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# Topological Origin of Non-Hermitian Skin Effects

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A unique feature of non-Hermitian systems is the skin effect, which is the extreme sensitivity to the boundary conditions. Here, we reveal that the skin effect originates from intrinsic non-Hermitian topology. Such a topological origin not merely explains the universal feature of the known skin effect, but also leads to new types of the skin effects — symmetry-protected skin effects. In particular, we discover the  $\mathbb{Z}_2$  skin effect protected by time-reversal symmetry. On the basis of topological classification, we also discuss possible other skin effects in arbitrary dimensions. Our work provides a unified understanding about the bulk-boundary correspondence and the skin effects in non-Hermitian systems.

Recently, non-Hermitian Hamiltonians [1–7] have been extensively studied in open classical [8–14] and quantum [15–20] systems as well as disordered or correlated solids with finite-lifetime quasiparticles [21–27]. In particular, much research has focused on distinctive characteristics of non-Hermitian topological phases [28–60]. The rich non-Hermitian topology is attributed to the complex-valued nature of the spectrum, which enables two types of complex-energy gaps [56]: line gap and point gap. Since a non-Hermitian Hamiltonian with a line gap is continuously deformed to a Hermitian one without closing the line gap [56], topology for a line gap describes the persistence of conventional topological phases against non-Hermitian perturbations, which is relevant to topological lasers [40–44], for example. On the other hand, a non-Hermitian Hamiltonian with a point gap is allowed to be deformed to a unitary one [46, 56]. As a result, point-gapped topological phases cannot always be continuously deformed into any Hermitian counterparts; topology for a point gap is intrinsic to non-Hermitian systems in sharp contrast to a line gap. A point gap describes unique non-Hermitian topological phenomena such as localization transitions [1, 2, 46, 52, 58] and emergence of exceptional points [21–26, 34, 37, 49, 50, 53, 55].

A hallmark of topological phases is the presence of the localized states at the boundaries as a result of non-trivial topology of the bulk [61–63]. Remarkably, non-Hermiticity alters the nature of the bulk-boundary correspondence (BBC) [64–92]. The critical distinction is the extreme sensitivity of the bulk to the boundary conditions, which is called the non-Hermitian skin effect [68]. It accompanies the localization of bulk eigenstates as well as the dramatic difference of bulk spectra according to the boundary conditions, which forces us to redefine the bulk topology so as to be suitable for the open boundary condition [67, 68, 78, 80]. The BBC persists in the presence of a line gap since non-Hermitian Hamiltonians with a line gap can be continuously deformed to Hermitian ones. However, the BBC for a point gap has still remained unclear. Since a point gap describes intrinsic non-Hermitian topology, the nature of the BBC may be

disparate from the Hermitian counterpart. In fact, even when a point gap is open under the periodic boundary condition, it can be close under the open boundary condition [46, 66, 76]. Thus, the non-Hermitian skin effect obscures point-gap topology.

This Letter provides a unified understanding about the BBC and the skin effect in non-Hermitian systems. We show that the BBC holds even for a point gap in semi-infinite systems with only one boundary. In finite systems with open boundaries, by contrast, we demonstrate that the point-gap topology inevitably induces the non-Hermitian skin effect and results in the absence of topologically protected boundary states due to a point gap. On the basis of such a topological origin, new types of the skin effects are revealed, including the  $\mathbb{Z}_2$  skin effect protected by time-reversal symmetry. We also elucidate the relationship between point and line gaps for the BBC.

*Bulk-boundary correspondence in semi-infinite systems.* — A non-Hermitian Hamiltonian  $H$  is defined to have a point gap if and only if its complex spectrum does not cross a reference point  $E \in \mathbb{C}$ , i.e.,  $\det(H - E) \neq 0$  [46, 56]. The simplest nontrivial example of the point-gapped topological phases appears in one-dimensional systems with no symmetry. Whereas  $\det(H - E)$  is always real for Hermitian  $H$ , it can be complex for non-Hermitian  $H$ , by which the following winding number  $W(E) \in \mathbb{Z}$  is defined:

$$W(E) := \int_0^{2\pi} \frac{dk}{2\pi i} \frac{d}{dk} \log \det(H(k) - E), \quad (1)$$

where  $H(k)$  is the non-Hermitian Bloch Hamiltonian in momentum space with the finite number of bands ( $k \in [0, 2\pi]$ ). Topological phases are absent in one-dimensional Hermitian systems without symmetry protection [61–63]; the point-gap topology characterized by  $W(E)$  is intrinsic to non-Hermitian systems.

Corresponding to  $W(E) \neq 0$ , the boundary modes with the eigenenergy  $E$  can appear in semi-infinite systems with only one boundary. Suppose the non-Hermitian system has a boundary on the left but no boundary on the right (the same semi-infinite boundary

condition is chosen below unless otherwise stated). An important observation is that the Hermitian Hamiltonian  $\tilde{H}$  is obtained by [46, 56]

$$\tilde{H} := \begin{pmatrix} 0 & H - E \\ H^\dagger - E^* & 0 \end{pmatrix}. \quad (2)$$

Under the periodic boundary condition, when a point gap is open for non-Hermitian  $H(k)$ , a real energy gap is also open for Hermitian  $\tilde{H}(k)$ , and vice versa. In addition,  $\tilde{H}$  respects additional chiral symmetry by construction:  $\Gamma \tilde{H} \Gamma^{-1} = -\tilde{H}$  with  $\Gamma := \sigma_z$ . As a result of the conventional BBC for Hermitian Hamiltonians,  $\tilde{H}$  with the semi-infinite boundary possesses topologically protected zero modes localized at the boundary [61–63] in a similar manner to the Su-Schrieffer-Heeger model [93]. The corresponding topological invariant coincides with  $W(E)$  in Eq. (1). For  $W(E) < 0$ , there appear boundary modes  $(0 |E\rangle)^T$  with negative chirality [i.e.,  $\Gamma(0 |E\rangle)^T = -(0 |E\rangle)^T$ ], which implies that  $|E\rangle$  is a right eigenstate of non-Hermitian  $H$  (i.e.,  $H|E\rangle = E|E\rangle$ ) localized at the boundary. For  $W(E) > 0$ , on the other hand, the boundary modes  $(|E\rangle 0)^T$  have positive chirality [i.e.,  $\Gamma(|E\rangle 0)^T = +(|E\rangle 0)^T$ ], which in turn implies that  $|E\rangle$  is a right eigenstate of  $H^\dagger$ , i.e., a left eigenstate of  $H$  (i.e.,  $\langle E|H = \langle E|E$ ) [94].

The above discussion is valid for arbitrary  $E \in \mathbb{C}$  in the complex-energy plane satisfying  $W(E) \neq 0$ . Thus, in semi-infinite systems  $H_{\text{SIBC}}$ , the infinite number of boundary modes with eigenenergies  $E$  emerges as a result of the nontrivial winding number  $W(E) \neq 0$ . This conclusion leads to the following theorem (index theorem in spectral theory [95–100]):

**Theorem I** Let  $\sigma(H(k))$  be the spectrum of  $H(k)$  with  $k \in [0, 2\pi]$ , which forms closed curves in the complex-energy plane (Fig. 1). Then, the spectrum of semi-infinite  $H_{\text{SIBC}}$  with only one boundary is equal to  $\sigma(H(k))$  together with the whole area of  $E \in \mathbb{C}$  enclosed by  $\sigma(H(k))$  with  $W(E) \neq 0$ . For  $W(E) < 0$  ( $W(E) > 0$ ),  $|E\rangle$  is a right (left) eigenstate of  $H_{\text{SIBC}}$  localized at the boundary [i.e.,  $H_{\text{SIBC}}|E\rangle = E|E\rangle$  ( $\langle E|H_{\text{SIBC}} = \langle E|E$ )].

Theorem I is illustrated with the Hatano-Nelson model [1, 2] without disorder, which is given by

$$H^{(\text{HN})} := \sum_i \left[ (t+g) c_{i+1}^\dagger c_i + (t-g) c_i^\dagger c_{i+1} \right] \quad (3)$$

with  $t > 0$  and  $g \in \mathbb{R}$ . The spectrum of the Bloch Hamiltonian  $H^{(\text{HN})}(k) = (t+g)e^{ik} + (t-g)e^{-ik}$  forms an ellipse in the complex-energy plane, and we have  $W(E) = \text{sgn}(g)$  for  $E \in \mathbb{C}$  inside this ellipse. In fact, the hopping from right to left dominates that from left to right for  $g < 0$ , which leads to the emergence of the boundary modes [100].

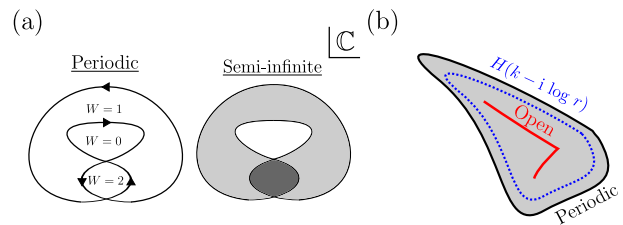


FIG. 1. Complex spectra of non-Hermitian systems with periodic, open, and semi-infinite boundaries. (a) A semi-infinite system possesses the infinite number of boundary modes due to the nonzero winding number  $W \neq 0$  in the corresponding periodic system. (b) The spectrum of a semi-infinite system shrinks through the imaginary gauge transformation, resulting in an arc of the open-boundary system.

*Skin effect as point-gap topology.* — The above discussion breaks down in finite systems with open boundaries. In fact, the infinite number of boundary modes is impossible in finite systems. Furthermore, the additional boundary condition is imposed because of the other boundary, which may forbid some of the boundary states appearing in semi-infinite systems. For example, the spectrum of the Hatano-Nelson model  $H_{\text{OBC}}^{(\text{HN})}$  with open boundaries forms not a loop but a line on the real axis in the complex-energy plane, which signals the non-Hermitian skin effect. In fact, using an imaginary gauge transformation [1, 2, 68, 76]

$$V_r^{-1} c_i^\dagger V_r = r^i c_i^\dagger, \quad V_r^{-1} c_i V_r = r^{-i} c_i, \quad (0 < r < \infty) \quad (4)$$

we have a Hermitian Hamiltonian  $\bar{H} := V_r^{-1} H_{\text{OBC}}^{(\text{HN})} V_r$  for  $r := \sqrt{|(t-g)/(t+g)|}$ . Here, Eq. (4) shifts the momentum from  $k$  to  $k - i \log r$ . Since this similarity transformation does not change the spectrum,  $H_{\text{OBC}}^{(\text{HN})}$  has the entirely real spectrum and hence no longer retains the point gap. Saliiently, such a non-Hermitian skin effect is a general non-Hermitian topological phenomenon as a direct consequence of point-gap topology, as summarized in the following theorem:

**Theorem II** Finite  $H_{\text{OBC}}$  with open boundaries is always topologically trivial in terms of a point gap. Consequently, if  $H(k)$  under the periodic boundary condition is point-gapped topological, the non-Hermitian skin effect inevitably occurs with a topological phase transition.

To see this theorem, we begin with

$$\lim_{N \rightarrow \infty} \sigma(H_{\text{OBC}}) \subset \sigma(H_{\text{SIBC}}), \quad (5)$$

where  $\sigma(H_{\text{OBC}})$  is the spectrum of a non-Hermitian system  $H_{\text{OBC}}$  with open boundaries and  $N$  unit cells, and  $\sigma(H_{\text{SIBC}})$  is the spectrum of the corresponding semi-infinite system  $H_{\text{SIBC}}$ . In fact, an approximate eigenstate

of  $H_{\text{SIBC}}$  can be obtained from an eigenstate of  $H_{\text{OBC}}$ , which becomes an exact eigenstate for  $N \rightarrow \infty$  [100]. The contrary is not always true: even if an approximate eigenstate of  $H_{\text{OBC}}$  is constructed from an eigenstate of  $H_{\text{SIBC}}$ , it is not necessarily an exact eigenstate of  $H_{\text{OBC}}$ .

A crucial step is again the imaginary gauge transformation:  $H_{\text{OBC}} \rightarrow V_r^{-1} H_{\text{OBC}} V_r$  and  $H_{\text{SIBC}} \rightarrow V_r^{-1} H_{\text{SIBC}} V_r$  with  $r \in (0, \infty)$ . For each transformation, we still have the inclusion in Eq. (5):

$$\lim_{N \rightarrow \infty} \sigma(V_r^{-1} H_{\text{OBC}} V_r) \subset \sigma(V_r^{-1} H_{\text{SIBC}} V_r). \quad (6)$$

This imaginary gauge transformation does not change the spectrum of  $H_{\text{OBC}}$ . However, it changes the spectrum of  $H_{\text{SIBC}}$  since  $H_{\text{SIBC}}$  has no boundary on the right because of the semi-infinite nature [Fig. 1(b)]. In fact,  $H(k)$  changes to  $H(k - i \log r)$  through  $V_r$ . Nevertheless, Eq. (6) implies that the transformed semi-infinite spectrum includes the spectrum of  $H_{\text{OBC}}$  for any transformation  $V_r$ . Thus, we have

$$\lim_{N \rightarrow \infty} \sigma(H_{\text{OBC}}) \subset \bigcap_{r \in (0, \infty)} \sigma(V_r^{-1} H_{\text{SIBC}} V_r). \quad (7)$$

Because of Theorem I, when  $H(k)$  has a point gap and  $W(E) < 0$  ( $W(E) > 0$ ), right (left) boundary modes with eigenenergy  $E$  appear in the semi-infinite system. Let us choose an appropriate imaginary gauge  $V_r$  such that these boundary modes are transformed to delocalized bulk modes. Then,  $E$  is on the edges of  $\sigma(V_r^{-1} H_{\text{SIBC}} V_r)$ , whereas it is originally located inside  $\sigma(H_{\text{SIBC}})$ . Thus, the intersection of  $\sigma(H_{\text{SIBC}})$  and  $\sigma(V_r^{-1} H_{\text{SIBC}} V_r)$  is strictly smaller than  $\sigma(H_{\text{SIBC}})$  [100]. Repeating this procedure for all  $V_r$  with  $r \in (0, \infty)$ , the right-hand side of Eq. (7) reaches an open curve or a topologically trivial area of which interior satisfies  $W(E) = 0$ , otherwise a contradiction arises [100]. Since this region includes  $\lim_{N \rightarrow \infty} \sigma(H_{\text{OBC}})$  because of Eq. (7),  $H_{\text{OBC}}$  is also topologically trivial and different from  $H(k)$  with nontrivial topology. Furthermore,  $\sigma(H_{\text{OBC}})$  is indeed distinct from  $\sigma(H(k))$ , which implies the inevitable occurrence of the non-Hermitian skin effect due to the point-gap topology.

Remarkably, Refs. [67, 68, 78, 80] determine the conditions for the spectra of open-boundary systems and develop the non-Bloch band theory of non-Hermitian systems. Their conditions are actually equivalent to the set in the right-hand side of Eq. (7) [100]. An observation similar to our Theorem II is also made in Ref. [76], which is made rigorous by our results. Moreover, we identify the non-Hermitian skin effect as the point-gap topology [101]. Such a topological origin constitutes a universal feature of the non-Hermitian skin effect. Furthermore, new types of the skin effects — symmetry-protected skin effects — are discovered, as illustrated below.

$\mathbb{Z}_2$  non-Hermitian skin effect. — The point-gap topology and the corresponding skin effect are enriched

by symmetry. Here, we consider time-reversal symmetry defined in terms of transposition [56]:

$$T H^T(k) T^{-1} = H(-k), \quad T T^* = -1, \quad (8)$$

where  $T$  is a unitary operator. This symmetry is fundamental as reciprocity in non-Hermitian spinful systems and naturally appears, for example, in mesoscopic systems [102, 103] and open quantum systems [104–106].

In conventional quantum spin Hall insulators, the integer Chern number vanishes but the Kane-Mele  $\mathbb{Z}_2$  one becomes nontrivial because of time-reversal symmetry [61–63]. Similarly, Eq. (8) trivializes the winding number in Eq. (1), but instead, it supplies a  $\mathbb{Z}_2$  invariant. The  $\mathbb{Z}_2$  topological invariant  $\nu(E) \in \{0, 1\}$  for a reference point  $E \in \mathbb{C}$  is given by [56]

$$(-1)^{\nu(E)} := \text{sgn} \left\{ \frac{\text{Pf}[(H(\pi) - E)T]}{\text{Pf}[(H(0) - E)T]} \times \exp \left[ -\frac{1}{2} \int_{k=0}^{k=\pi} d \log \det [(H(k) - E)T] \right] \right\}. \quad (9)$$

Corresponding to the  $\mathbb{Z}_2$  topological invariant  $\nu(E)$ , we have an index theorem similar to Theorem I for semi-infinite systems [100]. A clear distinction from Theorem I is the Kramers degeneracy due to Eq. (8) [30, 51, 56]. The extended Hermitian Hamiltonian  $\tilde{H}$  in Eq. (2) respects time-reversal as well as the additional chiral symmetry  $\Gamma$ , analogous to time-reversal-invariant topological superconductors [107–110]. The index theorem states that the semi-infinite system  $\tilde{H}$  hosts an odd number of boundary Majorana Kramers pairs for each  $E$  with  $\nu(E) = 1$ . In terms of the original non-Hermitian Hamiltonian  $H$ , the Kramers pair reduces to a pair of right and left eigenstates of  $H$  localized at the same boundary. Using the transposition version of time reversal in Eq. (8), we can convert the left eigenmode into a right one in the oppositely extended semi-infinite system (i.e., semi-infinite system with a boundary only on the right). As a result, finite systems with open boundaries host localized modes at both ends, as explicitly shown in the following model.

We recall that a quantum spin Hall insulator [111, 112] can be constructed from a pair of time-reversed quantum Hall insulators [113] with the spin-orbit coupling. Similarly, combining the Hatano-Nelson model  $H^{(\text{HN})}(k)$  in Eq. (3) and its time-reversed partner  $(H^{(\text{HN})})^T(-k)$ , we have a canonical model that exhibits the  $\mathbb{Z}_2$  skin effect:

$$H(k) = \begin{pmatrix} H^{(\text{HN})}(k) & 2\Delta \sin k \\ 2\Delta \sin k & (H^{(\text{HN})})^T(-k) \end{pmatrix} \\ = 2t \cos k + 2\Delta (\sin k) \sigma_x + 2ig (\sin k) \sigma_z, \quad (10)$$

with  $t, g, \Delta \geq 0$ . It indeed respects time-reversal symmetry with  $T = i\sigma_y$ , and its spectrum is given as  $E_{\pm}(k) = 2t \cos k \pm 2i\sqrt{g^2 - \Delta^2} \sin k$ . Thus,  $H(k)$  for  $g > \Delta$  retains a point gap. Since it can be continuously deformed to  $H(k)$  with  $t = g, \Delta = 0$  while keeping

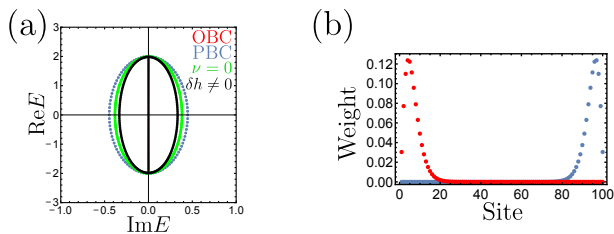


FIG. 2.  $\mathbb{Z}_2$  non-Hermitian skin effect. (a) Energy spectra of the non-Hermitian Hamiltonian in Eq. (10) under the various boundary conditions ( $t = 1$ ,  $g = 0.3$ ,  $\Delta = 0.2$ ,  $\delta = (1, 1, 1) \times 10^{-2}$ ,  $\delta h = 10^{-3}$ ,  $N = 100$ ). (b) Kramers doublet with  $E = 1.948$ , one of which is localized at the left boundary and the other of which at the right boundary.

the point gap, the  $\mathbb{Z}_2$  invariant in Eq. (9) is obtained as  $\nu(E) = 1$  when  $E$  is in the area enclosed by  $\sigma(H(k))$ .

The spectrum of Eq. (10) is shown in Fig. 2(a). The open-boundary spectrum is clearly different from the periodic-boundary counterpart, which indicates the non-Hermitian skin effect. Each complex eigenenergy consists of a Kramers pair, one of which is localized at the left boundary and the other of which at the right boundary [Fig. 2(b)]. Because of the  $\mathbb{Z}_2$  nature, the point-gap topology becomes trivial and no skin effect occurs if the two nontrivial systems are stacked. Figure 2(a) also shows the spectrum of such a stacked system

$$H^{\text{stack}}(k) = \begin{pmatrix} H(k) & i\delta \cdot \sigma \\ -i\delta \cdot \sigma & H(k) \end{pmatrix}, \quad (11)$$

where the off-diagonal terms are symmetry-preserving couplings. Consistently, the non-Hermitian skin effect no longer survives.

Since the  $\mathbb{Z}_2$  skin effect is topologically protected by time-reversal symmetry, it breaks down by a symmetry-breaking perturbation including  $(\delta h)\sigma_z$  [Fig. 2(a)]. In particular, such a local perturbation, which does not connect the ends, may be infinitesimal for the breakdown of the skin effect [81]. This local infinitesimal instability is unique to symmetry-protected non-Hermitian skin effects.

*Bulk-boundary correspondence in finite systems.*—

General theories on the BBC in non-Hermitian systems have recently been developed [67, 68, 78, 80]. These theories implicitly consider non-Hermitian topology for a line gap [56]. A non-Hermitian Hamiltonian  $H$  is defined to have a line gap if and only if its spectrum does not cross a reference line in the complex-energy plane. The modified BBC persists because a non-Hermitian Hamiltonian with a line gap can be continuously deformed to a Hermitian one [56]. On the other hand, we develop a theory of the BBC for a point gap, which complements Refs. [67, 68, 78, 80].

A prototypical example is a non-Hermitian extension

of the Su-Schrieffer-Heeger model [93] with asymmetric hopping [64, 67, 68, 80, 100]. It exhibits the skin effect under the open boundary condition due to the point-gap topology characterized by Eq. (1) under the periodic boundary condition [100]. Still, a line gap can be open and the corresponding topological invariant protected by sublattice symmetry can be well defined under the open boundary condition. As a result, topologically protected zero modes can emerge because of this line-gap topology.

Importantly, point and line gaps are not necessarily independent of each other. In fact, if a line gap is open, a point gap is also open with a reference point on the reference line. Hence, a reminiscence of line-gap topology may survive in the presence of a point gap even if the line gap is closed. A prime example includes non-Hermitian superconductors in one dimension without time-reversal symmetry. In this case, particle-hole symmetry

$$CH^T(k)C^{-1} = H(-k), \quad CC^* = +1 \quad (12)$$

makes zero energy a special point in the complex-energy plane in contrast to time-reversal symmetry. As a result, non-Hermitian systems have the  $\mathbb{Z}_2$  topological phases for both point and line gaps, and their topological invariants coincide with each other [56]. The Majorana zero modes in Hermitian topological superconductors survive as long as the point gap at  $E = 0$  is open. Correspondingly, an index theorem states the emergence of the zero modes localized at the boundary [100]. A concrete model of such a non-Hermitian  $s$ -wave topological superconductor is provided in Ref. [81]. To characterize this type of point-gap topology in a general manner, Refs. [100, 114] classify the homomorphisms from line-gap topology to point-gap topology for all the 38-fold internal symmetry class in arbitrary spatial dimensions.

*Higher-dimensional skin effects.*— By contrast, point-gap topology can be nontrivial even if line-gap topology is trivial. For example, whereas line-gap topology is absent in one dimension with and without time-reversal symmetry [56], the point-gap topology characterized by Eqs. (1) and (9) is present. As shown in this Letter, such intrinsic point-gap topology in finite systems leads to not the BBC but the skin effect. References [100, 114] also classify the non-Hermitian topology unique to a point gap. This classification allows us to know possible types of symmetry-protected skin effects for general symmetry classes and arbitrary dimensions. Like surface Dirac fermions in topological insulators, higher-dimensional skin modes appear in any boundary of the system under a proper boundary condition [115].

For example, a two-dimensional variant of the  $\mathbb{Z}_2$  skin effect is investigated in Ref. [100]. There, skin modes coexist with bulk modes under the open boundary condition in one direction and the periodic boundary condition in the other direction, which is the “proper boundary condition” in this system. Remarkably, only  $\mathcal{O}(L)$  skin

modes appear from all the  $\mathcal{O}(L^2)$  modes in this model ( $L$  denotes the length in one direction), which is unfeasible for the skin effects in one dimension.

*Discussion.* — The non-Hermitian skin effect has recently been observed in electrical circuits [89, 92], a mechanical metamaterial [90], and quantum walk [91], all of which we identify are intrinsic non-Hermitian topological phenomena. Beyond the observed one, this Letter predicts novel types of skin effects enabled by symmetry protection. It merits further research to investigate a variety of symmetry-protected non-Hermitian skin effects and their new physics.

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*Note added.* — After completion of this work, we became aware of a recent related work [116].

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