Comment on “Inverse Square Lévy Walks are not Optimal Search Strategies for math xmlns="http://www.w3.org/1998/Math/MathML" display="inline">mi>d/mi>mo>≥/mo>mn>2/mn>/math> ”

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Comment on "Inverse Square Lévy Walks are not Optimal Search Strategies for d $\geq$ 2"

It is widely accepted that "inverse square Lévy walks are optimal search strategies because they maximize the encounter rate with sparse, randomly distributed, replenishable targets" [1], when the search restarts in the vicinity of the previously visited target, which becomes revisitable again with high probability, i.e., non-destructive foraging [2]. Three objecting claims are raised in Ref. [1] for $d \geq 2$: (i) the capture rate η has linear dependence on the target density ρ for all values of the Lévy index α; (ii) "the gain $\eta_{\max}/\eta$ achieved by varying α is bounded even in the limit $\rho \to 0$" so that "tuning α can only yield a marginal gain"; (iii) depending on the values of the radius of detection a, the restarting distance l_c and the scale parameter s, the optimum is realized for a range of α.

We agree with claim (i), but as we will see, it is not relevant in $d \geq 2$ to whether or not inverse square Lévy walk searches are optimal for non-destructive foraging. Claim (iii) is also correct, however this claim was made already in Refs. [2–5]. In particular Ref. [2] showed that $\alpha = 1$ is optimal only in the limit $l_c \to a$, which is the main condition of non-destructive foraging, with the quantity l_c in Ref. [1] being none other than the distance r_o in Ref. [2]. Otherwise for large l_c the optimal strategy in the limit $\rho \to 0$ is to go along straight lines, i.e. $\alpha \to 0$. Moreover, it is known since 2003 that a range of α can be optimal (see Fig. 1 of Ref. [3], Figs. 2–3 of Ref. [4] and Figs. 1 and S1 of Ref. [6], none of which are cited in Ref. [1]). Crucially, claims (i) and (iii) do not *per se* contradict the main finding of Ref. [2] that $\alpha = 1$ is optimal under the specific conditions of non-destructive foraging (or of destructive foraging in patchy landscapes) [2–10].

To test claim (ii), we have numerically simulated the identical model proposed in Ref. [1] (see Fig. 1). The scaling for η with ρ proposed in Ref. [2] and proved in Ref. [8] for $d = 1$ does not hold in $d = 2$, in agreement with Ref. [1]. However, we find, for small enough $\delta = l_c/a - 1$, that η develops a maximum at $\alpha = 1$ with an arbitrarily large gain relative to the ballistic ($\alpha \to 0$) and Brownian ($\alpha = 2$) limits, contradicting claim (ii) about "marginal gain" in Ref. [1].

The main problem with Ref. [1] is that Eq. (3) fails in the limit $l_c \to a$ of non-destructing foraging. Eq. (3) yields a gain $K_d \sim 1/[A(a^\beta - B\, l_c^\beta)]$ in Eq. (5), with $\beta = -1$ for $\alpha < 1$ and $\beta = \alpha - 2$ for $\alpha > 1$. This gain, which agrees with claim (ii), is wrong in the limit $l_c \to a$.

Finally, we present a heuristic argument for the correct scaling of K_d for $d = 2$ when $l_c \to a$. Note that l_c is the distance at which the target stops hiding. The limit $\delta \to 0$ has biological relevance in this "hide-and-seek" model [10]. Let $\sigma = s/a$ and $\eta_0(\alpha, \delta, \rho, \sigma) = \eta/(\rho a)$. When $\delta \to 0$, the (radial) motion of the forager near the border of the detection circle is essentially one dimensional, hence the rigorous theory of the Riesz operator [8] on the interval of length L with absorbing ends becomes applicable. For $\sigma > \delta$ the efficiency increases when σ decreases because there are fewer large jumps away from the previous target that make re-encountering it difficult. When $\sigma \approx \delta$ the efficiency reaches its maximum. In the limit $\sigma \approx \delta \to 0$ we expect the same scaling behavior as in $d = 1$: $\eta_0 \sim \delta^{-\alpha/2}$ for $\alpha < 1$ and $\eta_0 \sim \delta^{-1+\alpha/2}$ for $\alpha > 1$. Hence η_0 has an arbitrarily strong maximum at $\alpha = 1$ when $\sigma \approx \delta \to 0$, in agreement with Fig. 1, and in disagreement with the title and claim (ii) of Ref. [1], restoring thus the original result for non-destructive foraging in Ref. [2] of the optimality of inverse square Lévy flights.

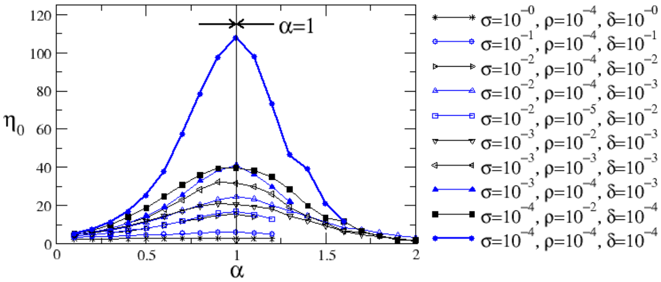

Figure 1. $\eta_0 = \eta/(\rho a)$ vs. α, for $N = 10^6$ Poisson distributed targets on a square of size $\sqrt{N/\rho}$ with periodic boundary conditions, averaged over 10^5 targets found.

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