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# A chain rule for the quantum relative entropy

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The chain rule for the classical relative entropy ensures that the relative entropy between probability distributions on multipartite systems can be decomposed into a sum of relative entropies of suitably chosen conditional distributions on the individual systems. Here, we prove a chain rule inequality for the quantum relative entropy. The new chain rule allows us to solve an open problem in the context of asymptotic quantum channel discrimination: surprisingly, adaptive protocols cannot improve the error rate for asymmetric channel discrimination compared to non-adaptive strategies.

*Introduction.*— The von Neumann entropy  $H(A)$  of a quantum system  $A$  is a fundamental measure of uncertainty. For example, it characterizes the optimal rates for basic information-theoretic tasks such as compression or entanglement manipulation [1] and it can be used to quantify entanglement and topological order in condensed matter systems [2–4]. The conditional von Neumann entropy  $H(A_1|A_2)$  is defined via the relation

$$H(A_1A_2) = H(A_1) + H(A_2|A_1). \quad (1)$$

Iterative use of this defining relation yields expressions such as

$$H(A^n|B) = \sum_{i=1}^n H(A_i|A^{i-1}B) \quad (2)$$

as visualized in Figure 1. This relation, called chain rule, allows us to view the entropy of a large composite system as a sum of entropies of its subsystems. The chain rule property is also crucial in the definition of entanglement measures such as the squashed entanglement [5].

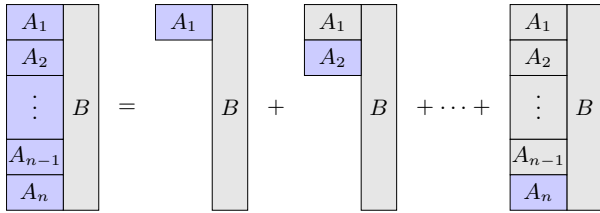


FIG. 1. A graphical visualization of the chain rule for the von Neumann entropy given in (2). The entropy is taken with respect to the blue systems conditioned on the gray systems.

In this Letter we propose a chain rule for the relative entropy defined as

$$D(\rho||\sigma) := \begin{cases} \text{tr } \rho (\log \rho - \log \sigma) & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise,} \end{cases}$$

for any states  $\rho$  and  $\sigma$  (where the latter does not need to be normalized). The relative entropy is a more general entropic quantity than the von Neumann entropy. It contains the latter and other information measures, such as the mutual information, as a special case. It can be seen as a dissimilarity measure between quantum states and is used to define various important quantities such as the relative entropy of entanglement [6]. The relative entropy characterizes the error exponent for asymmetric hypothesis testing [7] or quantifies the amount of resource in a resource theory [8, 9].

So far no chain rule for the quantum relative entropy has been proven. This is in sharp contrast with the classical case where a chain rule is known [10, Theorem 2.5.3] for the relative entropy (also called the Kullback-Leibler divergence). For a pair of discrete random variables  $(X, Y)$  with alphabet  $\mathcal{X} \times \mathcal{Y}$ , we have

$$\begin{aligned} D(P_{XY}||Q_{XY}) \\ = D(P_X||Q_X) + \sum_{x \in \mathcal{X}} P_X(x) D(P_{Y|X=x}||Q_{Y|X=x}), \end{aligned}$$

where  $P_{XY}$  and  $Q_{XY}$  are joint probability distributions, but  $Q_{XY}$  does not need to be normalized. No quantum analogue of such a chain rule is known, even if we relax the equality with the following inequalities

$$\begin{aligned} D(P_X||Q_X) + \min_{x \in \mathcal{X}} D(P_{Y|X=x}||Q_{Y|X=x}) \\ \leq D(P_{XY}||Q_{XY}) \\ \leq D(P_X||Q_X) + \max_{x \in \mathcal{X}} D(P_{Y|X=x}||Q_{Y|X=x}). \end{aligned} \quad (3)$$

In this Letter, we prove a quantum version of the upper bound (3) (see Theorem 2). We also show that it is tight in the sense that there exist non-trivial scenarios where the chain rule is an equality (see Corollary 3).

To model the quantum setting, the conditional distributions are replaced by trace-preserving completely positive maps  $\mathcal{E}$  and  $\mathcal{F}$  from  $A$  to  $B$  and the initial states are density operators  $\rho_{RA}$  and  $\sigma_{RA}$ , where  $R$  denotes a reference system. To express the very last term in (3)

in the quantum mechanical case we use a quantity called *non-stabilized channel relative entropy*, which is defined by

$$\bar{D}(\mathcal{E}\|\mathcal{F}) := \max_{\phi_A \in \mathcal{S}(A)} D(\mathcal{E}(\phi_A)\|\mathcal{F}(\phi_A)),$$

where  $\mathcal{S}(A)$  denotes the set of density operators on  $A$ . Its *stabilized* counterpart [11] is defined by  $D(\mathcal{E}\|\mathcal{F}) := \bar{D}(\mathcal{I}_A \otimes \mathcal{E}\|\mathcal{I}_A \otimes \mathcal{F})$ , where  $\mathcal{I}_A$  denotes the identity map on  $A$ . In the following we will omit identity maps if they are clear from the context. Motivated by the classical case (3), it is natural to ask whether the following chain rule

$$D(\mathcal{E}(\rho_{RA})\|\mathcal{F}(\sigma_{RA})) \stackrel{?}{\leq} D(\rho_{RA}\|\sigma_{RA}) + D(\mathcal{E}\|\mathcal{F}) \quad (4)$$

is correct [12].

*Limitations on a chain rule.*— It turns out that (4) does not hold in general because the channel relative entropy is not additive under the tensor product as shown next.

**Proposition 1.** *There exist trace-preserving completely positive maps  $\mathcal{E}, \mathcal{F}$  such that*

$$D(\mathcal{E} \otimes \mathcal{E}\|\mathcal{F} \otimes \mathcal{F}) > 2D(\mathcal{E}\|\mathcal{F}). \quad (5)$$

*This implies that there exist density operators  $\rho_{RA}, \sigma_{RA}$  for some finite-dimensional system  $R$  such that*

$$D(\mathcal{E}(\rho_{RA})\|\mathcal{F}(\sigma_{RA})) > D(\rho_{RA}\|\sigma_{RA}) + D(\mathcal{E}\|\mathcal{F}). \quad (6)$$

Before proving the assertion we introduce another quantity called *amortized channel relative entropy* [13] defined by

$$D^A(\mathcal{E}\|\mathcal{F}) := \sup_{\phi_{RA}, \omega_{RA} \in \mathcal{S}(R \otimes A)} \{D(\mathcal{E}(\phi_{RA})\|\mathcal{F}(\omega_{RA})) - D(\phi_{RA}\|\omega_{RA})\}, \quad (7)$$

where  $R$  is a reference system whose dimension is not constrained [14].

*Proof of Proposition 1.* We start by proving that (5) implies (6). It is known [15, Theorem 3 and 6] that

$$D(\mathcal{E}\|\mathcal{F}) \leq D^{\text{reg}}(\mathcal{E}\|\mathcal{F}) \leq D^A(\mathcal{E}\|\mathcal{F}),$$

where  $D^{\text{reg}}(\mathcal{E}\|\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{1}{n} D(\mathcal{E}^{\otimes n}\|\mathcal{F}^{\otimes n})$ . The statement (5) implies that the first inequality can be strict. By definition of the amortized channel relative entropy (7) this directly implies (6).

It thus remains to prove (5). To do so we construct an example of two trace-preserving completely positive maps  $\mathcal{E}$  and  $\mathcal{F}$  on qubits that satisfy (5). Consider the generalized amplitude damping channel  $\mathcal{A}_{\gamma, \beta}(\rho) = \sum_{i=1}^4 A_i \rho A_i^\dagger$  for  $\gamma, \beta \in [0, 1]$  with the Kraus operators  $A_1 = \sqrt{1-\beta}(|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|)$ ,  $A_2 = \sqrt{\gamma(1-\beta)}|0\rangle\langle 1|$ ,  $A_3 = \sqrt{\beta}(\sqrt{1-\gamma}|0\rangle\langle 0| + |1\rangle\langle 1|)$ , and  $A_4 = \sqrt{\gamma\beta}|1\rangle\langle 0|$ . For the two channels  $\mathcal{E} = \mathcal{A}_{0.3, 0}$  and

$\mathcal{F} = \mathcal{A}_{0.5, 0.9}$  their corresponding Choi matrices are given with respect to the computational basis by

$$J_{RB}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{0.7} \\ 0 & 0 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ \sqrt{0.7} & 0 & 0 & 0.7 \end{pmatrix} \quad J_{RB}^{\mathcal{F}} = \begin{pmatrix} 0.55 & 0 & 0 & \sqrt{0.5} \\ 0 & 0.45 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ \sqrt{0.5} & 0 & 0 & 0.95 \end{pmatrix}.$$

Due to the joint-convexity of the relative entropy, the maximization of the channel relative entropy can be taken over all pure states. For an arbitrary density matrix  $\rho \in \mathcal{S}(A)$  let  $|\phi\rangle_{RA} = (\sqrt{\rho_R^\top} \otimes \text{id}_A)|\Omega\rangle_{RA}$  be its purification where  $|\Omega\rangle_{RA} = \sum_i |i\rangle_R |i\rangle_A$  and where  $R$  is isomorphic to  $A$  and thus

$$\begin{aligned} \mathcal{E}(\phi_{RA}) &= (\mathcal{I} \otimes \mathcal{E})\left((\sqrt{\rho_R^\top} \otimes \text{id}_A)|\Omega\rangle\langle\Omega|_{RA}(\sqrt{\rho_R^\top} \otimes \text{id}_A)\right) \\ &= \sqrt{\rho_R^\top} J_{RB}^{\mathcal{E}} \sqrt{\rho_R^\top}. \end{aligned}$$

Hence we find

$$\begin{aligned} D(\mathcal{E}\|\mathcal{F}) &= \max_{\rho_R \in \mathcal{S}(R)} D(\sqrt{\rho_R^\top} J_{RB}^{\mathcal{E}} \sqrt{\rho_R^\top} \|\sqrt{\rho_R^\top} J_{RB}^{\mathcal{F}} \sqrt{\rho_R^\top}) \\ &= \max_{\rho_R = \text{diag}(p, 1-p)} D(\sqrt{\rho_R} J_{RB}^{\mathcal{E}} \sqrt{\rho_R} \|\sqrt{\rho_R} J_{RB}^{\mathcal{F}} \sqrt{\rho_R}). \end{aligned}$$

The final step follows since both  $\mathcal{E}$  and  $\mathcal{F}$  are covariant with respect to the Pauli- $Z$  operator. Thus it suffices to perform the maximization over input states with respect to the one-parameter family of states  $\rho_R = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$  (see e.g. [16, Proposition II.4.]). Using the `fminbnd` function in Matlab, we find  $D(\mathcal{E}\|\mathcal{F}) = 0.9176$  for an optimizer  $\rho_R = \text{diag}(0.8355, 1 - 0.8355)$ . This can also be seen by plotting the value of the relative entropy over the interval  $p \in [0, 1]$  as shown in Figure 2.

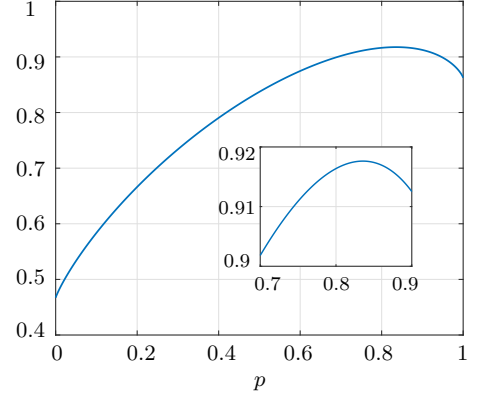


FIG. 2. Plot of the value  $D(\sqrt{\rho_R} J_{RB}^{\mathcal{E}} \sqrt{\rho_R} \|\sqrt{\rho_R} J_{RB}^{\mathcal{F}} \sqrt{\rho_R})$  with respect to the input state  $\rho_R = \text{diag}(p, 1-p)$ . The subfigure is a zoom of the large plot. It is evident that  $D(\mathcal{E}\|\mathcal{F})$  cannot be larger than 0.92.

On the other hand, if we choose the input state  $\rho_{R_1 R_2} = \text{diag}(0.8, 0, 0, 0.2)$  we have

$$\begin{aligned} D(\mathcal{E} \otimes \mathcal{E}\|\mathcal{F} \otimes \mathcal{F}) &\geq D(\sqrt{\rho_{R_1 R_2}} (J_{RB}^{\mathcal{E}})^{\otimes 2} \sqrt{\rho_{R_1 R_2}} \|\sqrt{\rho_{R_1 R_2}} (J_{RB}^{\mathcal{F}})^{\otimes 2} \sqrt{\rho_{R_1 R_2}}) \\ &= 1.9362 > 2 \times 0.92 > 2D(\mathcal{E}\|\mathcal{F}), \end{aligned}$$

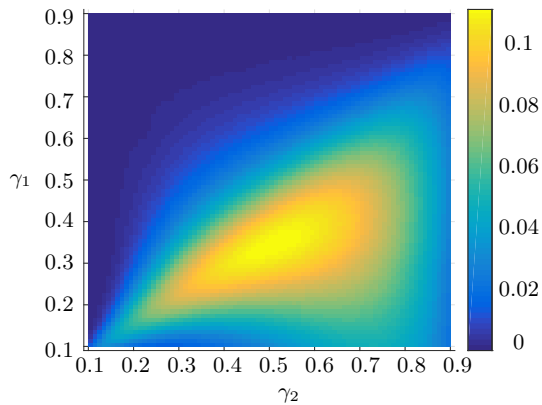


FIG. 3. A heat map of the value  $D(\mathcal{A}_{\gamma_1,0}^{\otimes 2} || \mathcal{A}_{\gamma_2,0.9}^{\otimes 2}) - 2D(\mathcal{A}_{\gamma_1,0} || \mathcal{A}_{\gamma_2,0.9})$  where  $\gamma_1, \gamma_2 \in [0.1, 0.9]$ . This shows that the non-additivity behavior of the stabilized channel relative entropy under tensor products occurs for many channels.

showing that the stabilized channel relative entropy is not additive under the tensor product.  $\square$

More generally, as shown in Figure 3, we can plot the difference  $D(\mathcal{A}_{\gamma_1,0}^{\otimes 2} || \mathcal{A}_{\gamma_2,0.9}^{\otimes 2}) - 2D(\mathcal{A}_{\gamma_1,0} || \mathcal{A}_{\gamma_2,0.9})$  for a wide range of  $\gamma_1, \gamma_2$ . Exploiting the symmetry that  $\mathcal{A}_{\gamma,\beta}$  is covariant with respect to the Pauli-Z operator and the tensor product channel  $\mathcal{A}_{\gamma,\beta}^{\otimes 2}$  is also covariant under permutation, we can restrict the computation of  $D(\mathcal{A}_{\gamma_1,0}^{\otimes 2} || \mathcal{A}_{\gamma_2,0.9}^{\otimes 2})$  to a two-parameter state  $\rho_{R_1 R_2} = \text{diag}(p_1, p_2, p_2, 1-p_1-2p_2)$  (see e.g. [16, Proposition II.4.]) and we utilize the function `quantum_rel_entr` from CVXQUAD [17]. We observe that the relative entropy is not additive for a wide range of parameters.

Proposition 1 justifies the definition of a (non-stabilized) *regularized channel relative entropy* as  $\bar{D}^{\text{reg}}(\mathcal{E} || \mathcal{F}) := \lim_{n \rightarrow \infty} \frac{1}{n} \bar{D}(\mathcal{E}^{\otimes n} || \mathcal{F}^{\otimes n})$  and similarly for the stabilized quantity. This contrasts with the relative entropy for states that is additive under the tensor product.

*Chain rule.*— The main result of this Letter ensures that (4) becomes valid if we replace the channel relative entropy term with its regularized version. More precisely, the inequality is correct whenever  $D_{\max}(\mathcal{E} || \mathcal{F}) := \max_{\phi_{RA} \in \mathcal{S}(A \otimes A)} D_{\max}(\mathcal{E}(\phi_{RA}) || \mathcal{F}(\phi_{RA}))$  is finite, where  $D_{\max}(\rho || \sigma) := \inf\{\lambda \in \mathbb{R} : \rho \leq 2^\lambda \sigma\}$  is the max-relative entropy [18, 19].

**Theorem 2** (Chain rule for relative entropy). *Let  $\rho_A, \sigma_A$  be density operators and  $\mathcal{E}, \mathcal{F}$  be trace-preserving completely positive maps such that  $D_{\max}(\mathcal{E} || \mathcal{F}) < \infty$ . Then*

$$D(\mathcal{E}(\rho_A) || \mathcal{F}(\sigma_A)) \leq D(\rho_A || \sigma_A) + \bar{D}^{\text{reg}}(\mathcal{E} || \mathcal{F}). \quad (8)$$

*In addition, in case  $A = A_1 \otimes A_2$  and  $\mathcal{F}$  is such that its output does not depend on the input on  $A_1$ , this inequality can be strengthened to*

$$D(\mathcal{E}(\rho_{A_1 A_2}) || \mathcal{F}(\sigma_{A_1 A_2})) \leq D(\rho_{A_2} || \sigma_{A_2}) + \bar{D}^{\text{reg}}(\mathcal{E} || \mathcal{F}). \quad (9)$$

Normalization properties of the relative entropy ensure that the chain rule remains valid if  $\sigma$  is not normalized and  $\mathcal{F}$  is not trace-preserving.

*Proof sketch.* The full proof can be found in the Supplemental Material which includes references [20–24]. Instead we sketch the proof idea. We start with the observation that the chain rule follows for the max-relative entropy from the triangle inequality [25, Lemma 2.1] — a property that does not hold for the relative entropy. To see this suppose  $\mathcal{F} = \mathcal{R} \circ \mathcal{G}$  for some channels  $\mathcal{R}$  and  $\mathcal{G}$ . Then the triangle inequality together with the data-processing inequality for the max-relative entropy [26] give

$$\begin{aligned} D_{\max}(\mathcal{E}(\rho) || \mathcal{F}(\sigma)) &\leq D_{\max}(\mathcal{E}(\rho) || \mathcal{F}(\rho)) + D_{\max}(\mathcal{F}(\rho) || \mathcal{F}(\sigma)) \\ &= D_{\max}(\mathcal{E}(\rho) || \mathcal{F}(\rho)) + D_{\max}(\mathcal{R} \circ \mathcal{G}(\rho) || \mathcal{R} \circ \mathcal{G}(\sigma)) \\ &\leq D_{\max}(\mathcal{G}(\rho) || \mathcal{G}(\sigma)) + D_{\max}(\mathcal{E}(\rho) || \mathcal{F}(\rho)). \end{aligned} \quad (10)$$

In case  $A = A_1 \otimes A_2$  and  $\mathcal{G}$  is the partial trace over  $A_1$  this has the form of (9). Based on that insight we prove a variant of (10), where all three terms are replaced by smooth-max relative entropies together with an additive error term that depends on the smoothing parameter. Finally we use the asymptotic equipartition property [26, 27], which ensures that for  $n$ -fold product states the smooth-max relative entropy converges to the relative entropy as  $n \rightarrow \infty$ , to obtain (9) from which (8) follows as the special case where  $A_1$  is trivial.  $\square$

In case  $\mathcal{E} = \mathcal{F}$ , inequality (8) simplifies to the data-processing inequality, i.e.,  $D(\mathcal{E}(\rho) || \mathcal{E}(\sigma)) \leq D(\rho || \sigma)$ . In this sense (8) may be viewed as a generalized data-processing inequality where not necessarily the same channel is applied to both arguments, which then is compensated with the regularized channel relative entropy term.

Note that, in the chain rule (1) the term  $H(A_2 | A_1)$  still depends on the state of  $A_1$ . However, if one instead considers the implication for any fixed  $\rho_{A_1 A_2}$

$$H(A_1 A_2)_\rho \geq H(A_1)_\rho + \min_{\nu_{A_2 | A_1} = \rho_{A_2 | A_1}} H(A_2 | A_1)_\nu, \quad (11)$$

where the minimization is over all  $\nu$  whose conditional state is identical to to the conditional state of  $\rho$ , i.e.,  $\rho_{A_2 | A_1} := \rho_{A_1}^{-1/2} \rho_{A_1 A_2} \rho_{A_1}^{-1/2}$ , the second term becomes independent of the marginal state of  $A_1$ . This is particularly desirable for an iterative version analogous to (2), as the terms in the sum then only depend on the correlations between a subsystem  $A_i$  and the rest, but not on the state of the rest. One can now see that (11) indeed can be retrieved from our relative entropy chain rule (9) by expressing the conditional von Neumann entropy in terms of the relative entropy, i.e.,  $H(A_2 | A_1) = -D(\rho_{A_1 A_2} || \rho_{A_1} \otimes \text{id}_{A_2})$  (see Supplemental Material for details).

An important corollary to Theorem 2 is when the maps are of the form  $\mathcal{I} \otimes \mathcal{E}$  and  $\mathcal{I} \otimes \mathcal{F}$ . The right hand side then simplifies to the more common (stabilized) relative entropy between channels.

**Corollary 3.** *Let  $\mathcal{E}$  be a trace-preserving completely positive map and  $\mathcal{F}$  be a completely positive map. Then*

$$D^A(\mathcal{E}\|\mathcal{F}) = D^{\text{reg}}(\mathcal{E}\|\mathcal{F}). \quad (12)$$

*Proof.* It is known [15, Theorem 3 and 6] that  $D^{\text{reg}}(\mathcal{E}\|\mathcal{F}) \leq D^A(\mathcal{E}\|\mathcal{F})$ . Theorem 2 applied to channels  $\mathcal{I}_R \otimes \mathcal{E}$  and  $\mathcal{I}_R \otimes \mathcal{F}$  shows that

$$\begin{aligned} D(\mathcal{E}(\rho_{RA})\|\mathcal{F}(\sigma_{RA})) - D(\rho_{RA}\|\sigma_{RA}) \\ \leq \bar{D}^{\text{reg}}(\mathcal{I}_R \otimes \mathcal{E}\|\mathcal{I}_R \otimes \mathcal{F}). \end{aligned}$$

To conclude it suffices to observe that for any system  $R$  we have  $\bar{D}^{\text{reg}}(\mathcal{I}_R \otimes \mathcal{E}\|\mathcal{I}_R \otimes \mathcal{F}) \leq D^{\text{reg}}(\mathcal{E}\|\mathcal{F})$ .  $\square$

Corollary 3 shows that for any trace-preserving completely positive map  $\mathcal{E}$  and any completely positive map  $\mathcal{F}$  there exist states  $\rho_{RA}$  and  $\sigma_{RA}$  such that the chain rule holds with equality.

The regularization of relative entropy term in Theorem 2 is necessary in full generality, as shown by Proposition 1. However, for channels with a specific structure their stabilized channel relative entropy is additive under the tensor product which implies that  $D^{\text{reg}}(\mathcal{E}\|\mathcal{F}) = D(\mathcal{E}\|\mathcal{F})$ . Examples of such channels are

- (i) classical-quantum channels [13, Lemma 25]
- (ii) covariant channels with respect to the unitary group [16, Corollary II.5]
- (iii)  $\mathcal{E}$  arbitrary and  $\mathcal{F}$  a replacer channel (i.e.,  $\mathcal{F}(X) = \omega \text{tr} X$  for  $\omega \in \mathcal{S}(B)$ ) [11, 13]
- (iv) environment-seizable channels [15]

We can single-letterize the chain rule from Theorem 2 by replacing the regularized channel relative entropy term with the Belavkin-Staszewski channel relative entropy defined by  $\hat{D}(\mathcal{E}\|\mathcal{F}) := \max_{\phi \in \mathcal{S}(A \otimes A)} \text{tr} \mathcal{E}(\phi) \log(\mathcal{E}(\phi)^{\frac{1}{2}} \mathcal{F}(\phi)^{-1} \mathcal{E}(\phi)^{\frac{1}{2}})$ . We note that the logarithmic trace inequality [28, 29] (see also [30, Theorem 4.6]) ensures that  $D(\mathcal{E}\|\mathcal{F}) \leq \hat{D}(\mathcal{E}\|\mathcal{F})$ . Furthermore, the Belavkin-Staszewski channel relative entropy is additive under tensor products [31, Lemma 6]. Another benefit from this relaxation is the fact that  $\hat{D}(\mathcal{A}\|\mathcal{B})$  has an explicit form and is thus efficiently computable [31, Lemma 5].

*Asymptotic quantum channel discrimination.*— A fundamental task in quantum information theory is to distinguish between two quantum channels  $\mathcal{E}, \mathcal{F}$ . For this problem one usually differentiates between two different classes of strategies:

*Non-adaptive strategies* (also called parallel strategies): Here we are given “black-box” access to  $n$  uses of a channel  $\mathcal{G}$ , which is either  $\mathcal{E}$  or  $\mathcal{F}$ , that can be used in parallel

before performing a measurement. More precisely, for an arbitrary state  $\rho_{A^n R} \in \mathcal{S}(A_1 \otimes \dots \otimes A_n \otimes R)$  with a reference system  $R$  we create the state  $\sigma_{B^n R} = \mathcal{G}^{\otimes n}(\rho_{A^n R})$  and perform a measurement on  $\sigma_{B^n R}$ . Based on the measurement outcome we try to guess if  $\mathcal{G} = \mathcal{E}$  or  $\mathcal{G} = \mathcal{F}$ . The protocol is depicted in Figure 4. It has been shown recently [15, Theorem 3] that in the asymmetric regime where we fix the type-I error to be bounded by  $\varepsilon$ , the asymptotic optimal rate of the type-II error exponent is given by  $D^{\text{reg}}(\mathcal{E}\|\mathcal{F})$ , when  $\varepsilon$  goes to 0. A type-I error is the rejection of a true null-hypothesis while a type-II error is the non-rejection of a false null-hypothesis.

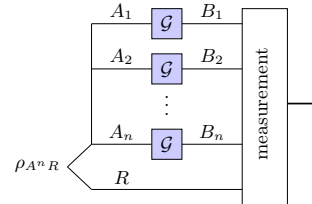


FIG. 4. General protocol for non-adaptive strategies for the task of channel discrimination. The channel  $\mathcal{G}$  is either  $\mathcal{E}$  or  $\mathcal{F}$  and the task is to distinguish between these two cases.

*Adaptive strategies* (also called sequential strategies): Here we are also given “black-box” access to  $n$  uses of a channel  $\mathcal{G}$  which is either  $\mathcal{E}$  or  $\mathcal{F}$ . However unlike in the non-adaptive scenario, after each use of a channel we are allowed to perform an adaptive trace-preserving completely positive map  $\mathcal{N}_k$  before we perform a measurement at the end. More precisely, for an arbitrary state  $\rho_{AR_0}^{(0)} \in \mathcal{S}(A \otimes R_0)$  we create  $\rho_{AR_k}^{(k)} = (\mathcal{N}_k \circ \mathcal{G})(\rho_{AR_{k-1}}^{(k-1)})$  for  $k = 1, \dots, n$ . Finally we perform a measurement on  $\rho_{AR_n}^{(n)}$  and based on the outcome try to guess if  $\mathcal{G} = \mathcal{E}$  or  $\mathcal{G} = \mathcal{F}$ . The strategy is depicted in Figure 5. The asymptotically optimal rate of the type-II error exponent for this strategy is given by  $D^A(\mathcal{E}\|\mathcal{F})$  [15, Theorem 6].

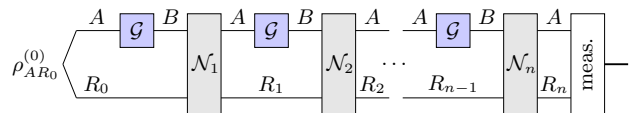


FIG. 5. General protocol for adaptive strategies for the task of channel discrimination.

Because a non-adaptive strategy can be viewed as a particular instance of an adaptive strategy [32] it follows that adaptive strategies are clearly as powerful as non-adaptive ones, which in technical terms means

$$D^{\text{reg}}(\mathcal{E}\|\mathcal{F}) \leq D^A(\mathcal{E}\|\mathcal{F}).$$

It has been an open question if adaptive strategies can be more powerful for the task of asymptotic quantum channel discrimination [13, 15, 33, 34]. For some special classes of channels, such as classical and classical-quantum channels it has been shown that adaptive pro-



protocols cannot improve the error rate for asymmetric channel discrimination [13, 33]. Corollary 3 now proves that this is the case for all quantum channels because

$$D^{\text{reg}}(\mathcal{E}||\mathcal{F}) = D^A(\mathcal{E}||\mathcal{F}).$$

We note that this is surprising for various reasons. In the symmetric Chernoff setting [35–37] adaptive protocols offer an advantage over non-adaptive ones. Furthermore, in the non-asymptotic setting adaptive protocols also outperform non-adaptive strategies [35, 36, 38]. Apart from its fundamental importance the channel discrimination problem features immediate applications in various areas ranging from quantum metrology [39] to the study of resource theories [40].

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- [1] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, 2013).
- [2] M. B. Hastings, *Journal of Statistical Mechanics: Theory and Experiment* **2007**, P08024 (2007).
- [3] F. G. Brandão and M. Horodecki, *Nature Physics* **9**, 721 (2013).
- [4] A. Kitaev and J. Preskill, *Physical Review Letters* **96**, 110404 (2006).
- [5] M. Christandl and A. Winter, *Journal of Mathematical Physics* **45**, 829 (2004).
- [6] V. Vedral, *Reviews on Modern Physics* **74**, 197 (2002).
- [7] F. Hiai and D. Petz, *Communications in Mathematical Physics* **143**, 99 (1991).
- [8] M. Berta and C. Majenz, *Physical Review Letters* **121**, 190503 (2018).
- [9] A. Anshu, M.-H. Hsieh, and R. Jain, *Physical Review Letters* **121**, 190504 (2018).
- [10] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley Interscience, 2006).
- [11] T. Cooney, M. Mosonyi, and M. M. Wilde, *Communications in Mathematical Physics* **344**, 797 (2016).
- [12] One may also ask whether a converse of (4), i.e., an inequality of the form  $D(\mathcal{E}(\rho_{RA})||\mathcal{F}(\sigma_{RA})) \geq D(\rho_{RA}||\sigma_{RA}) + f(\mathcal{E}, \mathcal{F})$ , for some appropriately chosen  $f$ , could hold. It follows however from the data processing inequality that  $f(\mathcal{E}, \mathcal{E})$  cannot be positive, and hence cannot be a relative entropy.
- [13] M. Berta, C. Hirche, E. Kaur, and M. M. Wilde, “Amortized channel divergence for asymptotic quantum channel discrimination,” (2018), [arXiv:1808.01498](https://arxiv.org/abs/1808.01498).
- [14] We note that this is in contrast to the stabilized channel relative entropy, where the reference system  $R$  can be assumed to be isomorphic to the input system  $A$ . It is unclear if this assumption can be made for the amortized channel relative entropy.
- [15] X. Wang and M. M. Wilde, “Resource theory of asymmetric distinguishability for quantum channels,” (2019), [arXiv:1907.06306](https://arxiv.org/abs/1907.06306).
- [16] F. Leditzky, E. Kaur, N. Datta, and M. M. Wilde, *Physical Review A* **97**, 012332 (2018).
- [17] H. Fawzi, J. Saunderson, and P. A. Parrilo, *Foundations of Computational Mathematics* **19**, 259 (2019), package [cvxquad](https://github.com/hfawzi/cvxquad) at <https://github.com/hfawzi/cvxquad>.
- [18] R. Renner, PhD thesis, ETH Zurich (2005), [arXiv:0512258](https://arxiv.org/abs/0512258).
- [19] N. Datta, *IEEE Transactions on Information Theory* **55**, 2816 (2009).
- [20] R. Bhatia, *Matrix Analysis* (Springer, 1997).
- [21] O. Fawzi and R. Renner, *Communications in Mathematical Physics* **340**, 575 (2015).
- [22] M. Tomamichel, PhD thesis, ETH Zurich (2012), [arXiv:1203.2142](https://arxiv.org/abs/1203.2142).
- [23] M. Tomamichel, R. Colbeck, and R. Renner, *IEEE Transactions on Information Theory* **56**, 4674 (2010).
- [24] M. Tomamichel and M. Hayashi, *IEEE Transactions on Information Theory* **59**, 7693 (2013).
- [25] D. Sutter and R. Renner, *Annales Henri Poincaré* **19**, 3007 (2018).
- [26] M. Tomamichel, *Quantum Information Processing with Finite Resources*, SpringerBriefs in Mathematical Physics, Vol. 5 (Springer, 2015).
- [27] M. Tomamichel, R. Colbeck, and R. Renner, *IEEE Transactions on Information Theory* **55**, 5840 (2009).
- [28] F. Hiai and D. Petz, *Linear Algebra and its Applications* **181**, 153 (1993).
- [29] T. Ando and F. Hiai, *Linear Algebra and its Applications* **197**, 113 (1994).
- [30] D. Sutter, *Approximate Quantum Markov Chains*, SpringerBriefs in Mathematical Physics, Vol. 28 (Springer, 2018).
- [31] K. Fang and H. Fawzi, “Geometric Rényi divergence and its applications in quantum channel capacities,” (2019), [arXiv:1909.05758](https://arxiv.org/abs/1909.05758).
- [32] G. Chiribella, G. M. D’Ariano, and P. Perinotti, *Physical Review Letters* **101**, 180501 (2008).
- [33] M. Hayashi, *IEEE Transactions on Information Theory* **55**, 3807 (2009).
- [34] M. M. Wilde, “Open problems session at the Banff workshop about algebraic and statistical ways into quantum resource theories,” (2019), <https://www.birs.ca/workshops/2019/19w5120/files/19w5120-OpenProblems-20190723.mp4>.
- [35] R. Duan, Y. Feng, and M. Ying, *Physical Review Letters* **103**, 210501 (2009).

- [36] A. W. Harrow, A. Hassidim, D. W. Leung, and J. Watrous, [Physical Review A \*\*81\*\*, 032339 \(2010\)](#).
- [37] R. Duan, C. Guo, C. Li, and Y. Li, in [2016 IEEE International Symposium on Information Theory \(ISIT\)](#) (2016) pp. 2259–2263.
- [38] D. Puzzuoli and J. Watrous, [Annales Henri Poincaré \*\*18\*\*, 1153 \(2017\)](#).
- [39] S. Pirandola, R. Laurenza, C. Lupo, and J. L. Pereira, [npj Quantum Information \*\*5\*\* \(2019\), 10.1038/s41534-019-0162-y](#).
- [40] R. Takagi and B. Regula, [Physical Review X \*\*9\*\*, 031053 \(2019\)](#).