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Phys. Rev. Lett. **126**, 030601 — Published 20 January 2021

DOI: 10.1103/PhysRevLett.126.030601

Information Scrambling over Bipartitions: Equilibration, Entropy Production, and Typicality

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(Dated: December 7, 2020)

In recent years, the out-of-time-order correlator (OTOC) has emerged as a diagnostic tool for information scrambling in quantum many-body systems. Here, we present exact analytical results for the OTOC for a typical pair of random local operators supported over two regions of a bipartition. Quite remarkably, we show that this "bipartite OTOC" is equal to the operator entanglement of the evolution and we determine its interplay with entangling power. Furthermore, we compute long-time averages of the OTOC and reveal their connection with eigenstate entanglement. For Hamiltonian systems, we uncover a hierarchy of constraints over the structure of the spectrum and elucidate how this affects the equilibration value of the OTOC. Finally, we provide operational significance to this bipartite OTOC by unraveling intimate connections with average entropy production and scrambling of information at the level of quantum channels.

Introduction.— A characteristic feature of certain quantum many-body systems is their ability to quickly spread "localized" information over subsystems, thereby making it inaccessible to local observables. Although unitary evolution retains all information, this local inaccessibility manifests itself as equilibration in closed systems, and has been termed "information scrambling" [1–5].

For Hamiltonian quantum dynamics, scrambling can be probed by examining the overlap of a time-evolved local operator $V(t) \coloneqq U_t^\dagger V U_t$ with a second static operator W. This overlap is commonly quantified via the strength of the commutator[1]

$$C_{V,W}(t) \coloneqq \frac{1}{2} \operatorname{Tr}\left([V(t), W]^\dagger \, [V(t), W] \, \rho_\beta \right) \tag{1}$$

where ρ_β denotes the thermal state at inverse-temperature β. From the perspective of information spreading, $C_{V,W}(t)$ is a natural quantity to consider since it constitutes a state-dependent variant of the Lieb-Robinson scheme; the latter enforces a fundamental restriction on the speed of correlations spreading in non-relativistic quantum systems [6–9]. In Eq. (1), it is convenient to consider pairs of operators V, W which at $t = 0$ act nontrivially on different subsystems, thus commute; we follow this convention here.

The commutator $C_{V,W}(t)$ is intimately linked to the out-of-time-order correlator (OTOC) [10, 11] which is a 4-point function with an unconventional time-ordering

$$F_{V,W}(t) \coloneqq \operatorname{Tr}\left(V^\dagger(t) W^\dagger V(t) W \rho_\beta \right). \tag{2}$$

The connection between the two arises when V, W are unitary; Eq. (1) then immediately reduces to $C_{V,W}(t) = 1 - \operatorname{Re}[F_{V,W}(t)]$. In this paper we focus on the infinite temperature, $\beta = 0$ case.

Through the years, several key signatures of quantum chaos [12–15] have been introduced. The initial exponential growth of the OTOC was proposed as a diagnostic of quantum chaos [16–23]. However, a careful analysis has revealed that information scrambling does not always necessitate chaos [24–29].

Per se, the OTOC's ability to probe dynamical features clearly depends on the choice of operators V, W. However, it is desirable to be able to capture these features as independently as possible from the specific choice of operators. This insensitivity can be achieved by averaging over a set of operators, a strategy also considered in Refs. [22, 30–35]. It is crucial to remark that for the averaged OTOC to faithfully capture information spreading, the averaging process must *preserve the initial locality of the system*, i.e., which subsystems V, W initially act upon — an observation that was quintessential in revealing the correct behavior of the OTOC and its connection with Loschmidt echo [35].

Given a bipartition of a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, we will henceforth focus on averaging $C_{V_A,W_B}(t)$ over the (independent) unitary operators V_A and W_B, whose support is over subsystems A and B, respectively. The resulting quantity

$$G(t) \coloneqq 1 - \frac{1}{d} \operatorname{Re} \int dV dW \operatorname{Tr}\left(V_A^\dagger(t) W_B^\dagger V_A(t) W_B \right), \tag{3}$$

depends only on the dynamics and the Hilbert space cut, where we denote $V_A = V \otimes I_B$, $W_B = I_A \otimes W$ and the averaging is performed according to the Haar measure [36]. We will refer to $G(t)$ for brevity as the *bipartite OTOC*, and analyzing its properties will be the focus of the present paper.

It was recently shown in Ref. [35], where $G(t)$ was first introduced, that under the assumptions of (i) weak coupling between A and B, and (ii) Markovianity, that

[1] In fact, $C_{V,W}(t) = \frac{1}{2} \| [V(t), W] \|^2$ for the norm associated with the inner product $\langle X, Y \rangle_\beta = \operatorname{Tr}(X^\dagger Y \rho_\beta)$, $\beta < \infty$.

$G(t)$ exhibits a close connection with the Loschmidt echo [37, 38]; the latter has been widely employed to characterize chaos [39, 40]. Here, we first show, without any of the previous assumptions, that $G(t)$ is, in fact, amenable to exact analytical treatment, and we uncover its direct relation with entropy production, information spreading, and entanglement. We also rigorously prove that the average case is also the typical one, hence justifying the averaging process. Our main results are stated in the theorems that follow. All proofs of the claims appearing in the text can be found in the Supplemental Material [41].

The bipartite OTOC.— We begin by bringing $G(t)$ in a more explicit form which will be the starting point for a sequence of results. This can be achieved by working on the doubled space $\mathcal{H}\otimes\mathcal{H}'$, where $\mathcal{H}'=\mathcal{H}_{A'}\otimes\mathcal{H}_{B'}$ is a replica of the original Hilbert space.

Theorem 1. *Let $S_{AA'}$ be the operator over $\mathcal{H}\otimes\mathcal{H}'$ that swaps A with its replica A' and $d=\dim(\mathcal{H})$. Then*

$$G(t)=1-\frac{1}{d^2}\operatorname{Tr}\left(S_{AA'}U_t^{\otimes 2}S_{AA'}U_t^{\dagger\otimes 2}\right). \tag{4}$$

The analogous expression for BB' also holds.

The above formula immediately exposes a connection between the bipartite OTOC and the *operator entanglement* of the evolution $E_{\rm op}(U_t)$, as defined in Ref. [42] (see also [41] for the relevant definitions). The two quantities, remarkably, coincide exactly. This observation also allows one to express the *entangling power* [43] $e_{\rm P}(U_t)$ as a function of the bipartite OTOC for the symmetric case $d_A=d_B$. The former quantifies the average entanglement produced by the evolution and has been established as an indicator of global chaos in few-body systems [44–47].

Theorem 2. *Let G_U denote the bipartite OTOC for the evolution U. Then, (i) $E_{\rm op}(U_t)=G_{U_t}$, and (ii) for a symmetric bipartition $d_A=d_B$,*

$$e_{\rm P}(U_t)=\frac{d}{(\sqrt{d}+1)^2}\left(G_{U_t}+G_{U_tS_{AB}}-G_{S_{AB}}\right). \tag{5}$$

For the finite temperature case, Eq. (4) admits a straightforward generalization which we report in [41]. However, a direct connection with operator entanglement and entangling power may not be so simple.

How informative is the average $G(t)$?— Usually, one is interested in behavior of the OTOC for a typical choice of random unitary operators. Due to measure concentration [48], we prove that the two essentially coincide, i.e., the probability that a random instance deviates significantly from the mean is exponentially suppressed as the dimension of either of the subsystems A and B grows large.

Proposition 3. *Let $P(\epsilon)$ be the probability that a random instance of $C_{V_A,W_B}(t)$ deviates from its Haar average $G(t)$ more than ϵ. Then,*

$$P(\epsilon)\leq 2\exp\left(-\frac{\epsilon^2 d_{\max}}{64}\right), \tag{6}$$

where $d_{\max}=\max\{d_A,d_B\}$.

In the definition of the bipartite OTOC and to obtain the replica formula Eq. (4), we have so far considered averaging over the uniform (Haar) ensemble which continuously extends over the whole unitary group. Although natural from a mathematical viewpoint, this choice can turn out to be rather complicated on physical and numerical grounds [49]. Nonetheless, we show in [41] that Haar averaging can be replaced by any unitary ensemble that forms a 1-design [50–53] without altering $G(t)$. Such ensembles mimic the Haar randomness only up to the first moment, which is the depth of randomness that the OTOC can probe [22]. The latter assumption is thus much weaker than Haar randomnsess. For instance, consider the case of a spin-1/2 many-body system split into two parts, A and B. Instead of averaging over Haar random unitaries V_A and W_B, that typically do not factor, the 1-design (equivalent) picture prescribes to instead consider only fully factorized unitaries with support over A and B, e.g., products of local Pauli matrices.

Time-averaging the bipartite OTOC.— In finite dimensional quantum systems, nontrivial quantum expectation values or quantities such as $C_{V,W}(t)$ do not converge to a limit for $t\to\infty$. Instead, after a long time they typically oscillate around an equilibrium value [54–59] which can be extracted by time-averaging $\overline{X(t)}:=\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,X(t)$. We now turn to examine this long-time behavior $\overline{G(t)}$ of the bipartite OTOC as a function of the Hamiltonian and the Hilbert space cut.

Let us begin with the case of a chaotic dynamics, which entails level repulsion statistics [15] and an "incommensurable" relation among the energy levels. As such, chaotic Hamiltonians satisfy (either exactly or to very good approximation) the no-resonance condition (NRC): The energy levels and energy gaps feature nondegeneracy. This has important implications for the long-time behavior of their bipartite OTOC, as we will see soon.

Let us spectrally decompose $H=\sum_k E_k\,|\phi_k\rangle\langle\phi_k|$ and use $\rho_k^{(\chi)}:=\operatorname{Tr}_{\overline{\chi}}(|\phi_k\rangle\langle\phi_k|)$ to denote the reduced density operator over $\chi=A,B$ corresponding to the $k^{\rm th}$ Hamiltonian eigenstate ($\overline{\chi}$ corresponds to the complement). Below, $\langle X,Y\rangle:=\operatorname{Tr}(X^\dagger Y)$ denotes the Hilbert-Schmidt inner product [60], which gives rise to the operator 2-norm $\|X\|_2:=\sqrt{\langle X,X\rangle}$.

Proposition 4. *Consider a Hamiltonian satisfying the NRC. Then*

$$\overline{G(t)}^{\rm NRC}=1-\frac{1}{d^2}\sum_{\chi\in\{A,B\}}\left(\left\|R^{(\chi)}\right\|_2^2-\frac{1}{2}\left\|R_D^{(\chi)}\right\|_2^2\right) \tag{7}$$

where $R^{(\chi)}$ is the Gram matrix of the reduced Hamiltonian eigenstates $\{\rho_k^{(\chi)}\}_{k=1}^d$, i.e.,

$$R_{kl}^{(\chi)} := \langle \rho_k^{(\chi)}, \rho_l^{(\chi)} \rangle \tag{8}$$

while $\left(R_D^{(\chi)}\right)_{kl} := R_{kl}^{(\chi)} \delta_{kl}$.

Let us first point out some basic, yet important properties of the above formula. The matrix $R^{(\chi)}$ is real and symmetric, while $R_D^{(\chi)}$ is positive-semidefinite and diagonal. Moreover, the completeness of the Hamiltonian eigenvectors imposes $\sum_k \rho_k^{(\chi)} = d_{\overline{\chi}} I$, thus the rescaled $\tilde{R}^{(\chi)} := R^{(\chi)}/d_{\overline{\chi}}$ are doubly stochastic, i.e., $\sum_i \tilde{R}_{ij}^{(\chi)} = \sum_i \tilde{R}_{ji}^{(\chi)} = 1 \; \forall j$. As $\tilde{R}^{(\chi)}$ is a (rescaled) Gram matrix, its eigevalues are nonnegative, upper bounded by 1, and at most d_χ^2 of them are nonzero [60]. This last property follows from the fact that $\operatorname{Rank} \tilde{R}^{(\chi)} = \dim \operatorname{Span}\{\rho_k^{(\chi)}\}_k \leq d_\chi^2$. Observe also that $\left\| R_D^{(A)} \right\|_2^2 = \left\| R_D^{(B)} \right\|_2^2$ as two states $\rho_k^{(A)}$ and $\rho_k^{(B)}$ always have the same spectrum (up to irrelevant zeroes).

Bipartite OTOC and entanglement.— Proposition 4 makes it possible to bridge the long-time behavior of the bipartite OTOC with the entanglement structure of the Hamiltonian eigenstates. Let us begin with the symmetric case where $d_A = d_B$ and all $|\phi_k\rangle$ are maximally entangled with respect to the A-B Hilbert space cut. This limit uniquely determines the time-average for the NRC case, regardless of the exact Hamiltonian eigenbasis. In general, however, knowledge of the entanglement is not enough to uniquely determine the equilibration value; the inner products $R_{kl}^{(\chi)}$ go beyond probing just the spectrum of the reduced states. A simple substitution in Eq. (7) gives for the maximally entangled case $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} = (1 - 1/d)^2$. We will later show the upper bound $G(t) \leq 1 - 1/d_{\min}^2$, therefore the equilibrium value for the bipartite OTOC in this case is nearly maximal, as expected for highly entangled models (e.g., [61, 62]).

How robust is this conclusion for chaotic Hamiltonians with a possibly asymmetric bipartition? Typical eigenstates of chaotic Hamiltonians, as also predicted by the eigenstate thermalization hypothesis [63–65], are believed to obey a volume law for the entanglement entropy. Moreover, their entanglement properties in the bulk resemble those of Haar random pure states [66–68]. We will now show that high entanglement for the Hamiltonian eigenstates necessarily implies that the deviation of the actual equilibration value from $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$ is small.

It is convenient for this purpose to quantify the amount of entanglement via the linear entropy [69, 70] of the reduced state $E(|\psi_{AB}\rangle) := S_{\mathrm{lin}}\left(\operatorname{Tr}_\chi |\psi_{AB}\rangle\langle\psi_{AB}|\right)$, where $S_{\mathrm{lin}}(\rho) := 1 - \operatorname{Tr}(\rho^2)$. The latter will also emerge naturally later when we express the bipartite OTOC in terms of entropy production. Notice that $E \leq 1 - 1/d_{\max} := E_{\max}$, which is achievable only for $d_A = d_B$.

Proposition 5. *If $E_{\max} - E(|\phi_k\rangle) \leq \epsilon$ holds for at least a fraction α of the Hamiltonian eigenstates, then $\left|\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} - \overline{G(t)}^{\mathrm{NRC}}\right| \leq \alpha J + (1-\alpha) K$, where*

$$J := \frac{6\epsilon}{d_{\min}} + \frac{5\epsilon^2}{2} + 2\frac{\lambda^2 - 1}{d_{\max}^2} \tag{9a}$$

$$K := \left(1 + \frac{2}{d_{\min}}\right)(1-\alpha) + \frac{2}{d} + 4(\epsilon + \sqrt{\epsilon}) \tag{9b}$$

and $\lambda = d_{\max}/d_{\min}$.

The above bound provides a sufficient condition such that the bipartite OTOC equilibrates around $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$. It is expressed in terms of the fraction α of the highly entangled eigenstates, their entanglement and the asymmetry of the A-B bipartition. Notice that the bound simplifies considerably for the case $\alpha = 1$ and $d_{\min} = d_{\max} = \sqrt{d}$, that is, $\left|\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} - \overline{G(t)}^{\mathrm{NRC}}\right| \leq \epsilon(6/\sqrt{d} + 5\epsilon/2)$ which should hold to a good approximation for Hamiltonians with high entanglement in the bulk of the energies. Applied to chaotic Hamiltonians[2], the bound of Proposition 5 indicates that the bipartite OTOC will equilibrate near $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$, with deviations up to $O(1/d_{\min}^2)$. For a fixed ratio λ and as d grows, $\overline{G(t)}^{\mathrm{NRC}}$ hence converges to $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$ for all chaotic systems. Since $G(t) \leq 1 - 1/d_{\min}^2$, fluctuations around the time-average are necessarily insignificant, justifying the term equilibration.

Beyond chaotic Hamiltonians.— We now relax the "strong" level repulsion, i.e., NRC, criterion and uncover how a hierarchy of constraints, each implying a different strength of chaos, is reflected in the equilibration value of the bipartite OTOC.

Integrable models, which possess a structured spectrum, are expected to violate the NRC. Nevertheless, notice that Eq. (7), although derived under the NRC, can still be evaluated for an (arbitrary) choice of orthonormal eigenvectors of the Hamiltonian. We will refer to the resulting value as the *NRC estimate* of the time-average and we will shortly show that this estimate always constitutes an upper bound of the actual equilibration value (and coincides with it for chaotic Hamiltonians). This is both of conceptual and practical importance, as evaluating the NRC estimate is considerably less intensive than calculating the exact value.

In fact, one can make a broader claim. For that, we first sketch three types of averaging processes over G, increasingly shifting away from the strong chaoticity limit. Each of them gives rise to a corresponding estimate for the (exact) equilibration time-average value $\overline{G(t)}$. (i) $\overline{G}^{\mathrm{Haar}}$: Averaging over (global) Haar random unitary operators $U \in U(d)$ in place of the time-evolution. This averaging process is "beyond chaos", in

[2] Here chaoticity concretely means that the Hamiltonian spectrum satisfies the NRC and that the entanglement of the typical eigenvectors in the bulk, which determine the equilibration value, resembles that of Haar random vectors [71, 72], i.e., $\operatorname{Tr}(\rho_\chi^2) \approx (d_A + d_B)/(d+1)$ thus $\epsilon = O(1/d_{\min})$ and $\alpha \approx 1$.

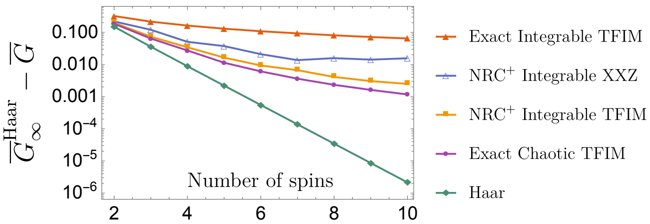

FIG. 1. Logarithmic plot of various $\overline{G}$ estimates, along with the exact time-average, for fixed $d_A = 2$ as a function of the total number of spins n. $\overline{G}_\infty^{\text{Haar}} = 3/4$ corresponds to the Haar estimate for $n \to \infty$. For the chaotic phase of the TFIM ($g = -1.05$, $h = 0.5$), the NRC constitutes a satisfactory, though imperfect, approximation. The chaotic and integrable phases ($h = 0$) can be clearly distinguished through the equilibration behavior of the bipartite OTOC. For the integrable XXZ model (we set $J = 0.4$, $\Delta = 2.5$), the NRC$^+$ estimate coincides (up to numerical error) with the exact time-average. Inequality (11) holds valid in all cases.

the sense that it does not conserve energy, in contrast with time-averaging over any Hamiltonian evolutions. Its estimate (only a function of the dimension) is given later in Eq. (10). (ii) $\overline{G(t)}^{\text{NRC}}$: Time-average, assuming the Hamiltonian has nondegenerate energy levels and nondegenerate energy gaps. The corresponding estimate is Eq. (7). (iii) $\overline{G(t)}^{\text{NRC}^+}$: As before, but assuming the Hamiltonian may have degenerate spectrum, but the energy gaps (between the different levels) are nondegenerate. Its estimate depends only on the eigenprojectors of the Hamiltonian and can be found in [41].

The value of the Haar average can be performed exactly, with result

$$\overline{G}^{\text{Haar}} = \frac{(d_A^2 - 1)(d_B^2 - 1)}{d^2 - 1}\,. \tag{10}$$

The following ordering holds.

Theorem 6. *For any given Hamiltonian, the corresponding estimates are related with the exact time-average* $\overline{G(t)}$ *as*

$$\overline{G}^{\text{Haar}} \geq \overline{G(t)}^{\text{NRC}} \geq \overline{G(t)}^{\text{NRC}^+} \geq \overline{G(t)}\,. \tag{11}$$

The above constitutes a proof that coincidences in the spectrum of a Hamiltonian up to the "gaps of gaps" (i.e., degeneracy over the energy levels and their gaps) always *reduce* the equilibration value of the bipartite OTOC.

Let us now numerically compare each of the estimates for two models of spin-1/2 chains with open-boundary conditions: (i) transverse-field Ising model (TFIM) with nearest neighbour interaction, $H_{\text{I}} = -\sum_i (\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x + h\sigma_i^z)$ (ii) nearest-neighbor XXZ interaction $H_{\text{XXZ}} = -J\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z)$. Recall that H_{I} for $h = 0$ is integrable in terms of free-fermions, while H_{XXZ} by Bethe Ansatz techniques. The two types of solutions yield qualitatively different spectra; free fermion solutions necessarily violate nondegeneracy of the gaps. This is reflected in the accuracy of the estimates (see Figure 1). Although the NRC estimate provides essentially the exact equilibration values for the chaotic phase of the TFIM, it overestimates them in the integrable phase. On the other hand, NRC$^+$ is essentially exact for the integrable case of the H_{XXZ} due to the lack of coincidences in the gaps. The results obtained here corroborate existing studies in the literature, where the (short- and) long-time behavior of the OTOC was studied for various many-body systems, see Refs. [73–75].

Bipartite OTOC and subsystem evolution.— We have so far focused on examining the behavior of the bipartite OTOC from the perspective of closed systems, i.e., over the full bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. One can instead express $G(t)$ as a function of the reduced time-dynamics over only either $\mathcal{H}_A$ or $\mathcal{H}_B$ (and the corresponding duplicate), at the expense of giving up unitarity. This can be easily realized by formally performing a partial trace in Eq. (4), which immediately results in the following equivalent expression for the bipartite OTOC.

Proposition 7. *Let* $\Lambda_t^{(A)}(\rho_A) \coloneqq \text{Tr}_B \left[U_t \left(\rho_A \otimes \frac{I_B}{d_B} \right) U_t^\dagger \right]$ *be the reduced dynamics over A when the environment B is initialized in a maximally mixed state. Then,*

$$G(t) = 1 - \frac{1}{d_A^2} \text{Tr}\left[S_{AA'} \left(\Lambda_t^{(A)}\right)^{\otimes 2} (S_{AA'}) \right]. \tag{12}$$

The analogous expression for BB' also holds.

The quantum map $\Lambda_t^{(\chi)}$ is unital, i.e., the maximally mixed state is a fixed point. As such, the transformation $\rho_\chi \mapsto \Lambda_t^{(\chi)}(\rho_\chi)$ results always in an output state whose spectrum is more disordered than the input one [76]. As a result, when ρ_χ is pure, the effect of the reduced time-dynamics is to scramble and hence produce entropy. Let us now turn to examine this connection more closely.

Bipartite OTOC as entropy production.— We now show that the bipartite OTOC $G(t)$ is nothing but a measure of the average entropy production over pure states, with the latter quantified by linear entropy S_{lin}.

Theorem 8.

$$G(t) = \frac{d_\chi + 1}{d_\chi} \int dU\, S_{\text{lin}} \left[\Lambda_t^{(\chi)}(|\psi_U\rangle\langle\psi_U|) \right] \tag{13}$$

where $\chi = A, B$ and $|\psi_U\rangle \coloneqq U|\psi_0\rangle$ corresponds to Haar random pure states over $\mathcal{H}_\chi$.

In this manner, the bipartite OTOC can be fully characterized by linear entropy measurements over any of the A, B subsystems. To obtain a satisfactory estimate of the mean in the RHS of Eq. (13), one does not, in practice, need to sample over the full Haar ensemble. An adequate estimate can be obtained with a rapidly decreasing number of necessary samples, as the dimension d_χ grows. More precisely, let $\tilde{P}(\epsilon)$ be the probability of the entropy

$S_{\rm lin}\left[\Lambda_t^{(\chi)}\left(|\psi\rangle\langle\psi|\right)\right]$ deviating from $\frac{d_\chi}{d_\chi+1}G(t)$ more than ϵ for an instance of a random state. We show in [41] that

$$\tilde{P}(\epsilon) \leq \exp\left(-\frac{d_\chi \epsilon^2}{64}\right). \tag{14}$$

The linear entropy, although, per se, a nonlinear functional, can be turned into an ordinary expectation value if two (uncorrelated) copies of the quantum state are simultaneously available, $1 - S_{\rm lin} = \operatorname{Tr}\left(S\rho^{\otimes 2}\right)$ for $S = S_{AA'}S_{BB'}$. This fact can be exploited simplify its experimental accessibility [77–81]. More recently, protocols based on correlating measurements over random bases have also been developed to measure entropies [82–85], as well as OTOCs [86, 87]. As a result, Theorem 8 and the typicality result Eq. (14) suggest that the bipartite OTOC is, in turn, tractable via linear entropy measurements. We provide more details in [41].

From Eq. (13) one can also infer the upper bound $G(t) \leq 1 - 1/d_\chi^2 := G_{\max}^{(\chi)}$ announced earlier that follows from the range of the linear entropy function. The bound is thus achievable only when $\Lambda_t^{(\chi)}$ is equal to the completely depolarizing map $\mathcal{T}^{(\chi)}(\cdot) := \operatorname{Tr}(\cdot)\dfrac{I_\chi}{d_\chi}$.

Finally, we remark that linear entropy occurs rather naturally in relation with the bipartite OTOC, as demonstrated by Theorem 2 (where it lies implicitly in the definition of operator entanglement and entangling power) and Theorem 8. This fact has its roots in the definition of the OTOC, which is intimately related to the Frobenious norm. Relevant relations for the linear entropy have been also reported in [31]. Starting from the inequality $S_{\rm lin}(\rho) \leq S(\rho)$ between the linear and von Neumann entropies ($S(\rho) := -\operatorname{Tr}[\rho\log(\rho)]$), one can also obtain the corresponding estimates for the latter.

Bipartite OTOC and information spreading.— The bipartite OTOC measures the average ability of the reduced time-evolution to erase information, as captured by the entropy production over a random pure state. This naturally raises the question as to whether $G(t)$ can also be understood as a measure of distance between $\Lambda_t^{(\chi)}$ and the depolarizing map $\mathcal{T}^{(\chi)}$, that is, in the space of quantum channels (i.e., Completely Positive and Trace Preserving (CPTP) maps [88]).

A straightforward answer can be obtained by resorting to the duality between quantum states and operations [88]. Let $\rho_{\mathcal{E}} := \mathcal{E} \otimes \mathcal{I}(|\phi^+\rangle\langle\phi^+|)$ denote the (Choi) state corresponding to the CPTP map $\mathcal{E}$, where $|\phi^+\rangle := d^{-1/2}\sum_{i=1}^d |ii\rangle$ is a maximally entangled state.

Proposition 9. *The bipartite OTOC is a measure of the distance between the reduced time-evolution and the depolarizing map:*

$$G(t) = G_{\max}^{(\chi)} - \left\|\rho_{\Lambda_t^{(\chi)}} - \rho_{\mathcal{T}^{(\chi)}}\right\|_2^2. \tag{15}$$

As an application, the proposition above can be utilized to bound the distance $\left\|\Lambda_t^{(\chi)} - \mathcal{T}^{(\chi)}\right\|_\diamond$ given by the diamond norm [89, 90]; the latter is a well-established measure of distance between quantum channels[3] since it admits an operational interpretation in terms of discrimination on the level of quantum processes [91]. The distinguishability of the two operations satisfies $\left\|\Lambda_t^{(\chi)} - \mathcal{T}^{(\chi)}\right\|_\diamond \leq d_\chi^{3/2}\sqrt{G_{\max}^{(\chi)} - G(t)}$ (see [41]), therefore if $G_{\max}^{(\chi)} - G(t)$ decays faster than d_χ^{-3}, then asymptotically the two channels are essentially indistinguishable.

Summary.— We showed that the bipartite OTOC is amenable to exact analytical treatment and, quite remarkably, is equal to the operator entanglement of the dynamics. This identity allows one to establish a rigorous quantitative connection between the OTOC and the notion of entangling power, a well-established quantifier of few-body chaos. This may provide insights into recent work involving "dual-unitaries" and many-body chaos [92–95]; the latter maximize operator entanglement [95, 96]. We then turned to late-time averages of the bipartite OTOC and provided a hierarchy of estimates for systems that violate the conditions of a "generic spectrum". Finally, we unraveled the operational significance of the OTOC by establishing intimate connections with entropy production and information scrambling at the level of quantum channels. Possible future directions include applying further these theoretical tools to concrete many-body systems and uncovering relations with thermalization, localization, and other many-body phenomena.

Acknowledgments.— G.S. is grateful to N.A. Rodríguez Briones for the interesting discussions and to the Louisa house in Kitchener for the hospitality. P.Z. acknowledges partial support from the NSF award PHY-1819189. Research was sponsored by the Army Research Office and was accomplished under Grant Number W911NF-20-1-0075. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Office or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation herein.

[3] Bounding the difference in terms of the quantum processes also constraints the distinguishability in terms of states: $\left\|\mathcal{E}_1(\rho) - \mathcal{E}_2(\rho)\right\|_1 \leq \left\|\mathcal{E}_1 - \mathcal{E}_2\right\|_\diamond$ for all states and quantum processes.

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