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Phys. Rev. Lett. **124**, 110605 — Published 19 March 2020

DOI: 10.1103/PhysRevLett.124.110605

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(Dated: February 26, 2020)

We give rigorous analytical results on the temporal behavior of two-point correlation functions –also known as dynamical response functions or Green's functions– in closed many-body quantum systems. We show that in a large class of translation-invariant models the correlation functions factorize at late times $\langle A(t)B\rangle_\beta \rightarrow \langle A\rangle_\beta \langle B\rangle_\beta$, thus proving that dissipation emerges out of the unitary dynamics of the system. We also show that for systems with a generic spectrum the fluctuations around this late-time value are bounded by the purity of the thermal ensemble, which generally decays exponentially with system size. For auto-correlation functions we provide an upper bound on the timescale at which they reach the factorized late time value. Remarkably, this bound is only a function of local expectation values, and does not increase with system size. We give numerical examples that show that this bound is a good estimate in non-integrable models, and argue that the timescale that appears can be understood in terms of an emergent fluctuation-dissipation theorem. Our study extends to further classes of two point functions such as the symmetrized ones and the Kubo function that appears in linear response theory, for which we give analogous results.

Two-point correlation functions –or dynamical response/Green's functions– are the central object of the theory of linear response [1], and appear in the characterization of a wide range of non-equilibrium and statistical phenomena in the study of quantum many-body systems and condensed matter physics [2]. This includes different types of scattering and spectroscopy experiments [3], quantum transport [4, 5], and fluctuation-dissipation relations [6–8]. They have also appeared in the characterization of topological [9] and crystalline ordering [10], of quantum-chaotic systems [11, 12] and of different notions of ergodicity in quantum and classical systems [13, 14].

Here we study the time evolution of such correlation functions in isolated systems evolving under unitary dynamics. More precisely, we focus on functions of the form

$$C^{AB}(t) \equiv \langle A(t)B\rangle_\beta = \mathrm{Tr}\left(\rho A(t)B\right), \tag{1}$$

where the evolution is generated by a time-independent Hamiltonian H, $\rho \equiv e^{-\beta H}/Z_\beta$ is a thermal state at inverse temperature β with partition function Z_β, and $A(t) = e^{iHt}Ae^{-iHt}$ is the evolved observable in the Heisenberg picture. Both A and B are usually taken to be either local (such as a single-site spin) or extensive operators (such as a global current or magnetization).

Two-point correlation functions have been widely studied before, mostly through numerical methods such as exact diagonalization [15], QMC [16] and tensor networks [17–23], and analytically for specific models, e.g. [24–29]. Also, a number of experimental schemes to measure it directly have been proposed [30–34], which manage to circumvent the obstacle of having to measure two non-commuting observables on a single system. Here, we give rigorous analytical results on their dynamical behavior with as few assumptions on the Hamiltonian as possible. Our results apply to most translation-invariant non-integrable Hamiltonians, in which the degeneracy of the energy spectrum is small.

First, for arbitrary local observables A and B we prove that, for late times, the following signature of dissipation occurs in a large class of translation-invariant models

$$\langle A(t)B\rangle_\beta \xrightarrow{t\rightarrow\infty} \langle A\rangle_\beta \langle B\rangle_\beta. \tag{2}$$

Moreover, we show that the fluctuations around the late-time value are in fact bounded by the effective dimension $d_{\text{eff}}^{-1} \equiv \mathrm{Tr}\left(\rho^2\right)$ of the ensemble, which decays quickly with system size.

For the case of auto-correlation functions, when $A = B$, we also derive an upper bound on the timescale at which the factorization of Eq. (2) happens, which, remarkably, is independent of the size of the system. We provide numerical evidence showing that the bound is in fact a good estimate even for moderate system sizes, and becomes tighter as the size increases.

Our study can be extended to a large class of 2-point correlation functions. For instance, for symmetric correlation functions $C_s^A(t) \equiv \langle A(t)A\rangle_\beta + \langle AA(t)\rangle_\beta$, we find that evolution is dominated by a timescale which is at most of order $t^2 \sim \frac{\langle A^2\rangle_\beta}{\langle [A,H][H,A]\rangle_\beta}$. We argue that this can be interpreted in terms of a fluctuation-dissipation theorem that arises from the unitary dynamics of the system. Finally, we consider the timescales of evolution of the

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Kubo correlation function that appears in linear response theory [1, 7], which dictates the response of a system at equilibrium to a perturbation in its Hamiltonian.

Late-time behaviour — We now show the rigorous formulation of the late-time factorization of 2-point functions. First, we need the following definition.

Definition 1 (Clustering of correlations). *A state ρ on an Euclidean lattice $\mathbb{Z}^D$ has finite correlation length $\xi > 0$ if it holds that*

$$\max_{X\in M, Y\in N} \frac{|\operatorname{Tr}(\rho X\otimes Y) - \operatorname{Tr}(\rho X)\operatorname{Tr}(\rho Y)|}{||X||||Y||} \leq e^{-\frac{\operatorname{dist}(M,N)}{\xi}}, \tag{3}$$

where M, N are regions on the lattice separated by a distance of at least $\operatorname{dist}(M,N)$, and X, Y are arbitrary operators with support on each region.

This condition is generic of thermal states at finite temperature away from a phase transition. It has been proven at least for 1D systems [35] and arbitrary models above a threshold temperature [36]. In order to prove factorization at late times, we focus on systems on states that show clustering of correlations, and whose Hamiltonians are k-local, i.e. which can be written as $H = \sum_j h_j$, where h_j couples at most k closest neighbors.

Given that evolution is unitary and the system is finite-dimensional, limits such as $\lim_{t\to\infty} C^{AB}(t)$ are not well-defined. Hence, we consider the late-time behaviour under *infinite-time averages* of the correlation functions $\lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T} C^{AB}(t)$. With these considerations, our first main result is the following.

Theorem 1. *Let H be a k-local, translation-invariant, non-degenerate Hamiltonian on a D-dimensional Euclidean lattice of N sites, and let $[\rho, H] = 0$ be an equilibrium ensemble (such as a thermal state) of finite correlation length $\xi > 0$. Let A, B be local observables with support on at most $\mathcal{O}(N^{\frac{1}{D+1}-\nu})$ sites, with $\nu > 0$. Then*

$$\lim_{T\to\infty}\int_0^T C^{AB}(t)\frac{dt}{T} = \operatorname{Tr}(\rho A)\operatorname{Tr}(\rho B) + \mathcal{O}\left(\xi^{\frac{2D}{D+1}}\log^2(N) N^{-\frac{2}{D+1}}\right) \tag{4}$$

This guarantees that all operators supported on a region with size scaling like any function smaller than $\mathcal{O}(N^{\frac{1}{D+1}})$, satisfy the assumptions of the theorem. The proof, found in [37], relies on a weak form of the Eigenstate Thermalization Hypothesis (ETH) shown in [38], which is itself based on previous works on large deviation theory for lattice models [39, 40]. This shows that, in fact, any model obeying the weak ETH and without too many degeneracies will display identical factorization of correlation functions at long times [41].

Note that we assume that the energy spectrum is non-degenerate, which is accurate for systems without non-trivial symmetries or extensive number of conserved quantities. In particular, non-integrable systems usually display Wigner-Dyson statistics in their fine-grained spectrum, which imply level repulsion [8].

This factorization of the correlation function can be thought of as a signature of the emergence of dissipation due to unitary dynamics, since the lack of correlations at different times indicates the loss of information about an initial perturbation of B at time $t = 0$, as reflected in the observable A at time t [1].

Fluctuations around late-time value — For most times, the 2-point correlation function is in fact close to its late-time average, with small fluctuations around the equilibrium value. In order to prove this, one needs the extra assumption that the energy gaps are non-degenerate, which is again reasonable in non-integrable systems with connected Hamiltonians [8, 42], where it is generally expected to hold as random perturbations are sufficient to lift degeneracies in energy gaps [43].

Let us define $C^{AB}_\infty = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T} C^{AB}(t)$, and the average fluctuations around the late-time value as

$$\sigma_C^2 = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\left(C^{AB}(t) - C^{AB}_\infty\right)^2. \tag{5}$$

The following result puts an upper bound on average fluctuations.

Theorem 2. *Let $H = \sum_j E_j |j\rangle\langle j|$ be a Hamiltonian with non-degenerate energy gaps, such that*

$$E_j - E_k = E_m - E_l \;\Leftrightarrow j = m\;, k = l, \tag{6}$$

and let $[\rho, H] = 0$. It holds that

$$\sigma_C^2 \leq \|A\|\,\|B\| \max_{j\neq k}\{|A_{kj}B_{jk}|\}\operatorname{Tr}\left(\rho^2\right), \tag{7}$$

where A_{kj}, B_{jk} are matrix elements in the energy eigenbasis, $A = \sum_{jk} A_{jk}|j\rangle\langle k|$.

The proof can be found in [37]. It follows the same steps as the main result in [44]. Here, we also find that the purity $\operatorname{Tr}\left(\rho^2\right)$ of the equilibrium ensemble plays a key role. For a microcanonical ensemble $\operatorname{Tr}\left((\mathbb{1}/d)^2\right) = 1/d$, so the RHS of Eq. (7) is expected to decay exponentially with system size in most situations of interest. Also, notice that for a thermal state $\operatorname{Tr}\left(\rho_\beta^2\right) \leq 1/Z_\beta$. Moreover, the ETH predicts that $|A_{kj}B_{jk}| \sim 1/d$ [45].

Timescales of equilibration — Theorems 1 and 2 combined imply that correlation functions of the form $\langle A(t)B\rangle_\beta$ are, for most times $t \in [0,\infty]$, close to the uncorrelated average $\langle A\rangle_\beta\langle B\rangle_\beta$, for a wide class of translation-invariant systems. It is expected that the timescale at which this happens may depend on a number of factors, such as the distance between A and B. If the operators are far apart on the lattice the correlations are limited by the Lieb-Robinson bound [46, 47], and timescales associated with ballistic ($\propto N^{1/D}$) or diffusive ($\propto N^{2/D}$) processes may play a role. However, for the autocorrelation function $C^A(t) \equiv \langle A(t)A\rangle_\beta$, we can show that equilibration to the late-time value occurs in a short timescale, independent of system size. There may also be further effects at larger timescales, such as the Thouless time [48, 49], and for those effects our result limits their relative size.

Let us define $\rho = \sum_j \rho_{jj} |j\rangle\langle j|$, so that ρ_{jj} and A_{jk} are the matrix elements of ρ and A in the energy basis. We can then write

$$\frac{C^A(t)}{C^A(0)} = \sum_{jk} \frac{\rho_{jj}|A_{jk}|^2}{C^A(0)} e^{-i(E_j-E_k)t} \equiv \sum_\alpha v_\alpha e^{-iG_\alpha t}, \quad (8)$$

where we denote each pair of levels $\{i, j\}$ by a Greek index, and the corresponding energy gaps by $G_\alpha \equiv E_j - E_k$ (notice that both $E_j - E_k$ and $E_k - E_j$ appear in the sum). The normalized distribution $v_\alpha \equiv \frac{\rho_{jj}|A_{jk}|^2}{C^A(0)}$ is central to our proofs, since it contains all the relevant information about the state, observable and Hamiltonian, and determines which frequencies contribute to the dynamics of the autocorrelation function. Based on it, we define the following functions.

Definition 2. *Given a normalized distribution p_α over energy gaps G_α, we define $\xi_p(x)$ as the maximum weight that fits an interval of energy gaps with width x:*

$$\xi_p(x) \equiv \max_\lambda \sum_{\alpha: G_\alpha \in [G_\lambda, G_\lambda + x]} p_\alpha. \quad (9)$$

We also define

$$a(\epsilon) \equiv \frac{\xi_p(\epsilon)}{\epsilon}\sigma_G, \qquad \delta(\epsilon) \equiv \xi_p(\epsilon), \quad (10)$$

where $\sigma_G = \sqrt{\sum_\alpha p_\alpha G_\alpha^2 - \left(\sum_\alpha p_\alpha G_\alpha\right)^2}$ is the standard deviation of the distribution p_α over the energy gaps G_α.

The important point behind these definitions is that, for a sufficiently smooth and unimodal probability distribution, one can find an ϵ small enough such that $a(\epsilon) \sim \mathcal{O}(1)$ and $\delta(\epsilon) \ll 1$. In the following theorem, the relevant probability distribution is given by v_α. Our main result regarding the timescales of correlation functions, proven in [37], is:

Theorem 3. *For any Hamiltonian H and state ρ such that $[H, \rho] = 0$, and any observable A, the autocorrelation function $C^A(t) = \mathrm{Tr}\,(\rho A(t) A)$ satisfies*

$$\frac{1}{T}\int_0^T \frac{|C^A(t) - C^A_\infty|^2}{\left(C^A(0)\right)^2} dt \leq 3\pi \left(\frac{a(\epsilon)}{\sigma_G}\frac{1}{T} + \delta(\epsilon)\right) \quad (11)$$

for any $\epsilon > 0$. Here, $a(\epsilon)$ and $\delta(\epsilon)$ are as in Definition 2 for the normalized distribution $v_\alpha \equiv \frac{\rho_{jj}|A_{jk}|^2}{C^A(0)}$, and σ_G is given by

$$\sigma_G^2 = \frac{1}{C^A(0)}\,\mathrm{Tr}\,(\rho[A, H][H, A]) - \frac{\mathrm{Tr}\,(\rho[H, A]A)^2}{\left(C^A(0)\right)^2}. \quad (12)$$

Theorem [3] provides an upper bound of $T_{eq} \equiv \frac{3\pi\, a(\epsilon)}{\sigma_G}$ on the timescales under which autocorrelation functions approach their steady state value. To see this note that, if for a given T the RHS of Eq. (11) is small, $C^A(t)$ must have spent a significant amount of time during the interval $[0, T]$ near the late-time value C^A_∞ .

The crucial point is that for distributions v_α that are uniformly spread over many values of the gaps G_α, one can always find an ϵ such that $\delta \ll 1$. In that case, the right hand side of Eq. (11) becomes small on timescales $\mathcal{O}(T_{eq})$. As discussed in [50] and in [37], if one further assumes smooth unimodal distributions, typically one also finds that $a \sim \mathcal{O}(1)$. In that case, the timescale is governed by $1/\sigma_G$. Given that σ_G is a combination of expectation values of local observables, it does not scale with the system of the system. In fact, a result of [51] shows that a timescale of order $1/\sigma_G$ provides a lower bound to the timescale of equilibration, which strongly suggests that our upper bound is tight when the conditions of $a \sim \mathcal{O}(1)$ and $\delta \ll 1$ hold.

As a prime example, for local operators in non-integrable lattice models, in which (as per the ETH) $|A_{jk}|$ are uniformly distributed around a peak at zero energy gap [52, 53], one should be able to choose ϵ such that $a \sim \mathcal{O}(1)$ and $\delta \ll 1$. In Fig. 1 we numerically show that this is indeed the case in a non-integrable Ising model.

Theorem 3 does not make assumptions on the specifics of the Hamiltonian, the observable or the state, making it completely general. However, we do not expect the correlation functions to equilibrate well in all cases, as in some scenarios $a(\epsilon)$ and $\delta(\epsilon)$ will be large no matter what value of ϵ is chosen, in which case the RHS of Eq. (11) may not become small within reasonable timescales. This can happen, for instance, in models with highly degenerate energy spectrum. To illustrate this, in [37] we compute these parameters in an integrable model, where we see that the gap degeneracies of the model negatively affect the quantities $a(\epsilon)$ and $\delta(\epsilon)$, making the estimated equilibration timescales longer.

Symmetric correlation functions — The previous results can be extended to other correlation functions, such as

$$C_s^A(t) \equiv \frac{1}{2}\,\mathrm{Tr}\,(\rho\,\{A, A(t)\}) = \frac{C^A(t) + C^A(t)^*}{2}. \quad (13)$$

Along the same lines of Theorem [3], in [37] we prove the following.

Theorem 4. *For any Hamiltonian H and state ρ such that $[H, \rho] = 0$, and any observable A, the time correlation function $C_s^A(t) = \mathrm{Tr}\,(\rho\{A, A(t)\})$ satisfies*

$$\frac{1}{T}\int_0^T \frac{|C_s^A(t) - C^A_{s,\infty}|^2}{(C_s^A(0))^2} dt \leq 3\pi\left(\frac{a(\epsilon)}{\sigma_G}\frac{1}{T} + \delta(\epsilon)\right), \quad (14)$$

for any $\epsilon > 0$. Here, $a(\epsilon)$ and $\delta(\epsilon)$ are as in Definition [2] for the normalized distribution $v_\alpha^s \equiv \frac{\rho_{jj}+\rho_{kk}}{2}\frac{|A_{jk}|^2}{C_s^A(0)}$, *and*

$$\sigma_G^2 = \frac{1}{C_s^A(0)} \operatorname{Tr}\left(\rho[A_0, H][H, A_0]\right). \tag{15}$$

Thus an upper bound for the equilibration timescale is

$$T_{eq} = \frac{3\pi\, a(\epsilon)\, \sqrt{C_s^A(0)}}{\sqrt{\operatorname{Tr}\left(\rho[A,H][H,A]\right)}}, \tag{16}$$

where again we expect that for small enough ϵ, $a(\epsilon) \sim \mathcal{O}(1)$ and $\delta \ll 1$ for the same reasons as before. The denominator in T_{eq} can be seen as an "acceleration" of the symmetric autocorrelation function. Eq. (16) can in fact be written as

$$T_{eq} = \frac{3\pi\, a(\epsilon)\, \sqrt{C_s^A(0)}}{\sqrt{\left|\frac{d^2 C_s^A(t)}{dt^2}\big|_0\right|}}. \tag{17}$$

Such timescale turns out to be similar to that of a short-time analysis. A Taylor expansion gives

$$C_s^A(t) = C_s^A(0)\left(1 - \frac{1}{2C_s^A(0)}\frac{d^2 C_s^A(t)}{dt^2}\bigg|_0 t^2\right) + \mathcal{O}(t^3). \tag{18}$$

For early times, the above expression decays on a timescale $\tau = \frac{\sqrt{2}}{3\pi a(\epsilon)} T_{eq}$, identical to our upper bound Eq. (17) up to a prefactor.

The timescale of Eq. (16) suggests an interpretation in terms of an emergent fluctuation-dissipation theorem. Consider *i)* T_{eq} to be the timescale of dissipation of unitary dynamics, meaning that $\langle A(t)A\rangle_\beta \longrightarrow \langle A\rangle_\beta \langle A\rangle_\beta$ occurs, and *ii)* $C_s^A(0) = \operatorname{Tr}\left(\rho A^2\right)$ as a measure of the fluctuations of A. Then, Eq. (16) gives a proportionality relation between the strength of the fluctuations and the timescale of equilibration, in a similar spirit to what was found in [54] using random matrix theory arguments.

Linear response and the Kubo correlation function — As a further application of our methods, we study the evolution of a quantum system under a perturbation of its Hamiltonian. Let the system start in a thermal state, such that $\rho \propto e^{-\beta(H+\lambda A)}$. Subsequently, the Hamiltonian is slightly perturbed by λA, so that the evolved state is $\rho_t = e^{-itH}\rho e^{itH}$.

It was shown by Kubo [1] that, to leading order in λ, the expectation value of A satisfies $\operatorname{Tr}\left(\rho A(t)\right) = C_{\text{Kubo}}(t) \operatorname{Tr}\left(\rho A\right)$, where for thermal initial states ρ the Kubo correlation function can be written as

$$C_{\text{Kubo}}(t) \propto \sum_{j \neq k} \frac{e^{-\beta E_k} - e^{-\beta E_j}}{E_j - E_k} |A_{jk}|^2 e^{it(E_j - E_k)}. \tag{19}$$

Equilibration of $\operatorname{Tr}\left(\rho A(t)\right)$ is then equivalent to equilibration of the function $\mathrm{C}_{\text{Kubo}}(t)$, for which we prove in [37] that the following holds.

Theorem 5. *For any Hamiltonian H, thermal state $\rho \propto e^{-\beta(H+\lambda A)}$, and any observable A, the Kubo correlation function C_{Kubo} satisfies*

$$\frac{1}{T}\int_0^T \frac{|C_{\text{Kubo}}(t) - C_{\text{Kubo},\infty}|^2}{C_{\text{Kubo}}(0)^2} dt \leq 3\pi\left(\frac{a(\epsilon)}{\sigma_G}\frac{1}{T} + \delta(\epsilon)\right), \tag{20}$$

for any $\epsilon > 0$. Here, $a(\epsilon)$ and $\delta(\epsilon)$ are as in Definition [2] for the normalized distribution $w_\alpha \equiv \frac{e^{-\beta E_k} - e^{-\beta E_j}}{E_j - E_k}\frac{|A_{jk}|^2}{C_{\text{Kubo}}(0)}$, and

$$\sigma_G^2 = \frac{1}{\mathrm{C}_{\text{Kubo}}(0)} \operatorname{Tr}\left([A,\rho][A,H]\right). \tag{21}$$

This again implies an upper bound $T_{eq} = \frac{3\pi\, a(\epsilon)}{\sigma_G}$ on the equilibration timescale of C_{Kubo}, and therefore on the time to return to thermal equilibrium after a perturbation of the system Hamiltonian by A. The distribution w_α plays the same role as v_α and v_α^s before. If w_α is smoothly distributed and unimodal (which we expect for local observables in non-integrable models) then $a \sim \mathcal{O}(1)$ and $\delta(\epsilon) \ll 1$ holds (see [37]).

Simulations — We test Theorem [3] in a spin model governed by the Hamiltonian

$$H = \sum_{j=1}^{L}\left(\gamma\sigma_j^X + \lambda\sigma_j^Z\right) + J\sum_{j=1}^{L-1}\sigma_j^Z\sigma_{j+1}^Z + \alpha\sum_{j=1}^{L-2}\sigma_j^Z\sigma_{j+2}^Z, \tag{22}$$

where σ_j^Z and σ_j^X are the Pauli spin operators along Z and X directions for spin j, and we take open boundary conditions. The field and interaction coefficients $(\gamma, \lambda, J, \alpha)$ characterize the model. We focus on a case corresponding to a system satisfying ETH by choosing $(\gamma, \lambda, J, \alpha) = (0.8, 0.5, 1, 1)$ [55], and study the autocorrelation functions of the observable $A = \sigma_{\frac{L}{2}}^x$. For simplicity we set $\beta = 1$ in our numerics, though no significant changes were observed for $\beta \in [0.1, 5]$. Figure 1 depicts the functions $a(\epsilon)$ and $\delta(\epsilon)$ that appear in Theorem 3, confirming that there exist regions of ϵ such that $\delta \ll 1$, ensuring equilibration occurs, and $a \sim 0.4$. Importantly, this is increasingly the case as the size of the system grows.

Figure 2 compares the two sides in bound (11), showing that dynamics obtained from the upper bound differs from the actual dynamics by roughly an order of magnitude. Thus, the general, model-independent bounds obtained from Theorem 3 provide remarkably good estimates of the actual (simulated) dynamics. Note that the estimate becomes increasingly better as the size of the system increases. This discrepancy could, however, be a finite-size effect, which is also suggested by the lower bound obtained in [51]. Details of the simulations can be found in [37].

Discussion — We derived analytic results on the dynamical behavior of 2-point correlation functions in quantum

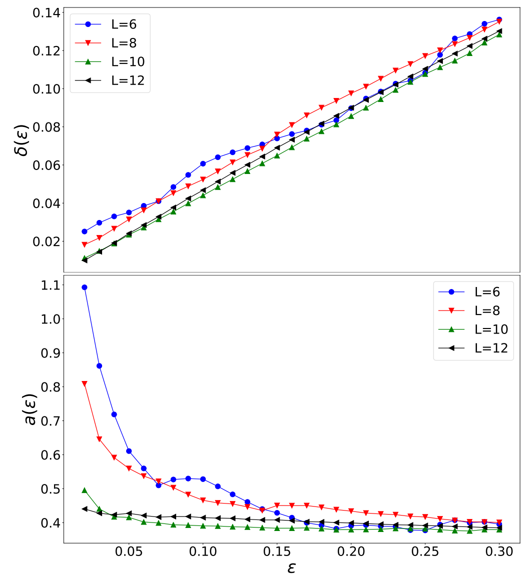

FIG. 1. Plots of $\delta(\epsilon)$ (top) and $a(\epsilon)$ (bottom) for distribution $v_\alpha \equiv \frac{\rho_{jj}|A_{jk}|^2}{C^A(0)}$ in Theorem 3, obtained by exact diagonalization and a Monte Carlo approximation. The plots were generated with $10,000$ sampled frequency intervals. Small values of δ imply equilibration occurs for long enough times, while the value of a controls the prefactor in the equilibration timescale $T_{eq} \equiv \frac{3\pi\, a(\epsilon)}{\sigma_G}$ derived from Eq. (11). For small ϵ one can satisfy both $\delta \ll 1$ and $a \sim \mathcal{O}(1)$, and this becomes increasingly so for larger system sizes.

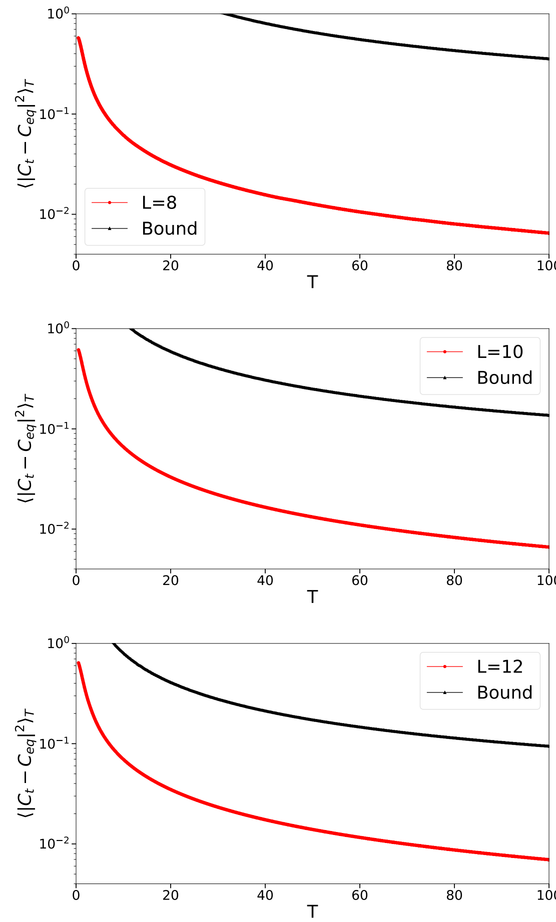

FIG. 2. Comparison of the upper bound in Eq. 11 (RHS) with the simulated evolution of the time-averaged correlation function (LHS) as a function of time, for increasing number of spins L. The evolution obtained from the upper bound approaches the exact dynamics of the system for larger system size.

systems. These include conditions that imply that time-correlation functions factorize for long times, as well as easy-to-estimate upper bounds on the timescales under which such process occurs which hold regardless of details of the model under consideration. Remarkably, our numerical findings show that the derived upper bounds can correctly estimate the actual dynamics of the system to within an order of magnitude, and become increasingly better estimates as the size of the system increases.

We used techniques previously applied in the context of equilibration of quenched quantum systems [50, 56, 57], for which finding rigorous estimates on the timescales is a largely open problem [58–61]. This connection is not surprising, specially considering that previous works [6, 62, 63] have argued that in some situations one can approximate the out of equilibrium dynamics with the autocorrelation functions covered here.

Given the importance of time-correlation functions in the analysis of a wide range of problems in many-body physics –for instance, in transport phenomena– we anticipate that our results will be useful in the description of closed system dynamics, whose study has surged in recent times due to enormous experimental advances in settings such as cold atoms or ion traps [64, 65].

ACKNOWLEDGMENTS

The authors acknowledge useful discussions with Anurag Anshu, Beni Yoshida, Charlie Nation and Jens Eisert. We are also thankful to Yichen Huang for pointing out an improvement to the error term in Theorem 1 from a previous version. This research was supported in part by the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research, Innovation and Science. This research was also supported by NSERC and enabled in part by support provided by (SHARCNET) (www.sharcnet.ca) and Compute/Calcul Canada (www.computecanada.ca). LPGP acknowledges support from the John Templeton Foundation, UMass Boston Project No. P20150000029279, DOE Grant No. DE-SC0019515, AFOSR MURI project "Scalable Certification of Quantum Computing Devices and Networks", DoE ASCR Quantum Testbed Pathfinder program (award No. DE-SC0019040), DoE BES QIS program (award No. DE-SC0019449), DoE ASCR FAR-QC (award No. DE-SC0020312), NSF PFCQC program, AFOSR, ARO MURI, ARL CDQI, and NSF PFC at JQI.

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