Critical Collapse of a Scalar Field in Semiclassical Loop Quantum Gravity

Florencia Benítez, Rodolfo Gambini, Luis Lehner, Steve Liebling, and Jorge Pullin

Phys. Rev. Lett. **124**, 071301 — Published 21 February 2020

DOI: 10.1103/PhysRevLett.124.071301

Critical collapse of a scalar field in semiclassical loop quantum gravity

Florencia Benítez[1], Rodolfo Gambini[1], Luis Lehner[2], Steve Liebling[3], Jorge Pullin[4]
1. Instituto de Física, Facultad de Ciencias, Iguá 4225, esq. Mataojo, 11400 Montevideo, Uruguay.
2. Perimeter Institute, 31 Caroline St, Waterloo, ON N2L 2Y5, Canada.
3. Long Island University, Brookville, NY 11548.
4. Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001

We study the collapse in spherical symmetry of a massless scalar field minimally coupled to gravity using the semiclassical equations that are expected from loop quantum gravity. We find critical behavior of the mass as a function of the parameters of the initial data similar to that found by Choptuik in classical general relativity for a large set of initial data and values of the polymerization parameter. Contrary to wide expectations for quantum gravity, our semiclassical field equations have an exact scale invariance, as do the classical field equations. As one would then expect, we numerically find that the phase transition is second order, again as in the classical case.

Choptuik [1] studied numerically the collapse of a massless, minimally coupled, scalar field coupled to general relativity. For a one parameter family of initial data he noted that there exists a critical value of the parameter. Below it, the scalar field disperses to infinity. Above it, a black hole forms through a second order phase transition. The dependence of the final mass of the black hole on the parameter of the initial data has a universal form $M_{\rm BH} \sim (p-p_*)^{\beta}$ where p_* is the critical value and $\beta \sim 0.37$ is a universal exponent, independent of the choice of parameter and initial data, provided p_* is non-vanishing. This critical behavior and universal scaling has been observed for several other systems (see [2] for a review). While it seemed likely that the transition would be second order as there was no natural length scale in the problem, before these numerical studies the order of the phase transition was unsettled [3]. This opens the question of how things could change in a quantum treatment of the collapse. Quantum gravity has a natural length scale, the Planck length. Indeed, previous studies of polymerized dynamics of metric general relativity seemed to suggest that the transition becomes first order [4]. Even today, a complete quantum treatment of the problem is not available.

Here we study the critical collapse of massless scalar fields, minimally coupled to the semi-classical equations that stem from loop quantum gravity with spherical symmetry [5]. In it, the classical variables for gravity are given by a the spherical remnants of the triads in the radial and transverse directions E^x and E^φ and their canonically conjugate momenta K_x and K_φ. The metric of space-time can be written as,

$$ds^2 = -\alpha^2 dt^2 + \Lambda^2 dr^2 + R^2 d\Omega^2 \tag{1}$$

and the relation to the loop quantum gravity triads are $\Lambda = E^\varphi/r$, $R^2 = |E^x|$ and to the extrinsic curvatures $K_{xx} = -\text{sign}(E^x)\,(E^\varphi)^2\, K_x/\sqrt{|E^x|}$ and $K_{\theta\theta} = -\sqrt{|E^x|}A_\varphi/(2\gamma)$ with γ the Immirzi parameter.

To try to stay as close to Choptuik's treatment as possible, we choose coordinates such that $E^x = r^2$. This corresponds to the usual Schwarzschild radial coordinate and eliminates K_x through the diffeomorphism constraint (one of the Einstein equations). His "polar condition" ($K^r_r = \text{Tr}(K)$) corresponds in these variables to $K_\varphi = 0$, which makes the metric diagonal. This has the unexpected effect of making the gravitational part of the semiclassical equations reduce to the classical form (both K_φ and K_x drop out from the equations and these would be the variables that would get polymerized in the semi-classical theory). The only effect of the loop quantization is in the polymerization of the scalar variables. The need to consider polymeric representations for scalar fields in loop quantum gravity was first pointed out by Thiemann [6] as a need to deal with diffeomorphism invariance and have a well defined measure in the space of matter fields. It might be possible that in a more complete treatment using different coordinates, effects from polymerization of the gravitational variables could potentially produce somewhat different results than those of this paper.

The system of classical equations (for $G = c = 1$) is [7],

$$\frac{\alpha'}{\alpha} - \frac{(E^\varphi)'}{E^\varphi} + \frac{2}{r} - \frac{(E^\varphi)^2}{r^3} = 0 \tag{2}$$

$$\frac{(E^\varphi)'}{E^\varphi} - \frac{3}{2r} + \frac{(E^\varphi)^2}{2r^3} - 2\pi r\left(\Pi^2 + \Phi^2\right) = 0 \tag{3}$$

$$\dot\Phi = \left(\frac{\alpha r}{E^\varphi}\Pi\right)' \tag{4}$$

$$\dot{\Pi}=\frac{r\alpha}{E^{\varphi}}\Phi' + \left(\frac{r\alpha'}{E^{\varphi}} + \frac{3\alpha}{E^{\varphi}} - \frac{r\alpha\left(E^{\varphi}\right)'}{\left(E^{\varphi}\right)^2}\right)\Phi \tag{5}$$

where $\Phi \equiv \phi'$ and $\Pi \equiv \frac{E^{\varphi}}{\alpha r}\dot{\phi}$. The first equation determines the lapse (α) and arises from imposing $K_{\varphi} = 0$. The second equation is the Hamiltonian constraint (another of the Einstein equations). The last two equations are the evolution equations (the rest of the Einstein equations).

To construct the semi-classical equations we polymerize the scalar field $\phi \rightarrow \frac{\sin(k\varphi)}{k}$, and its canonical momentum, $P_{\phi} \rightarrow P_{\varphi}$. A more detailed discussion of the polymerization of scalar fields can be seen in [8]. This is also a construction that has been extensively used in the context of loop quantum cosmology (see [9] for a review). k is the polymerization parameter. In the cases in which the variable being polymerized is a connection (as in the gravitational variables), its interpretation is associated with the loops appearing in the holonomies of the loop representation. The fundamental discreteness that appears at the quantum level establishes a lower bound for these parameters, of the order of the Planck scale (in our case the parameter has dimensions of length so the natural scale would be the Planck length). Notice that in this context the most natural thing is to polymerize the configuration variables, which in the gravitational case are connections. In the context of polymerized metric theories it is not clear which is a more natural choice, whether to polymerize the configuration variable or the momentum [4]. Polymerizing is not guaranteed to produce the correct semiclassical theory. In all examples studied up to now it has. Ideally one would derive a semiclassical theory from a full quantum theory of gravity, but unfortunately no such description is known even for this model. Instead, we settle for the polymerized theory as the best candidate available for a semiclassical theory.

The resulting semiclassical equations become,

$$\frac{\alpha'}{\alpha} - \frac{\left(E^{\varphi}\right)'}{E^{\varphi}} + \frac{2}{r} - \frac{\left(E^{\varphi}\right)^2}{r^3} = 0, \tag{6}$$

$$\frac{\left(E^{\varphi}\right)'}{E^{\varphi}} - \frac{3}{2r} + \frac{\left(E^{\varphi}\right)^2}{2r^3} - 2\pi r\left(\frac{\left(P_{\varphi}\right)^2}{r^4} + \left(\varphi'\right)^2\cos^2\left(k\varphi\right)\right) = 0, \tag{7}$$

$$\dot{\varphi} = \frac{\alpha}{E^{\varphi} r}P_{\varphi}, \tag{8}$$

$$\dot{P}_{\varphi} = \frac{r^2}{E^{\varphi}}\left[\left(\frac{3\alpha E^{\varphi} - r\alpha\left(E^{\varphi}\right)' + \alpha' E^{\varphi} r}{E^{\varphi}}\right)\varphi'\cos^2\left(k\varphi\right) + r\alpha\varphi''\cos^2\left(k\varphi\right) - r\alpha k\left(\varphi'\right)^2\cos\left(k\varphi\right)\sin\left(k\varphi\right)\right], \tag{9}$$

and one recovers the classical limit when $k \rightarrow 0$, one has that φ reduces to ϕ and P_{φ} to $r^2\Pi$ in that limit. To facilitate comparison with Choptuik's notation it should be noted that $E^{\varphi} = ra$ in terms of his variables. It should be noted that the polymerized equations retain the scaling symmetry of the classical equations $r \rightarrow cr$, $t \rightarrow ct$ with c a constant. This is a key difference with other polymerized treatments based on metric variables [4] which introduce a length scale dependent correction near the origin for the radial coordinate, and that found a mass gap. It should be noted that Garfinkle [10] pointed out that this symmetry is a necessary condition for the existence of a self similar critical solution and therefore a zero mass gap. And as we shall see, indeed no mass gap seems to develop.

We proceed to integrate the equations adapting a version of Choptuik's original code (available publicly at [11]). This paper can be seen as a first approach to the problem, in particular with a few exceptions (to confirm the observed behavior) we have not used adaptive mesh refinement as in Choptuik's studies. We choose as initial data family a set of Gaussians parameterized as $\phi(r) = \phi_0 \exp\left(-\left[(r - r_0)/\delta\right]^q\right)$. We will keep $r_0 = 25, \delta = 1.5$ and $q = 2$ fixed and vary ϕ_0, which we call the parameter p. The simulations show that there is a critical value p_* of the parameter below which no black hole forms and above it one sees the collapse of the lapse typical of the formation of a black hole, as shown in figure 1. The coordinate system we are using cannot penetrate the horizon. However, there are clear indications of the formation of a black hole *in appropriate regimes* (see further discussions below). For instance the mass aspect $m(r,t) = (1 - \Lambda^{-2})r/2$ is always smaller than $r/2$ and tends to that value when the lapse vanishes, as shown in figure 2. From this we can get an approximation to the mass of the black hole.

To find the critical value p_* we used a method of binary search in which one increases monotonically the value of the parameter until a black hole is formed, then one backtracks and brackets the critical value.

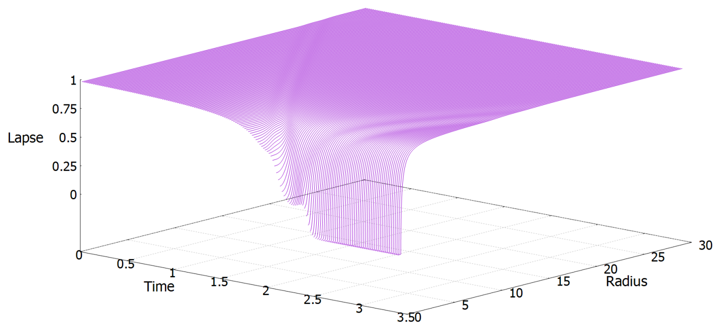

FIG. 1: Demonstration of the collapse of lapse signifying black hole formation. The lapse deviates from one in the region of the initial pulse which travels toward $r = 0$. As the pulse encounters the origin of coordinates, the lapse quickly decreases to zero indicating the formation of a black hole. An outgoing wave of scalar field is seen which is not captured by the black hole formation. For this evolution, the polymerization parameter k is unity, a value large given expectations of the parameter being of the Planck scale, and yet the dynamics is very similar to the classical case for which $k = 0$.

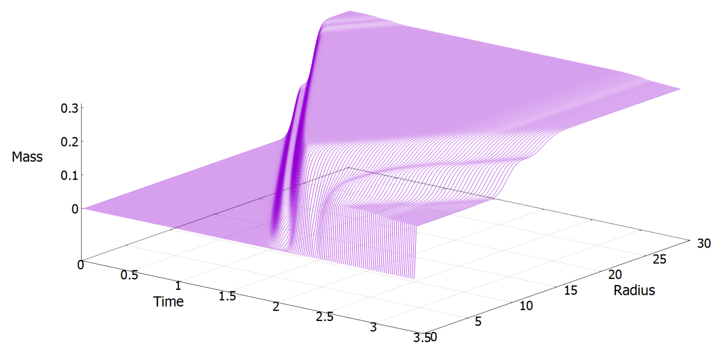

FIG. 2: The behavior of the mass aspect function $m(r,t)$ defined as $g_{rr}(r,t) = (1 - 2m(r,t)/r)^{-1}$ for the same evolution as shown in figure 1. It quickly settles to its final value after the formation of the black hole.

Figure 3 shows the behavior of the final black hole mass as a function of the parameter p, for various values of the polymerization parameter. We see the same behavior Choptuik encountered in the classical theory with the same universal exponent and a very mild dependence on the polymerization parameter (up to $k = 0.5$ the exponent remains the same, within numerical errors —we stress that physically this is an unrealistically large value—). The mass scales as $M_{\rm BH} = C(p - p_*)^\gamma$ where C depends on the value of k (and, as in the usual case, on the particulars of the initial data) but we do not detect significant deviations from the value of $\gamma \sim 0.37$ observed in the classical case unless we force very large values of the polymerization parameter (it should be remembered that it is supposed to be Planck scale). We have run tests with other families of initial data confirming its universality.

An interesting point to be discussed is that the polymerized theory has a maximum departure from the classical theory when $k\varphi \sim \pi/2$. By observing simulations close to criticality but sub-critical, we note that $k\varphi$ is always considerably smaller than $\pi/2$ for a given p in the domain covered (which in the case of black hole formation is only the black hole exterior), even for $k = 1$, which as we have argued, is already an unrealistically large value. What can be happening is that we are not getting close enough to criticality and if one did, regions with $k\varphi \sim \pi/2$ might occur at the origin, where one expects large curvatures to develop. This is in line with the expectation that the polymerized theory will depart from general relativity only close to where the singularity was supposed to be. Solutions of the quantum theory for eternal black holes reinforce this belief [12]. More careful analysis, perhaps with horizon penetrating coordinates and adaptive mesh refinement, will be needed to confirm these points and others, like the self-similar scaling seen in the classical case. We will study this in future work using adaptive mesh refinement.

If one considers this model simply as a dynamical system, then it is interesting to study the regime in which the product $k\varphi$ becomes dynamically large (that is, close to $\pi/2$). It should be noted that the equations for the

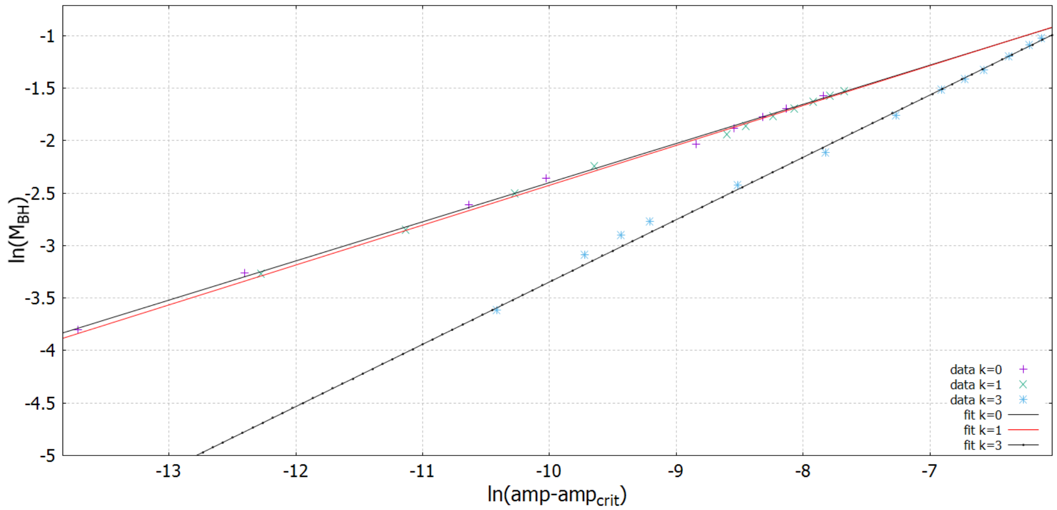

FIG. 3: Plots of the final black hole mass as a function of the parameter of the initial data for different values of the polymerization parameter k. They show the scaling observed by Choptuik, $M_{BH} = C(p - p_*)^\gamma$ where C depends on the value of the polymerization parameter k and only a mild deviation from the value of $\gamma \sim 0.37$ as a function of k. It should be recalled that k is supposed to be Planck scale, so values like $k = 1$ are already quite exaggerated. Plus signs correspond to $k = 0$ (the case studied by Choptuik), x corresponds to $k = 1$, stars to $k = 3$.

polymerized scalar field allow for the formation of shocks/rarefaction as propagation speeds depend on $\cos(k\varphi)$. Such phenomena requires additional conditions to pick a unique solution and there is not a well developed theory for handling them in this case. Certain initial choices of $k\varphi$ can lead to rather complex behavior in timescales shorter than potential black hole formation developing features which are hard to follow even with adaptive mesh refinement. Within families with large initial $k\varphi$ one may find "islands" where the behavior is similar to the one we observed for small k's and is analogous to the classical Choptuik behavior, but that are surrounded by initial data that may not even form black holes. More study is needed to understand the full phase space of initial data when $k\varphi$ may be allowed to be large. It also urges some caution to conclude things about the interior, which our code cannot cover. It should also be noted that, although potentially interesting from a mathematical point of view, solutions with large $k\varphi$ are really beyond the realm of physical applicability of the semiclassical theory we are considering. It is well known that the Choptuik phenomenon close to criticality generates large curvatures near the horizon, and it is widely expected that large curvature regions require full quantum gravity for their description. Although it has been observed, in the context of loop quantum cosmology, that the semiclassical theory works well even in the deep quantum regime [13], we have no reason to expect something similar in our case. Nevertheless, the fact that propagation speeds are dependent on the value of φ when $k \neq 0$ could have potentially observable consequences even in a regime where a semiclassical approach would apply. The potential reach of this observation should be explored. Another point to be considered is that we have considered a polymerization with a constant parameter. In loop quantum cosmology at least, it has proven more physically correct to use polymerization parameters that depend on the dynamical variables [9]. This issue has not been significantly explored out of the cosmology context and may wish to be considered in future analysis of the situation studied in this paper.

Summarizing, we have studied the critical collapse of a massless, minimally coupled, scalar field in a version of semiclassical, spherically symmetric loop quantum gravity. We find that the results for the scaling of the mass agree with those of classical general relativity with very mild dependence on the polymerization parameter and no mass gap (minimum value of the black hole mass). We plan on carrying further studies of the echos that are present near the critical solution in a forthcoming paper using adaptive mesh refinement to see if the self-similarity observed in the classical case persists. We would also like to probe whether the wiggles [14] that appear in the exponent also appear. We also wish to probe closer to where the singularity would be in the classical theory to see if the behavior observed there of the curvature [15] is present or is modified by the polymerization. We would like to probe better the case in which departures from the classical theory are large already at the level of the initial data.

Acknowledgment

We wish to thank Javier Olmedo for discussions. This work was supported in part by Grants NSF-PHY-1603630, 1903799, 1827573 and 1912769, funds of the Hearne Institute for Theoretical Physics, CCT-LSU, fqxi.org and Pedeciba. LL was supported in part by NSERC, and CIFAR. FB thanks the Perimeter Institute for Theoretical Physics for hospitality. Research at Perimeter Institute is supported by the Government of Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science.

---

[1] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993). doi:10.1103/PhysRevLett.70.9.
[2] C. Gundlach and J. M. Martin-Garcia, Living Rev. Rel. **10**, 5 (2007) doi:10.12942/lrr-2007-5 [arXiv:0711.4620 [gr-qc]].
[3] D. Christodolou, Commun. Math. Phys. 105, 337 (1986); 106, 587 (1986); 109, 591 (1987); 109, 613 (1987)
[4] V. Husain, Adv. Sci. Lett. 2, 214 (2009) [arXiv:0808.0949 [gr-qc]]; J. Ziprick and G. Kunstatter, Phys. Rev. D **80**, 024032 (2009) doi:10.1103/PhysRevD.80.024032 [arXiv:0902.3224 [gr-qc]].
[5] R. Gambini, J. Olmedo and J. Pullin, Class. Quant. Grav. **31**, 095009 (2014) doi:10.1088/0264-9381/31/9/095009 [arXiv:1310.5996 [gr-qc]].
[6] T. Thiemann, Class. Quant. Grav. **15**, 1487 (1998) doi:10.1088/0264-9381/15/6/006 [gr-qc/9705021].
[7] R. Gambini, J. Pullin and S. Rastgoo, Class. Quant. Grav. **26**, 215011 (2009) doi:10.1088/0264-9381/26/21/215011 [arXiv:0906.1774 [gr-qc]].
[8] W. Kaminski, J. Lewandowski and M. Bobienski, Class. Quant. Grav. **23**, 2761 (2006) doi:10.1088/0264-9381/23/9/001 [gr-qc/0508091]; W. Kaminski, J. Lewandowski and A. Okolow, Class. Quant. Grav. **23**, 5547 (2006) doi:10.1088/0264-9381/23/18/005 [gr-qc/0604112].
[9] A. Ashtekar and P. Singh, Class. Quant. Grav. **28**, 213001 (2011) doi:10.1088/0264-9381/28/21/213001 [arXiv:1108.0893 [gr-qc]].
[10] D. Garfinkle, Phys. Rev. D **56**, 3169 (1997) doi:10.1103/PhysRevD.56.R3169 [gr-qc/9612015].
[11] ftp://laplace.physics.ubc.ca/pub/
[12] R. Gambini and J. Pullin, Phys. Rev. Lett. **110**, no. 21, 211301 (2013) doi:10.1103/PhysRevLett.110.211301 [arXiv:1302.5265 [gr-qc]].
[13] C. Rovelli and E. Wilson-Ewing, Phys. Rev. D **90**, no. 2, 023538 (2014) doi:10.1103/PhysRevD.90.023538 [arXiv:1310.8654 [gr-qc]].
[14] S. Hod and T. Piran, Phys. Rev. D **55**, 440 (1997) doi:10.1103/PhysRevD.55.R440 [gr-qc/9606087].
[15] D. Garfinkle and G. C. Duncan, Phys. Rev. D **58**, 064024 (1998) doi:10.1103/PhysRevD.58.064024 [gr-qc/9802061]; C. Gundlach, Living Rev. Rel. **2**, 4 (1999) [gr-qc/0001046].