Application of the Resource Theory of Channels to Communication Scenarios

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Application of the resource theory of channels to communication scenarios

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We introduce a resource theory of channels relevant to communication via quantum channels, in which the set of constant channels — useless channels for communication tasks — are considered as free resources. We find that our theory with such a simple structure is useful to address central problems in quantum Shannon theory — in particular, we provide a converse bound for the one-shot non-signalling assisted classical capacity that naturally leads to its strong converse property, as well as obtain the one-shot channel simulation cost with non-signalling assistance. We clarify an intimate connection between the non-signalling assistance and our formalism by identifying the non-signalling assisted channel coding with the channel transformation under the maximal set of resource nongenerating superchannels, providing a physical characterization of the latter. Our results provide new perspectives and concise arguments to those problems, connecting the recently developed fields of resource theories to 'classic' settings in quantum information theory and shedding light on the validity of resource theories of channels as effective tools to address practical problems.

Introduction. — A central problem addressed in quantum Shannon theory is to understand how much of the resources are required to accomplish the desired communication tasks and how one can efficiently use them. In general, the idea of distinguishing costly resources and free resources is helpful for articulating the problem to address, and it has been employed in a series of works in quantum information theory.

Once the precious resources are identified for the given setting, one is naturally motivated to consider quantifying and manipulating them in an appropriate manner, which leads to a general framework called *resource theories* [1]. The resource theoretic framework has been applied to various kinds of quantities [2–8] and has been employed to extract common features shared by a wide class of resources [9–21]. Recently, the framework has been extended beyond the consideration of static resources contained in quantum states to dynamic resources attributed to quantum measurements and channels, and it is currently under active investigation [15–17, 22–39].

Although the idea of resource theory has succeeded to provide much insight into the properties of the interested quantities, a common criticism is that the discussion often ends up with a formalistic level not solving other existing problems, apart from a few attempts along this line for several resource theories of states [40–43]. In particular, it has been elusive whether the resource theory of channels would be helpful for answering concrete problems at all.

In this work, we take the first step in this direction. We introduce the *resource theory of communication*, a resource theory of channels relevant to communication via quantum channels. Unlike many resource theories of channels with underlying state theories [15, 26, 27, 29, 31–35], our setting is not equipped with any theory of states, making the consideration of the resource theories of channels crucial. We consider the *generalized robustness of communication* and another related quantity, *max-relative entropy of communication*, as resource quantifiers and provide them with an operational meaning in terms of state discrimination task. We set the maximal set of superchannels that do not create resourceful channels as free operations and show that they coincide with the non-signalling assisted channel transformations, which suggests an intimate connection between our framework and the non-signalling assisted communication. With this formalism, we address two important problems in quantum Shannon theory: strong converse property and channel simulation cost under resource assistance.

One of the most fundamental properties of quantum channels is the communication capacity, and it is said to have the strong converse property if the error rate necessarily approaches unit whenever the transmission rate exceeds its capacity [44–58]. The strong converse property sets a fundamental limitation on information transmission — one can never hope to succeed in sending information with the rates exceeding the capacity even allowing for a finite error — and thus provides a sharp notion to the capacity as a phase transition point in the information transmission. While entanglement assisted channels admit wider choices of encoding than the conventional classical-quantum channels, making the strong converse property appear to be even more subtle and surprising, the strong converse property for the entanglement assisted capacity has been shown by an operational argument by flipping the quantum reverse

Shannon theorem [53, 54]. Later, a more direct proof of the strong converse property for the entanglement assisted capacity without using the above operational argument but by directly evaluating the error at the rates above the capacity has been shown in Ref. [55], for which several involved techniques were employed. Here, combining our framework with recent progress in operational characterization of resource theories in terms of discrimination tasks [14, 16, 17, 24], we provide a concise proof by showing an even stronger claim: the strong converse property for non-signalling assisted capacity, which includes entanglement assisted communication as a special case. We note that the quantum reverse Shannon theorem under non-signalling assistance [56] allows one to prove the strong converse property by an operational argument that is analogous to the one used for the entanglement assisted case [53]. Thus, our proof can be seen as an alternative proof in a more direct approach in a similar sense of Ref. [55]. Our main result is a converse bound for the one-shot non-signalling assisted classical capacity that immediately leads to the strong converse property together with the recently shown asymptotic equipartition property of max-information of channels [56]. (See also Refs. [58–61] for related works on the non-signalling assisted capacity.)

Channel simulation is a reverse task of noiseless communication via noisy channels, in which one is to implement the given noisy channel from the noiseless channel using accessible free resources [26, 29, 53, 54, 56, 62]. Recently, the one-shot channel simulation cost with non-signalling assistance has been obtained by techniques based on the semidefinite programming [56]. Here, we propose a new approach — we obtain the one-shot non-signalling assisted simulation cost by casting it as a resource dilution problem in our framework, which is a subject well studied in the context of resource theories [18, 63–65].

Our results provide new perspective to the fundamental problems in quantum Shannon theory while lifting the resource theory of channels to effective tools to address concrete problems, and the arguments employed here are expected to be extendable to more generic settings thanks to the systematic nature of the resource-theoretic frameworks.

Resource theory of communication. — Let $\mathcal{T}(A,B)$ be the set of quantum channels with input system A and output system B, while we omit the specification of input/output systems when it is clear from the context. We would like to quantitatively understand how useful the given channel is for communication tasks, and a reasonable as well as operationally motivated approach for this purpose is to take "useless" channels for communication as the set of free channels. For classical communication over quantum channels, a natural choice for useless channels are constant channels [66, 67], which map any state to some fixed state. Namely, we choose the set of free channels as

$$\mathfrak{F} := \left\{ \Xi \in \mathcal{T} \;\middle|\; \exists \sigma \text{ s.t. } \Xi(\rho) = \sigma,\ \forall \rho \right\}. \tag{1}$$

It is clear that constant channels are not able to transmit any information and thus suitable for our choice of free resources — indeed a channel has zero classical capacity if and only if it is a constant channel. In other words, choosing a different set of free channels will allow for free communication, which is not appropriate to our setting.

Our goal is to gain ideas of usefulness of a given channel in this framework, which motivates us to quantify the resourcefulness of channels with respect to the set of constant channels. To this end, we consider robustness of communication, the generalized robustness measure [13, 14, 16, 31, 36, 68, 69] with respect to the set of constant channels defined for any channel $\mathcal{N}$ as [70]

$$R(\mathcal{N}) := \min \left\{ r \;\middle|\; \tfrac{\mathcal{N}+r\mathcal{L}}{1+r} \in \mathfrak{F},\ \mathcal{L} \in \mathcal{T} \right\}, \tag{2}$$

which was introduced in Ref. [71]. It is also convenient to consider the max-relative entropy of communication:

$$\mathfrak{D}_{\max}(\mathcal{N}) := \min \left\{ s \;\middle|\; \mathcal{N} \le 2^s \mathcal{L},\ \mathcal{L} \in \mathfrak{F} \right\}, \tag{3}$$

where the inequality is in terms of complete positiveness. Then, it is straightforward to see the relation $\mathfrak{D}_{\max}(\mathcal{N}) = \log(1 + R(\mathcal{N}))$. We discuss their properties in detail in Appendix A [72].

Besides the quantification of resources, another central theme that resource theories deal with is manipulation of resources. Since our resource objects are quantum channels, it is natural to consider channel transformations under superchannels [73, 74]. Let $\mathcal{S}(\{A,B\},\{A',B'\})$ be the set of superchannels that map channels in $\mathcal{T}(A,B)$ to channels in $\mathcal{T}(A',B')$. Of particular interest are channel transformations under "free" superchannels. The requirement for free superchannels is that they do not create resourceful channels out of free channels. Within this constraint, there is still much freedom in what additional constraints one should impose [36, 39]. Here, we will take a similar approach to Ref. [39], considering the maximal set of free superchannels (often called "resource non-generating") defined as [75]

$$\mathcal{O}_{\mathfrak{F}} := \left\{ \Theta \in \mathcal{S} \;\middle|\; \Theta[\Xi] \in \mathfrak{F},\ \forall \Xi \in \mathfrak{F} \right\}. \tag{4}$$

Although this type of "maximal" choice of free operations is motivated by a mathematical convenience and usually does not have a good characterization (e.g. separability preserving operations for bipartite entanglement [63], maximally incoherent operations for coherence [76]), it turns out that in our case this choice of free superchannels exactly corresponds to the communication setting with non-signalling assistance, which connects the mathematical formulation of resource theory and communication tasks. (See Refs. [77–79] for other resource assisted tasks considered in different settings.) More precisely, consider a channel transformation by a non-signalling bipartite channel $\Pi_{\rm NS} : A_i B_i \to A_o B_o$ converting channel $\mathcal{N} \in \mathcal{T}(A_o, B_i)$ to another channel $\mathcal{N}' \in \mathcal{T}(A_i, B_o)$ where $\Pi_{\rm NS}$ satisfies the non-signalling conditions [60]

$$\mathrm{Tr}_{A_o} \Pi_{\rm NS}(\rho^{(0)}_{A_i} \otimes \rho_{B_i}) = \mathrm{Tr}_{A_o} \Pi_{\rm NS}(\rho^{(1)}_{A_i} \otimes \rho_{B_i}) \tag{5}$$

$$\mathrm{Tr}_{B_o} \Pi_{\rm NS}(\rho_{A_i} \otimes \rho^{(0)}_{B_i}) = \mathrm{Tr}_{B_o} \Pi_{\rm NS}(\rho_{A_i} \otimes \rho^{(1)}_{B_i}) \tag{6}$$

for any state ρ_{A_i}, ρ_{B_i}, and any pair of states $\{\rho^{(j)}_{A_i}\}_{j=0}^1, \{\rho^{(j)}_{B_i}\}_{j=0}^1$. The subscript $i(o)$ in $A_i(B_o)$ indicates that the system is an input (output) system of the bipartite channel $\Pi_{\rm NS}$. Eq. (6) ensures that the bipartite operation $\Pi_{\rm NS}$ is "semicausal" from A to B [80], which is shown to be "semilocalizable" [81] and constructs a "quantum comb" with a causal order [73, 82] as shown in Fig. 1.

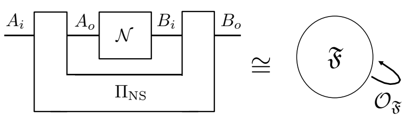

FIG. 1. Non-signalling bipartite channel $\Pi_{\rm NS}$ constructs a quantum comb (left). Proposition 1 shows that the non-signalling assisted channel transformation is equivalent to the maximal set of superchannels that preserves the set of constant channels (right).

Let $\mathcal{O}_{\rm NS}$ be the set of superchannels realized by non-signalling channels; i.e. $\mathcal{O}_{\rm NS} := \{\Theta \in \mathcal{S} \,|\, \Theta[\mathcal{N}] = \Pi_{\rm NS} \circ \mathcal{N}, \ \forall \mathcal{N} \in \mathcal{T}\}$ where $\Pi_{\rm NS}$ is a non-signalling channel satisfying (5) and (6) and the concatenation ($\circ$) refers to the comb structure in Fig. 1. Then, we have the following identification between two sets of channel transformations. (Proof in Appendix B.)

Proposition 1. *The set of resource non-generating free superchannels coincides with that of non-signalling assisted channel transformations, i.e.* $\mathcal{O}_{\mathfrak{F}} = \mathcal{O}_{\rm NS}$.

This result shows that the maximal set of free superchannels, which is motivated by a mathematical convenience, is characterized by the ultimate bipartite correlation respecting the causality in the theory of relativity [80] and thus gains a physical characterization. Furthermore, Proposition 1 allows us to consider the non-signalling assisted channel coding as a channel transformation under the maximal set of free superchannels, which will be useful for later discussions.

Finally, we show the monotonicity property of a "smoothed" version of the max-relative entropy measure under free superchannels, ensuring it to be a valid resource monotone. Moreover, the monotonicity holds for general resource theories, in which our theory is included as a special case. Define the smooth max-relative entropy measure with respect to an arbitrary set of free channels $\mathfrak{F}'$ as $\mathfrak{D}^{\epsilon}_{\max,\mathfrak{F}'}(\mathcal{N}) := \min_{\|\mathcal{N}'-\mathcal{N}\|_\diamond \le \epsilon} \mathfrak{D}_{\max,\mathfrak{F}'}(\mathcal{N}')$ where $\mathfrak{D}_{\max,\mathfrak{F}'}(\mathcal{N}')$ is the max-relative entropy measure with respect to $\mathfrak{F}'$ defined analogously to (3), and $\|\cdot\|_\diamond$ is the diamond norm. Then, we have the following general monotonicity property of the smoothed measure. (Proof in Appendix C.)

Lemma 2. *Let $\mathfrak{F}'$ be an arbitrary set of free channels, $\mathcal{O}_{\mathfrak{F}'}$ be the corresponding set of resource non-generating free superchannels. Then, for any channel $\mathcal{N}$ and free superchannel $\Theta \in \mathcal{O}_{\mathfrak{F}'}$, it holds that $\mathfrak{D}^{\epsilon}_{\max,\mathfrak{F}'}(\mathcal{N}) \ge \mathfrak{D}^{\epsilon}_{\max,\mathfrak{F}'}(\Theta[\mathcal{N}])$ for $\epsilon \ge 0$.*

In the next section, we see the relation between the resource measure developed here and operational advantage in state discrimination tasks, which is naturally connected to classical communication tasks we will discuss later.

Operational characterization of channel resources. — Characterizing an operational advantage enabled by the given resource is a central question in quantum information theory, and recent works have clarified that discrimination tasks are effective platforms to investigate for this purpose. More specifically, let $p_{\rm succ}(\mathcal{A}, \Lambda, \{M_i\}) := \sum_i p_i \,\mathrm{Tr}[\Lambda(\sigma_i) M_i]$ be the average success probability for discriminating the given state ensemble $\mathcal{A} = \{p_i, \sigma_i\}$ with the action of channel Λ. Then, the following relation has been shown to hold for general resource theories.

Lemma 3 ([16])**.** *For any convex and closed set of free channels $\mathfrak{F}'$, it holds that, for any channel $\mathcal{N} \in \mathcal{T}(A, B)$,*

$$\max_{\mathcal{A}, \{M_i\}} \frac{p_{\rm succ}(\mathcal{A}, \mathrm{id}_E \otimes \mathcal{N}, \{M_i\})}{\max_{\Xi \in \mathfrak{F}'} p_{\rm succ}(\mathcal{A}, \mathrm{id}_E \otimes \Xi, \{M_i\})} = 1 + R_{\mathfrak{F}'}(\mathcal{N})$$

where $\mathcal{A}$ is the state ensemble defined on the system EA with E being some quantum system, and $R_{\mathfrak{F}'}(\mathcal{N})$ is the generalized robustness defined with respect to the set of free channels $\mathfrak{F}'$.

Lemma 3 was shown aiming to characterize the operational advantage enabled by the given resourceful channel with respect to free resources in terms of the performance of state discrimination tasks at a high level of generality. However, to obtain such a general result, the same measurement strategy $\{M_i\}$ needs to be employed on comparing the resourceful case and resourceless case, which makes the task a little artificial. Interestingly, the simple structure of our theory allows for a more natural and convenient form in the left-hand side of the above result — in particular, the optimization over measurements can be taken separately in the numerator and denominator, and the denominator can be reduced to the form relevant to our communication setting. Let $p_{\rm succ}(\mathcal{A}, \mathcal{N}) := \max_{\{M_i\}} p_{\rm succ}(\mathcal{A}, \mathcal{N}, \{M_i\})$ be the optimal success probability, and $p_{\rm guess}(\mathcal{A}) := \max_i p_i$ be the success probability for the best random guess. Then, we obtain the following. (Proof in Appendix D.)

Theorem 4. *For any channel $\mathcal{N} \in \mathcal{T}(A, B)$, we get*

$$\max_{\mathcal{A} \in \mathscr{A}} \frac{p_{\rm succ}(\mathcal{A}, \mathrm{id}_E \otimes \mathcal{N})}{p_{\rm guess}(\mathcal{A})} = 1 + R(\mathcal{N}) \tag{7}$$

where $\mathscr{A} := \left\{ \{p_i, \sigma_i^{EA}\} \,\middle|\, \mathrm{Tr}_A[\sigma_i^{EA}] = \mathrm{Tr}_A[\sigma_j^{EA}],\ \forall i,j \right\}$. *On the other hand, for the case when the external system E is not considered, we have that for any ensemble $\mathcal{A}$ supported on system A,*

$$\frac{p_{\mathrm{succ}}(\mathcal{A},\mathcal{N})}{p_{\mathrm{guess}}(\mathcal{A})} \leq 1 + R(\mathcal{N}). \tag{8}$$

Now, we are in a position to consider communication tasks, for which we employ the relation between resource measure and state discrimination.

Converse bound for assisted capacity. — Let M be the number of classical messages Alice tries to send to Bob. Also, let A_i and B_o be the m-dimensional message space and $\Theta \in \mathcal{S}(\{A_o, B_i\}, \{A_i, B_o\})$ be a superchannel that is constructed by a non-signalling bipartite channel as in Fig. 1, i.e. $\Theta \in \mathcal{O}_{\mathrm{NS}}$. (We also call this *non-signalling assisted code* in the context of message transmission.) For a given superchannel, we write the average error probability of decoding as $\varepsilon[\Theta,\mathcal{N}] := 1 - \frac{1}{M}\sum_{m=0}^{M-1} \langle m|\,\Theta[\mathcal{N}](|m\rangle\langle m|)\,|m\rangle$. Then, the non-signalling assisted one-shot classical capacity with error ϵ is defined as [61]

$$C^{\epsilon}_{\mathrm{NS},(1)}(\mathcal{N}) := \sup_{\Theta} \left\{ \log M \,\middle|\, \varepsilon[\Theta,\mathcal{N}] \leq \epsilon \right\}. \tag{9}$$

Using our framework and above result, we can concisely show a converse bound for the one-shot non-signalling assisted classical capacity.

Theorem 5. *For $\delta \geq 0$ and $0 \leq \epsilon < 1 - \delta/2$, it holds that*

$$C^{\epsilon}_{\mathrm{NS},(1)}(\mathcal{N}) \leq \mathfrak{D}^{\delta}_{\max}(\mathcal{N}) + \log\left(\frac{1}{1-\epsilon-\delta/2}\right) \tag{10}$$

Proof. Let $\mathcal{A} = \{1/M, |m\rangle\langle m|\}_{m=0}^{M-1}$ and Θ be a non-signalling assisted code achieving $\varepsilon[\Theta,\mathcal{N}] = 1 - p_{\mathrm{succ}}(\mathcal{A},\Theta[\mathcal{N}]) \leq \epsilon$. Using Proposition 1, we have that $\Theta \in \mathcal{O}_{\mathfrak{F}}$. Now, take the channel $\mathcal{L}$ with $\|\mathcal{L} - \Theta[\mathcal{N}]\|_\diamond \leq \delta$ that satisfies $\mathfrak{D}_{\max}(\mathcal{L}) = \mathfrak{D}^{\delta}_{\max}(\Theta[\mathcal{N}])$. Using (8) in Theorem 4 and $p_{\mathrm{guess}}(\mathcal{A}) = 1/M$, we get

$$p_{\mathrm{succ}}(\mathcal{A},\mathcal{L}) \leq \frac{1 + R(\mathcal{L})}{M} = 2^{\mathfrak{D}^{\delta}_{\max}(\Theta[\mathcal{N}])}/M \tag{11}$$

where we used $\log(1 + R(\mathcal{L})) = \mathfrak{D}_{\max}(\mathcal{L}) = \mathfrak{D}^{\delta}_{\max}(\Theta[\mathcal{N}])$. Then, we use the following simple lemma, which we show in Appendix E.

Lemma 6. *For any state ensemble $\mathcal{A}$ and channels $\mathcal{L}, \mathcal{M} \in \mathcal{T}(A,B)$, we have $|p_{\mathrm{succ}}(\mathcal{A},\mathcal{M}) - p_{\mathrm{succ}}(\mathcal{A},\mathcal{L})| \leq \frac{1}{2}\|\mathcal{M}-\mathcal{L}\|_\diamond$.*

Applying Lemma 6 and the monotonicity of the smoothed max-relative entropy measure (Lemma 2) to (11), we reach $p_{\mathrm{succ}}(\mathcal{A},\Theta[\mathcal{N}]) - \delta/2 \leq 2^{\mathfrak{D}^{\delta}_{\max}(\mathcal{N})}/M$. The proof is completed by taking the logarithm on both sides and using $p_{\mathrm{succ}}(\mathcal{A},\Theta[\mathcal{N}]) = 1 - \varepsilon[\Theta,\mathcal{N}] \geq 1 - \epsilon$.

□

This result establishes a fundamental connection between the resourcefulness of the channel quantified in the resource-theoretic framework and its operational capability as a communication channel.

We remark that related bounds have been presented in Refs. [83, 84]. An advantage of our result over their bounds is that, besides the simplicity of its proof, it naturally leads to the strong converse property as we shall see below. Note also that by combining the relations presented in Refs. [58, 84, 85] one can alternatively reach the same one-shot bound in Theorem 5 [86].

Strong converse property for the assisted capacity. — Next, we discuss the advantage of Theorem 5 in the asymptotic setting. For this aim, we take into account the situation where multiple copies of the channel are in use. Consider a sequence of the message size $M^{(n)}$ and non-signalling assisted codes $\Theta^{(n)}$. Then, we can define the non-signalling assisted classical capacity as the maximum rate of the message transmission with vanishing error in the asymptotic limit: [87]

$$C_{\mathrm{NS}}(\mathcal{N}) := \sup_{\{\Theta^{(n)}\}} \left\{ \underline{\lim} \frac{\log M^{(n)}}{n} \,\middle|\, \lim_{n\to\infty} \varepsilon[\Theta^{(n)}, \mathcal{N}^{\otimes n}] = 0 \right\}. \tag{12}$$

We also introduce the non-signalling assisted strong converse capacity $C^{\dagger}_{\mathrm{NS}}(\mathcal{N})$ by replacing $\lim_{n\to\infty} \varepsilon[\Theta^{(n)},\mathcal{N}^{\otimes n}] = 0$ in (12) with $\lim_{n\to\infty} \varepsilon[\Theta^{(n)},\mathcal{N}^{\otimes n}] < 1$. By definition, it holds that $C_{\mathrm{NS}}(\mathcal{N}) \leq C^{\dagger}_{\mathrm{NS}}(\mathcal{N})$ for any $\mathcal{N}$. If $C_{\mathrm{NS}}(\mathcal{N}) = C^{\dagger}_{\mathrm{NS}}(\mathcal{N})$ also holds, we say that the strong converse property holds.

A direct proof of the strong converse property for the entanglement assisted capacity without employing the operational argument via quantum reverse Shannon theorem was reported in Ref. [55]. There, they first put an upper bound for the decoding success probability in terms of variants of mutual information derived from the α-sandwiched Rényi entropy [52, 88] using the meta-converse bound [84]. Then, they showed the additivity of the α-mutual information for $\alpha \in (1,\infty)$ using the multiplicativity of completely bounded p-norms [89], which allowed them to connect the α-mutual information to the usual mutual information, eventually proving the strong converse.

Here, we see that Theorem 5 naturally allows for this type of direct proof of an even stronger claim, the strong converse property for the non-signalling assisted capacity, without delving into involved steps such as the ones in Ref. [55]. The main idea is to combine Theorem 5 with the following asymptotic equipartition property [56],

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n}\mathfrak{D}^{\delta}_{\max}(\mathcal{N}^{\otimes n}) = I(\mathcal{N}) \tag{13}$$

where $I(\mathcal{N}) := \max_{|\psi\rangle} I(\rho_{AB})$ and $\rho_{AB} := \mathrm{id}\otimes\mathcal{N}(|\psi\rangle\langle\psi|)$ is the channel mutual information.

Corollary 7. *For any channel $\mathcal{N}$, the strong converse property holds for the non-signalling assisted capacity, i.e. $C_{\mathrm{NS}}(\mathcal{N}) = C^{\dagger}_{\mathrm{NS}}(\mathcal{N})$.*

Proof. Theorem 5 implies that for any n and $0 \le \epsilon < 1 - \delta/2$,

$$\frac{1}{n} C^{\epsilon}_{\mathrm{NS},(1)}(\mathcal{N}^{\otimes n}) \le \frac{1}{n} \mathfrak{D}^{\delta}_{\max}(\mathcal{N}^{\otimes n}) + \frac{1}{n} \log\left(\frac{1}{1-\epsilon-\delta/2}\right).$$

Taking $\lim_{\delta\to 0}\lim_{n\to\infty}$ in both sides and using (13), we obtain $\lim_{n\to\infty}\frac{1}{n}C^{\epsilon}_{\mathrm{NS},(1)}(\mathcal{N}^{\otimes n}) \le I(\mathcal{N}) = C_{\mathrm{EA}}(\mathcal{N})$ for any $0 \le \epsilon < 1$ where C_{EA} is the entanglement assisted classical capacity [90]. This proves $C_{\mathrm{NS}}(\mathcal{N}) \ge C_{\mathrm{EA}}(\mathcal{N}) \ge C^{\dagger}_{\mathrm{NS}}(\mathcal{N})$, showing the strong converse property. □

This result provides a new perspective to the ultimate communication capability with many channel uses in terms of the operational characterization of channel resources.

Channel simulation. — Proposition 1 also allows us to identify the non-signalling assisted channel simulation with the resource dilution problem in our resource theory. Specifically, let id_k be the identity channel acting on k-dimensional Hilbert space. We ask the minimum size of the identity channel needed to realize the desired channel by free superchannels. To this end, we define the one-shot dilution cost for given channel $\mathcal{N}$ and error ϵ as

$$C^{\epsilon}_{c,(1)}(\mathcal{N}) := \min\left\{ k \;\middle|\; \exists\Theta \in \mathcal{O}_{\mathfrak{F}} \text{ s.t. } \|\Theta[\mathrm{id}_k] - \mathcal{N}\|_{\diamond} \le \epsilon \right\}.$$

Then, we obtain the following. (Proof in Appendix F.)

Theorem 8.

$$C^{\epsilon}_{c,(1)}(\mathcal{N}) = \lceil 2^{\frac{1}{2}\mathfrak{D}^{\epsilon}_{\max}(\mathcal{N})} \rceil. \tag{14}$$

This result provides the generalized robustness/max-relative entropy measure with another operational meaning. As expected, our result coincides with the non-signalling assisted one-shot channel simulation cost obtained by a different approach [56]. Since our method is based on a systematic resource theoretic treatment, it will provide a useful tool with wide applicability. We also remark that because of (13), the asymptotic cost is characterized by the mutual information of the channel, which makes the channel transformation reversible at the asymptotic limit. Our resource-theoretic treatment makes the comparison to other reversible theories clearer — in our case the mutual information serves as the "potential" function that fully characterizes the resource transformability.

Conclusions. — We introduced a resource theory of channels relevant to communication scenarios where the set of constant channels serves as the free channels. We considered channel transformation under the maximal set of free superchannels and found that such channel transformation coincides with that under non-signalling assistance. Employing this identification, we applied our formalism to provide a converse bound for the one-shot non-signalling assisted classical capacity, which leads to the strong converse property for the non-signalling assisted capacity, as well as to obtain the one-shot channel simulation cost with non-signalling assistance by considering the resource dilution cost under free superchannels. Both of the quantities are characterized by the max-relative entropy measure with respect to our choice of free channels, endowing this measure with clear operational meanings.

Our results indicate the further potential of resource theoretic framework as effective tools to solve concrete problems. In this respect, an interesting future direction will be to adopt our method to encompass other communication settings such as non-assisted classical/quantum communication and communication with restricted quantum measurements.

Note added. — Recently, we became aware that the latest update of Ref. [56] has obtained a similar relation to the one in Lemma 2 for the case of non-signalling superchannels.

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