Compact math xmlns="http://www.w3.org/1998/Math/MathML" display="inline">mrow>msup>mrow>mi mathvariant="normal">U/mi>/mrow>mi>k/mi>/msup>mo stretchy="false">(/mo>mn>1/mn>mo stretchy="false">)/mo>/mrow>/math> Chern-Simons Theory as a Local Bosonic Lattice Model with Exact Discrete 1-Symmetries

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Compact $U^\kappa(1)$ Chern-Simons theory as local bosonic lattice model with exact discrete 1-symmetries

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We propose a bosonic $U^\kappa(1)$ rotor model on a three dimensional spacetime lattice. With the inclusion of a Maxwell term, we show that the low-energy properties of our model can be obtained reliably via a semi-classical approach. Those properties are the same as that of the Chern-Simons field theory, $S = \int \mathrm{d}^3x\ \frac{K_{IJ}}{4\pi} A_I \mathrm{d}A_J$. We require the lattice variables on each link to be compact (*i.e.* take values on circles), which enforces the quantization of the K-matrix as a symmetric integer matrix with even diagonals. Our lattice model also has exact 1-symmetries, which gives raise to the 1-form symmetry in the Chern-Simons field theory. In particular, some of those 1-symmetries are anomalous (*i.e.* non-on-site) in the expected way. The anomaly can be probed via the breaking of those lattice 1-symmetries by the boundaries.

Introduction: Chern-Simons (CS) field theory is a very important field theory with myriad applications from condensed matter to quantum gravity. Though well studied in the continuum as field theory, defining CS field theory on the lattice presents an opportunity to better tame the field integration measure as well as allowing us to consider non-smooth gauge fields with singularities. Furthermore, it is well known that it is quite non-trivial to define the action of CS field theory if the fiber bundle described by the gauge field is non-trivial on the spacetime,[1] which leads to an obstruction to have a globally defined gauge fields (the connection 1-forms). One way to address this problem is to define CS theory on a spacetime lattice where the lattice gauge fields for "distinct fiber bundles" are continuously connected. Then the lattice gauge field with monopoles/flux-lines is continuously connected to gauge field without monopoles/flux-lines. So once we have a spacetime lattice description of CS theory, the theory is automatically well defined for gauge fields of topologically non-trivial fiber bundle, as well as for gauge fields of monopoles and flux-lines. Certainly, the lattice description of CS field theory also remove the infinity problem of the field theory.

People have tried to put CS theory on lattice for a long time. In one approach, people try to construct local lattice models whose many-body Hilbert space admits tensor product decomposition $\mathcal{V} = \bigotimes_i \mathcal{V}_i$, where $\mathcal{V}_i$ is the local Hilbert space on site-i. The key is to find a proper local Hamiltonian H acting on $\mathcal{V}$ such that the low energy properties of H are fully described by a CS field theory.[2–10] However, those lattice models are usually not soluble. Given a lattice model, we usually do not know if it is in a quantum-Hall topologically ordered phase. We usually do not know if the lattice model produce a CS theory at low energy or not, and we do not know which CS theory it produces. So here we are looking for a better result, where we can derive, under a controlled approximation, the low energy effective CS field theory from the lattice model.

In another approach, people try to construct lattice gauge models that produce CS field theory at low energies.[11–14] The many-body Hilbert space $\mathcal{V}_{\text{gauge}}$ for the lattice gauge theory is formed by gauge invariant states, which is non-local, *i.e.* $\mathcal{V}_{\text{gauge}}$ does not admit the tensor product decomposition $\mathcal{V}_{\text{gauge}} \neq \bigotimes_i \mathcal{V}_i$. Ref. 11 and 12 proposed lattice gauge models with compact $U(1)$ gauge group, however the gauge field in each link is not compact. The compactness is inforce at global level. In contrast, the link variables in this paper are already in compact $U(1)$ groups. Ref. 13 and 14 proposed lattice gauge models with non-compact $U(1)$ gauge group (*i.e.* $\mathbb{R}$ gauge group), which is quite different form the compact $U(1)$ gauge theory studied in this paper.

In this paper, we try to realize the most general bosonic *compact* $U(1)$ CS theory via local bosonic lattice model with *compact* degrees of freedom on each link. In contrast to previous emergent CS field theory from local lattice model, we want our local lattice model to be "soluble", in the sense that we can determine its low energy effective theory reliably. We find such local bosonic model on spacetime lattice, which is given in eqn. (5). Under a controlled semi-classical approximation for small g in eqn. (5), we show that our spacetime lattice model can produce any even-K-matrix CS field theory[7] of *compact* $U(1)$'s in continuum limit (see eqn. (10)).

We will rely on the cochain theory familiar from algebraic topology [15] to construct our lattice model (see Supplemental Material). A striking character of our lattice model (5) is that the Lagrangian density is not a continuous function of the field values. Also the lattice model is defined for $\mathbb{R}/\mathbb{Z}$-valued fields and is not quadratic (*i.e.* corresponds to an interacting theory). But the weak fluctuations in small g limit are described by a quadratic action. This is why we can reliably obtain the low energy effective CS field theory in small g limit.

However, being able to reliably obtain the low energy effective theory is not the most important character of our constructed model (5). What really special of our lattice model is that it has many exact 1-symmetries. It was known that 2+1D Z_n gauge theory described by mutual CS theory has many 1-symmetries on lattice.[16–22] A generic $U(1)$ CS theory also has many 1-form symmetries in continuum.[23, 24] Our lattice realization of a generic $U(1)$ CS theory is a very special one, that those 1-form symmetries in continuum become the exact 1-symmetries[22] in our lattice model. In contrast, the realizations of CS theory by local lattice models in Ref. 2–10 do not have those 1-symmetries. Therefore, we can state our new result more precisely as the following:

We construct a local bosonic rotor model on spacetime lattice that, at low energies, realizes most general compact $U(1)$ CS field theory characterized by K-matrix, where the 1-form symmetries in the CS field theory are realized as the exact $Z_{k_1} \times Z_{k_2} \times \cdots$ 1-symmetries in our lattice model.[25] Here k_i are diagonal entries of the Smith normal form of the K-matrix. Some of the 1-symmetries are anomalous.[22, 26–28] Different lattice realizations of the same $U(1)$ CS field theory may lead to different anomalous 1-symmetries.

We like to stress that our model is a local bosonic lattice model, rather than a lattice gauge theory, since it is defined via a path integral that is a product of integrals of local variables. We build our model to be periodic in the lattice variables, and take the field integral over one period of each lattice variable. This periodic redundancy provides level quantization, without relying on the underlying topology of the manifold as "large gauge transformations" do. Since our lattice model is not a lattice gauge theory, its action does not has to be gauge invariant. Indeed, the action eqn. (5), on spacetime with boundary, is not invariant under the usual gauge redundancy $A \to A + \mathrm{d}\theta$, with θ a 0-cochain.

We also like to remark that our lattice model eqn. (5) is actually a tensor network path integral in spacetime.[29] Thus, we have found a tensor network path integral that realize a topologically ordered phase described by CS field theory. We note that the tensor in the tensor network is indexed by a $\mathbb{R}/\mathbb{Z}$-value. In other words, the dimension of the tensor is infinity.

Throughout, we will use the convention that the periods of lattice variables are quantized to unity. Thus the familiar "level one" CS action is $\pi \int a\,\mathrm{d}a$, rather than the more common $\frac{1}{4\pi}\int A\,\mathrm{d}A$. This will simplify our calculations; one may always return to the usual formulation by replacing $a \to A/2\pi$. In order to avoid introducing a spin structure, we confine our model to only bosonic theories. Thus, with the lattice variables of period 1 in their values, bosonic K-matrix theory in the continuum takes the form:

$$S = \pi \sum_{IJ} K_{IJ} \int a_I \,\mathrm{d}a_J \tag{1}$$

where K is a symmetric integer matrix with even diagonals.

Chern Simons Theory on Lattice: To construct our local bosonic spacetime lattice model, we will use a cochain formalism on a spacetime complex. A spacetime complex (lattice) is a triangulation of the three-dimensional spacetime *with a branching structure*,[30–32] which is denoted as $\mathcal{M}^3$. The spacetime complex is formed by simplices – the vertices, links, triangles, *etc.* We will use $i, j, \cdots$ to label vertices of the spacetime complex. The links of the complex (the 1-simplices) will be labeled by $\langle ij\rangle, \cdots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $\langle ijk\rangle, \cdots$. The degrees of freedom of lattice model live on the links of the spacetime complex: $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ on link $\langle ij\rangle$, $I = 1, 2, \cdots, \kappa$. $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ is $\mathbb{R}/\mathbb{Z}$-valued, *i.e.* $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ and $(\tilde a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ are equivalent if $(\tilde a_I^{\mathbb{R}/\mathbb{Z}})_{ij} - (a_I^{\mathbb{R}/\mathbb{Z}})_{ij} = 0$ mod 1. Such $\mathbb{R}/\mathbb{Z}$-valued fields on the links are simply the so called 1-cohains $a_I^{\mathbb{R}/\mathbb{Z}}$ on the spacetime complex $\mathcal{M}^3$. Here we have κ different 1-cochains $a_I^{\mathbb{R}/\mathbb{Z}}$ labeled by I. The lattice action of our bosonic model will be constructed from those 1-cochains using cup product and cochain derivative. For a more detailed introduction to the cochain formalism for defining local bosonic spacetime lattice models, see Ref. 33 and Supplemental Material.

We want to construct our lattice bosonic model in such a way that it is very similar to a CS theory. Hopefully, the resulting lattice bosonic model realizes a topologically ordered state described the CS topological quantum field theory. Due to the similarity between 1-cochain and differential 1-form, between the cup product for cochains and wedge product for differential forms, as well as the derivative d acting on them, naïvely, we would write the partition function for a bosonic lattice as:

$$Z = \int [\prod \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}]\, \mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}}, \tag{2}$$

which formally looks like the continuum CS field theory written in terms of differential 1-forms. Here k_{IJ} are integers, and $\int_{\mathcal{M}^3}$ means the sum over all 3-simplices in $\mathcal{M}^3$. Also $\int [\prod \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}] \equiv \prod_{\langle ij\rangle} \prod_I \int_{-\frac12}^{\frac12} \mathrm{d}(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ gives rise to the path integral, where $\prod_{\langle ij\rangle}$ is a product over all the links. Here we have lifted the $\mathbb{R}/\mathbb{Z}$-valued $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ to a $\mathbb{R}$-valued $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij} \in (-\frac12, \frac12]$ before we do the path integral.

Since $a_I^{\mathbb{R}/\mathbb{Z}}$ is $\mathbb{R}/\mathbb{Z}$-valued, we require the action amplitude in eqn. (2) to be invariant under the following "gauge" transformation

$$a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + n^I, \tag{3}$$

where n^I are arbitrary $\mathbb{Z}$-valued 1-cochains. But, the action amplitude $\mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}}$ is not gauge invariant and we need to fix it.

One way to fix this problem is modify the partition function as

$$\int [\prod \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}] \mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} (a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil) \mathrm{d}(a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil)},$$
$$= \int [\prod \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}] \mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}} \tag{4}$$
$$\mathrm{e}^{-\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}\lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil + \lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}} = Z,$$

where $\lfloor x \rceil$ denotes the nearest integer to x. The combination $a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ is invariant under the gauge transformation eqn. (3). However, in the weak-field-strength limit (*i.e.* in the continuum limit): $|\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}| \ll 1$, $a_J^{\mathbb{R}/\mathbb{Z}}$ can be large and the term $\mathrm{e}^{-\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}\lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil + \lfloor a_I^{\mathbb{R}/\mathbb{Z}} \rceil \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}}$ is not equal to 1. Thus, eqn. (4) does not reproduce the Chern-Simons path integral in the weak-field-strength limit.

This motivates us to consider the following modified partition function (which is the main result of this paper):

$$Z = \int [\prod \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}]\ \mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} \mathrm{d}\big(a_I^{\mathbb{R}/\mathbb{Z}} (a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil)\big)}$$
$$\mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} (\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil) - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil a_J^{\mathbb{R}/\mathbb{Z}}} \tag{5}$$
$$\mathrm{e}^{-\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil}\, \mathrm{e}^{-\int_{\mathcal{M}^3} \frac{|\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil|^2}{g}},$$

where $\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ is the 2-cochain whose value on the triangle $\langle ijk \rangle$ is given by $\lfloor (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}})_{ijk} \rceil$. The 1-cup product $\underset{1}{\smile}$ is defined in Supplemental Material.[34]

To see that the path integral (5) is invariant under gauge transformation (3) for $\mathcal{M}^3$ with boundary, we first note that $\mathrm{e}^{-\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil}$ and $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ are invariant under eqn. (3). Under eqn. (3), the term $\mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} (\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil) - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil a_J^{\mathbb{R}/\mathbb{Z}}}$ changes by a factor

$$\mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} n^I \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \mathrm{d}n^I a_J^{\mathbb{R}/\mathbb{Z}}}$$
$$= \mathrm{e}^{-\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\partial\mathcal{M}^3} n^I a_J^{\mathbb{R}/\mathbb{Z}}} \tag{6}$$

Such a factor is cancelled by the change of the term

$$\mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} \mathrm{d}\big(a_I^{\mathbb{R}/\mathbb{Z}} (a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil)\big)}$$
$$= \mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\partial\mathcal{M}^3} \big(a_I^{\mathbb{R}/\mathbb{Z}} (a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil)\big)}. \tag{7}$$

So the action amplitude of the above path integral is indeed invariant under (3) even when $\mathcal{M}^3$ has boundary.

Now, we like to argue that the bosonic lattice model (5) realizes a topological order described by $U^\kappa(1)$ CS topological quantum field theory, in the small g limit. In such a limit, $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ is close to an $\mathbb{Z}$-valued cocycle. On a local patch of spacetime, we use the gauge transformation eqn. (3) to make $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ to be near zero on the patch. In this case, the action amplitude in the path integral eqn. (5) becomes quadratic (*i.e.* non-interacting)

$$\mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}}. \tag{8}$$

Since $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ is close to zero, we can use a 1-form A^I to describe the 1-cochain $a_I^{\mathbb{R}/\mathbb{Z}}$:

$$\int_i^j A^I = 2\pi (a_I^{\mathbb{R}/\mathbb{Z}})_{ij} \tag{9}$$

Then the above action amplitude can be rewritten as

$$\mathrm{e}^{\,\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}} \approx \mathrm{e}^{\,\mathrm{i} \sum_{IJ} \frac{K_{IJ}}{4\pi} \int_{M^3} A^I \mathrm{d}A^J}$$
$$K_{IJ} = K_{JI} \equiv \begin{cases} 2k_{IJ}, & \text{if } I = J, \\ k_{IJ}, & \text{if } I < J, \end{cases} \tag{10}$$

in the small $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ limit when A^I is nearly constant on the lattice scale. Hence the low energy dynamics of our lattice bosonic model are described by a $U^\kappa(1)$ CS field theory (10) at low energies.

We like to remark that, when $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ is near integers, $\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ is a $\mathbb{Z}$-valued 2-cocycle. This is because if $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} = \epsilon + \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ where ϵ is small, then

$$\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil = -\mathrm{d}\epsilon + \mathrm{d}\,\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} = -\mathrm{d}\epsilon. \tag{11}$$

Since $\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ is quantized as integer, we have

$$\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil = 0. \tag{12}$$

Such a $\mathbb{Z}$-valued 2-cocycle $\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ characterize the $U^\kappa(1)$ principle bundle on the spacetime, since

$$\int_{\mathcal{M}^2} (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil) = -\int_{\mathcal{M}^2} \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil \tag{13}$$

for any closed $\mathcal{M}^2$. Note that $\int_{\mathcal{M}^2} (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)$ is the magnetic flux through $\mathcal{M}^2$ which is always quantized to be an integer. In other words, $-\int_{\mathcal{M}^2} \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ is the Chern number.

The above discussion of dynamics only apply when $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ is near integers, *i.e.* when g is small. When g is large, the large quantum fluctuations of $a_I^{\mathbb{R}/\mathbb{Z}}$ in the lattice bosonic model can go between configurations representing different $U^\kappa(1)$ principle bundles. The large g ground state of our model (5) may have a different topological order from the one described by the K-matrix CS theory.

What is really special about our constructed action is that it has many 1-symmetries. First, consider the model on a closed manifold, so that we may ignore the surface term. Then under the shift

$$a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + \beta_I^{\mathbb{R}/\mathbb{Z}}, \quad \sum_I \beta_I^{\mathbb{R}/\mathbb{Z}} K_{IJ} \in \mathbb{Z} \tag{14}$$

where $\beta_I^{\mathbb{R}/\mathbb{Z}}$ are $\mathbb{R}/\mathbb{Z}$-valued 1-cocycles, the exponentiated action is invariant so long as $\sum_I \beta_I^{\mathbb{R}/\mathbb{Z}} K_{IJ}$ are $\mathbb{Z}$-valued 1-cochain. Such the transformations (14) are the 1-symmetries of lattice model (5).

To see the above result, we first note that, under the transformation (14), the action amplitude in eqn. (5) on a closed manifold changes by a factor

$$\mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^3} \beta_I^{\mathbb{R}/\mathbb{Z}} (\mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}} \rceil) - \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil \beta_J^{\mathbb{R}/\mathbb{Z}}} \times \\ \mathrm{e}^{-\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^3} \beta_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d} \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \tag{15}$$

Because we may integrate by parts on a closed manifold and $\mathrm{d}\beta_I^{\mathbb{R}/\mathbb{Z}} = 0$, the change is of the form (see eqn. (10) in Supplemental Material):

$$\mathrm{e}^{-\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^3} \beta_I^{\mathbb{R}/\mathbb{Z}} \lfloor \mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}} \rceil + \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil \beta_J^{\mathbb{R}/\mathbb{Z}} + \beta_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d} \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \\ = \mathrm{e}^{-\mathrm{i}\, 2\pi \sum_{IJ} K_{IJ} \int_{\mathcal{M}^3} \beta_I^{\mathbb{R}/\mathbb{Z}} \lfloor \mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}} \rceil} \tag{16}$$

which remains unity for all $\lfloor \mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}} \rceil$ iff $\sum_I \beta_I^{\mathbb{R}/\mathbb{Z}} K_{IJ}$ are $\mathbb{Z}$-valued cochains. We see that, on a fixed link ij, the allowed values $(\beta_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ form the rational lattice K^{-1}. The 1-symmetries are given by the rational lattice K^{-1} mod out the integer lattice, which is same as integer lattice mod out lattice K. In other words, the 1-symmetries are $Z_{k_1} \times Z_{k_2} \times \cdots$ 1-symmetries with k_i being the diagonal entries of the Smith normal form of K.

For example, for $U(1)$ Chern Simons theory with $\kappa = 1$ and $K_{11} = 2k_{11} = k$, we have a $\mathbb{Z}_k$ 1-symmetry. For mutual CS theory (that describes a Z_n gauge theory), with $(K_{IJ}) = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$ we have a $Z_n \times Z_n$ 1-symmetry.

Some of the above 1-symmetries are anomalous. To see which 1-symmetries are anomalous, we need check which of the transformations in eqn. (14) changes the action amplitude when the spacetime has a boundary. Under the transformation (14), the action amplitude in eqn. (5) only changes by a factor defined on the boundary $\partial\mathcal{M}^3$:

$$\mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\partial\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} (\beta_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \beta_J^{\mathbb{R}/\mathbb{Z}} \rceil) + \beta_I^{\mathbb{R}/\mathbb{Z}} (a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil) + \beta_I^{\mathbb{R}/\mathbb{Z}} (\beta_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \beta_J^{\mathbb{R}/\mathbb{Z}} \rceil)}\, \mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\partial\mathcal{M}^3} \beta_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil - \beta_I^{\mathbb{R}/\mathbb{Z}} a_J^{\mathbb{R}/\mathbb{Z}}} \\ = \mathrm{e}^{\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\partial\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} (\beta_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \beta_J^{\mathbb{R}/\mathbb{Z}} \rceil) + \beta_I^{\mathbb{R}/\mathbb{Z}} (\beta_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \beta_J^{\mathbb{R}/\mathbb{Z}} \rceil) + \beta_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \lfloor \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \rceil}\, \mathrm{e}^{-\mathrm{i}\, 2\pi \sum_{I\leq J} k_{IJ} \int_{\partial\mathcal{M}^3} \beta_I^{\mathbb{R}/\mathbb{Z}} \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil} \tag{17}$$

We see that the transformations leave the action amplitude invariant if $\sum_{I\leq J} k_{IJ}\beta_J^{\mathbb{R}/\mathbb{Z}} = 0$ and $\sum_{I\leq J} k_{IJ}\beta_I^{\mathbb{R}/\mathbb{Z}} =$ integer. We note that $\beta_I^{\mathbb{R}/\mathbb{Z}}$ satisfy the condition $\sum_{IJ} K_{IJ}\beta_I^{\mathbb{R}/\mathbb{Z}} =$ integer. Thus the first equation implies the second one. We find that the 1-symmetry transformations in eqn. (14) are anomaly-free if

$$\sum_{I\leq J} k_{IJ}\beta_J^{\mathbb{R}/\mathbb{Z}} = 0 \tag{18}$$

For the level $k = K_{11}$ CS theory with a single $U(1)$ gauge field, this is simply the fact that the only Z_k 1-symmetry must break at the boundary and is anomalous. For the case of mutual CS theory (ie the Z_n gauge theory) with $Z_n \times Z_n$ 1-symmetry, this implies that one of the Z_n 1-symmetry must break at the boundary and is anomalous. The other Z_n 1-symmetry is anomaly-free. Note that the choice of lattice model automatically selects which of the Z_n 1-symmetry is anomalous; one can select the opposite by replacing all $\sum_{I\leq J}$ with $\sum_{I\geq J}$.

Framing anomaly: It is well known that the CS theory has a framing anomaly.[35, 36] In other words, after integrating out the physical degrees of freedom $a_I^{\mathbb{R}/\mathbb{Z}}$ in eqn. (5) in small g limit, we should get a partition function given by the 2+1D gravitational CS term:

$$Z(M^3, g_{\mu\nu}) \propto \mathrm{e}^{\mathrm{i}\, \frac{2\pi c}{24} \int_{M^3} \omega_3} \tag{19}$$

where the 3-form ω_3 satisfies $\mathrm{d}\omega_3 = p_1$ and p_1 is the first Pontryagin class for the tangent bundle. Here c is the chiral central charge – the difference between the numbers of positive and negative eigenvalues of the K-matrix. There is a framing anomaly when $c \neq 0$ mod 24.

One may wonder, if the framing anomaly prevents us to have a local lattice realization of chiral CS theory with a non-zero central charge $c \neq 0$. Our construction shows that chiral $U(1)$ CS theory can always be realized on any 2+1D spacetime lattice. We think that this is possible because our spacetime lattice has an extra structure – the branching structure.[30–32] It is possible that for the same spacetime lattice, if we choose different branching structures, the resulting partition function Z may be different. This branching structure dependence of partition function may represent the framing anomaly.

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