest quantum efficiencies. The $|\alpha|$-value for which the negativity with the maximal significance is obtained is marked by an arrow, and its position depends on $w_{\rm opt}$, which is independently optimized for each state. (a) Single photon; (b) Two photons. The thin dashed line corresponds to an error of one standard deviation, which is barely noticeable.

into our direct sampling formula [Eqs. (17) and (18)] to get the nonclassicality quasiprobabilities $P_\Omega(\alpha)$. In Figs. 3 (a) and (b) the nonclassicality quasiprobabilities corresponding to the optimal filter parameters are shown for the efficiencies considered, see Supplemental Materials [26] for more details. We certify nonclassicality for the efficiencies of about 0.4 and 0.3 for both the single-photon state and the two-photon state with a very high statistical significance of more than 11 standard deviations, see [26]. The results show that for the same number of data points a smaller quantum efficiency leads to a smaller maximal significance which is obtained for a larger optimal filter parameter. Since the statistical significance increases as the square root of the number of data points, by increasing the latter, i.e., enlarging the measurement time, it is possible for all efficiencies under study to arbitrarily reduce the statistical error of the negativities of the nonclassicality quasiprobabilities and, thus, to significantly certify nonclassicality.

Conclusions.— In conclusion, we demonstrated that nonclassicality quasiprobabilities clearly outperform s-parametrized quasiprobabilities in uncovering signatures of nonclassicality of light in phase space in the presence of high losses. The regular s-parametrized quasiprobabilities, being Gaussian filtered versions of the Glauber-Sudarshan P function, completely fail to indicate nonclassicality by negative values in many cases, such as for quantum efficiencies smaller than 0.5. By contrast, the nonclassicality quasiprobabilities, which are always regular functions obtained by proper non-Gaussian filtering of the P-function, enable a *universal* nonclassicality test. Our results are based on a real experiment where we generated single and two photon states with high losses and detected the produced light with phase-insensitive balanced homodyne detection. Specifically adapted to this measurement scenario we developed a direct sampling formula to retrieve the regular quasiprobabilities from a set of quadrature samples. The results of this work underline the supremacy of nonclassicality quasiprobabilities to regularize the P function and to certify all possible nonclassical effects of light. Note that it has been shown most recently that combining different s-parameterized quasiprobabilities can improve their potential to verify nonclassicality [27, 28].

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