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How fast do quantum walks mix?

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The fundamental problem of sampling from the limiting distribution of quantum walks on networks, known as *mixing*, finds widespread applications in several areas of quantum information and computation. Of particular interest in most of these applications, is the minimum time beyond which the instantaneous probability distribution of the quantum walk remains close to this limiting distribution, known as the *quantum mixing time*. However this quantity is only known for a handful of specific networks. In this letter, we prove an upper bound on the quantum mixing time for *almost all networks*, i.e. the fraction of networks for which our bound holds, goes to one in the asymptotic limit. To this end, using several results in random matrix theory, we find the quantum mixing time of Erdös-Renyi random networks: networks of n nodes where each edge exists with probability p independently. For example for dense random networks, where p is a constant, we show that the quantum mixing time is $\mathcal{O}(n^{3/2+o(1)})$. Besides opening avenues for the analytical study of quantum dynamics on random networks, our work could find applications beyond quantum information processing. Owing to the universality of Wigner random matrices, our results on the spectral properties of random graphs hold for general classes of random matrices that are ubiquitous in several areas of physics. In particular, our results could lead to novel insights into the equilibration times of isolated quantum systems defined by random Hamiltonians, a foundational problem in quantum statistical mechanics.

The quantum dynamics of any discrete system can be captured by a quantum walk on a network, which is a universal model for quantum computation [1]. Besides being a useful primitive to design quantum algorithms [2–6], quantum walks are a powerful tool to model transport in quantum systems such as the transfer of excitations in light-harvesting systems [7–9]. Studying the long-time dynamics of quantum walks on networks is crucial to the understanding of these diverse problems. As quantum evolutions are unitary and hence distance-preserving, quantum walks never converge to a limiting distribution, unlike their classical counterpart. However, given a network of n nodes, one can define the limiting distribution of quantum walk on the network as the long-time average probability distribution of finding the walker in each node [10]. Of particular interest is the *quantum mixing time*: starting from some initial state, the minimum time after which the underlying quantum walk remains close to its limiting distribution.

The importance of the problem of mixing for quantum walks cannot be overstated: this is at the heart of quantum speedups for a number of quantum algorithms [2, 11] and is also key to demonstrating the equivalence between the standard (circuit) and Hamiltonian-based models of quantum computation [12, 13]. Unfortunately, no general result exists for *quantum mixing time* on networks: it has been estimated for a handful of specific graphs (*graphs* and *networks* are used interchangeably throughout the letter) such as hypercubes, d-dimensional lattices etc., and is known to be slower than its classical counterpart for some graphs while faster in the case of others [10, 14–21]. In this letter we bound the quantum mixing time for *almost all* networks, i.e. the fraction of networks for which our result holds goes to one as n goes to infinity.

We prove this general result by studying the mixing of quantum walks on Erdös-Renyi random networks: networks of n nodes with an edge existing between any two nodes with probability p independently, denoted as $G(n,p)$ [22, 23]. It is important to note that our problem differs from designing quantum algorithms for classical mixing: preparing a coherent encoding of the stationary state of a classical random walk [24–26]. Such problems involve running quantum algorithms for finding a marked node, known as quantum spatial search, in reverse. In fact, it has already been established that the problem of spatial search by quantum walk is optimal for $G(n,p)$ [27–29]. Asymptotic dynamics of coined quantum walks on percolation graphs has been studied [30] while quantum dynamics on complex networks has also been numerically investigated [31–33].

In this work we prove that the quantum mixing time for $G(n,p)$ is in $\widetilde{\mathcal{O}}\big(n^{5/2-2\phi}\sqrt{p}\big)$, for $n^{-1/3} \leq p \leq 1-n^{-1/3}$ and $\phi = \frac{\log pn}{2\log n}$ [34]. We obtain this by using several results in random matrix theory to first compute upper bounds on sums of inverses of eigenvalue gaps of the adjacency matrix of $G(n,p)$, which is crucial to subsequently calculate bounds for the quantum mixing time.

For example, in the case of *dense random networks*, i.e. when p is a constant, we prove that the quantum mixing time is $\widetilde{\mathcal{O}}(n^{3/2})$ with probability that tends to one as n goes to infinity. Throughout the letter, we shall refer any such event that occurs with probability $1-o(1)$, as an event that occurs *almost surely*. It can be demonstrated that when $p = 1/2$, $G(n,1/2)$ is a network picked uniformly at random from the set of all networks. This im-

plies that our bound for the quantum mixing time holds for *almost all* networks from this set. Classical random walks on $G(n,p)$ mix quite fast: the classical mixing time is in $\widetilde{\mathcal{O}}(1)$ as long as $np \to \infty$ [35]. Thus, our bound for the quantum mixing time is slower than its classical counterpart, irrespective of p. We emphasize that although we focus on continuous-time quantum walks, our results also hold for discrete-time quantum walks, namely for coined quantum walks and quantum walks *à la* Szegedy [36].

An important direction of research in the emerging field of quantum networks is to study features of quantum dynamics on random networks and explore whether they are distinct from their classical counterparts. Thus besides finding the mixing time of quantum walks in a very general scenario, our work also opens up ways for analytically studying the difference between classical and quantum dynamics on random networks.

Mixing of quantum walks: Let G be a network with a set of nodes $V = \{1, \ldots, n\}$. We consider the Hilbert space spanned by the localized quantum states at the nodes of the network $\mathcal{H} = \text{span}\{|1\rangle, \ldots, |n\rangle\}$ and define the Hamiltonian corresponding to a quantum walk on G by its (rescaled) adjacency matrix γA_G. Then, starting from initial state $|\psi_0\rangle$, the state of the walker after some time t is governed by the Schrödinger Equation, i.e. $|\psi(t)\rangle = e^{-i\gamma A_G t}|\psi_0\rangle$. Note that γ is a parameter that controls the rate of propagation of the walker and is typically considered to be inverse of the spectral norm of A_G such that $\|\gamma A_G\| = 1$. This ensures that evolution for unit time corresponds to $\mathcal{O}(1)$ steps of its discrete-time counterpart. Henceforth we shall denote by $\bar{A}_G$, the normalized adjacency matrix of G, which implicitly assumes the choice of $\gamma = 1/\|A_G\|$. We shall assume that A_G has a simple spectrum, i.e. all eigenvalues of A_G are distinct. For our purposes it suffices as this is indeed the case for random graphs, almost surely [37]. So, let the spectral decomposition of $\bar{A}_G = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle v_i|$, such that $\lambda_n = 1 > \lambda_{n-1} > \cdots > \lambda_1 \geq -1$ are the eigenvalues of $\bar{A}_G$ and $|v_i\rangle$ is the eigenstate corresponding to λ_i.

As mentioned earlier, as quantum evolutions are unitary, quantum walks never "mix" unlike classical random walks. As such, the mixing of a quantum walk on a network G is defined in the following sense: starting from some arbitrary initial state say $|\psi_0\rangle = \sum_{l=1}^{n} c_l |v_l\rangle$ such that $0 \leq |c_l| < 1$, one can obtain the probability that the walker is localized in some node $|f\rangle$ after a time T which is picked uniformly at random in the interval $[0, T]$, i.e.

$$P_f(T) = \frac{1}{T}\int_0^T dt |\langle f|e^{-i\bar{A}_G t}|\psi_0\rangle|^2. \tag{1}$$

Besides resulting in a time-averaged probability distribution at any time T, this definition leads to a well defined *limiting probability distribution*,

$$P_f(T \to \infty) = \lim_{T\to\infty} P_f(T) = \sum_{i=1}^{n} |\langle f|v_i\rangle\langle v_i|\psi_0\rangle|^2, \tag{2}$$

as $T \to \infty$. In order to determine how fast the quantum walk on G converges to the limiting distribution, we need to bound the total distance between this distribution and the distribution at any time T, i.e. $D(P_T) = \|P_f(T) - P_f(T\to\infty)\|_1 = \sum_f |P_f(T) - P_f(T\to\infty)|$.
Evaluating the integral in Eq. (1) and subtracting out the expression for $P_f(T\to\infty)$, followed by some simplifications lead to the following upper bound

$$D(P_T) \leq \sum_{i\neq l} \frac{2|\langle v_i|\psi_0\rangle|.|\langle v_l|\psi_0\rangle|}{T|\lambda_i - \lambda_l|}. \tag{3}$$

As an aside, we would like to mention that in the case where $\bar{A}_G$ has repeated eigenvalues, the sum in Eq. (3) is over distinct eigenvalues $\lambda_i \neq \lambda_l$ and as such a finite value of T_{mix} is always obtained (See Supplemental Material for details [38]). Observe that there exists a time beyond which $D(P_T) \leq \epsilon$, i.e. for all times beyond this, the instantaneous distribution will remain ϵ-close (in total variation distance) to the limiting distribution. This allows us to obtain that the *quantum mixing time* is

$$T_{\text{mix}} = \mathcal{O}\left(\frac{1}{\epsilon}\sum_{i=1}^{n-1}\sum_{r=1}^{n-i} \frac{|\langle v_i|\psi_0\rangle|.|\langle\psi_0|v_{i+r}\rangle|}{|\lambda_{i+r} - \lambda_i|}\right). \tag{4}$$

Note that the mixing time of classical random walks on a network can drastically differ from its quantum counterpart. For example, while the former depends only on the inverse of the spectral gap (gap between two highest eigenvalues) of $\bar{A}_G$, the latter depends on the inverse of *all* eigenvalue gaps, as is evident from Eq. (4). Let us define $\Sigma_r = \sum_{i=1}^{n-r} |\lambda_{i+r} - \lambda_i|^{-1}$. Then the sums of the inverse of eigenvalue gaps appearing in the right hand side of Eq. (4) is given by Σ, where

$$\Sigma = \sum_{r=1}^{n-1} \Sigma_r = \sum_{r=1}^{n-1}\sum_{i=1}^{n-r} \frac{1}{|\lambda_{i+r} - \lambda_i|} \tag{5}$$

In fact, if $\Delta_{\min}$ denotes the minimum of all eigenvalue gaps of $\bar{A}_G$, then one obtains that $1/\Delta_{\min} \leq \Sigma \leq n^2/\Delta_{\min}$, where the lower bound is obtained by noting that for Σ_1 (i.e. when $r = 1$), there exists i such that $|\lambda_{i+1} - \lambda_i| = \Delta_{\min}$. On the other hand, the upper bound is obtained by simply replacing all terms of Σ by $\Delta_{\min}^{-1}$. Obtaining a tight bound on the aforementioned quantity is crucial to obtaining a good bound for the quantum mixing time. Next, by using several results in random matrix theory, we bound Σ for the adjacency matrix of $G(n,p)$ and consequently obtain a bound on the quantum mixing time.

Mixing time of quantum walks on $G(n,p)$: For a random network $G(n,p)$, its adjacency matrix, which we denote as $A_{G(n,p)}$, is the $n \times n$ symmetric matrix with each non-diagonal entry being 1 with probability p and 0 with probability $1-p$. All diagonal entries of $A_{G(n,p)}$ are 0. In order to obtain the normalized adjacency matrix $\bar{A}_{G(n,p)}$, the rescaling parameter should be appropriately chosen. As shown in Refs. [39, 40], as long as $p \geq \log^8(n)/n$, the highest eigenvalue of $A_{G(n,p)}$ is a Gaussian random variable with mean np and standard deviation $\sqrt{p(1-p)}$, *almost surely*. However, even if one does not have access to the random variable, the rescaling $\bar{A}_{G(n,p)} = A_{G(n,p)}/(np)$, suffices. In fact, we prove rigorously in the Supplemental Material [38] that $\|\bar{A}_{G(n,p)}\| \approx 1$.

As established previously, in order to find the quantum mixing time of $G(n,p)$, we require bounds on Σ_r for $\bar{A}_{G(n,p)}$. It is well known that as $np \to \infty$, the spectral density of $\bar{A}_{G(n,p)}$ converges to the so called Wigner's semicircle distribution. While the highest eigenvalue λ_n is isolated from the bulk, the second highest eigenvalue λ_{n-1} lies at the edge of the semicircle. We show in the Supplemental Material [38], using the results of Ref. [41], that the second highest eigenvalue of $\bar{A}_{G(n,p)}$ is upper bounded as $\lambda_{n-1} \leq 6/\sqrt{np} + \widetilde{\mathcal{O}}\big((np)^{-3/4}\big)$. Thus there exists a constant gap between the two highest eigenvalues of $\bar{A}_{G(n,p)}$, i.e. $\Delta = 1 - o(1)$, almost surely as long as $p \geq \log^8(n)/n$. This immediately implies that the classical mixing time is in $\widetilde{\mathcal{O}}(1/\Delta) = \widetilde{\mathcal{O}}(1)$.

For the quantum mixing time, *all* eigenvalue gaps are crucial. So, what about the other eigenvalue gaps? Note that for $p = 1$, the deterministic, all-to-all connected network (complete graph) has $(n-1)$-degenerate eigenvalues at -1, while the highest eigenvalue is 1. However for any $p < 1$, $G(n,p)$ can be considered as an all-to-all network affected by spatial disorder: with probability $1-p$, a link between any two nodes is removed. It is well known that the addition of even a small amount of disorder destroys the symmetry of the underlying structure. This is precisely the case for $\bar{A}_{G(n,p)}$. As such, obtaining a bound on Σ is a challenging problem in random matrix theory.

The Wigner's semicircle law implies that there are $\mathcal{O}(n)$-eigenvalues in the bulk, within a semicircle of radius $R = 2\sqrt{(1-p)/np}$. Thus, the average eigenvalue gap within the bulk scales as $\bar{\Delta} \sim n^{-3/2}p^{-1/2}$. However, this does not rule out the possibility of having gaps $\ll \bar{\Delta}$ and as such, to extract bounds on eigenvalue gaps one needs to look at the local spectral statistics of $\bar{A}_{G(n,p)}$.

It has been recently proven that $\bar{A}_{G(n,p)}$ has no degenerate eigenvalues (simple spectrum), almost surely as long as $C\log^6(n)/n \leq p \leq 1 - C\log^6(n)/n$ for some constant $C > 0$ [37, 42, 43]. Note that these bounds are quite tight: for $p = 1$, we know that $\bar{A}_{G(n,p)}$ has repeated eigenvalues while on the other hand for $p = o(\log(n)/n)$, the underlying random graph is disconnected, implying again that $\bar{A}_{G(n,p)}$ has repeated eigenvalues. Also, one can obtain tail-bounds for consecutive eigenvalue gaps of $G(n,p)$, i.e. $\delta_i = \lambda_{i+1} - \lambda_i$ which in turn leads to a lower bound on $\Delta_{\min}$ for $\bar{A}_{G(n,p)}$ as $\Delta_{\min} \geq n^{-5/2+o(1)}p^{-1/2}$, almost surely [43, 44]. This is the best known bound for this quantity for discrete random matrices.

Using these bounds on δ_i, we are able to show that Σ_1 is almost surely close to $1/\Delta_{\min}$. Formally, we prove in the Supplemental Material [38] that for $\bar{A}_{G(n,p)}$,

$$\Sigma_1 = \sum_{i=1}^{n-1} \frac{1}{\lambda_{i+1} - \lambda_i} \leq n^{5/2+o(1)}\sqrt{p}, \tag{6}$$

with probability $1 - o(1)$. The key observation is that most of the gaps are almost surely within an interval that is $1/\log n$ times the average gap $\bar{\Delta}$.

However, in order to obtain the quantum mixing time for $G(n,p)$, we require an upper bound on Σ instead of Σ_1, i.e. we need information about gaps of the form $\lambda_{i+r} - \lambda_i$. In order to obtain this, we combine two things: (a) the knowledge of the so-called *classical eigenvalue locations* as predicted by the Wigner's semicircle law (henceforth denoted by γ_i) and (b) the *eigenvalue rigidity criterion* which establishes that the location of the eigenvalues of $\bar{A}_{G(n,p)}$ in the bulk are likely to be close to their classical locations, predicted by the semicircle law [40].

From the semicircle distribution itself, one obtains a lower bound on the distances between the classical locations of eigenvalues. For $\bar{A}_{G(n,p)}$, we prove in the Supplemental Material [38] that for $i \leq n/2$, $r \leq n - 2i$ and some universal constant $c > 0$,

$$\gamma_{i+r} - \gamma_i \geq c\frac{r}{n^{7/6}i^{1/3}\sqrt{p}}. \tag{7}$$

An identical estimate holds for the other half of the spectrum by symmetry. On the other hand, we make use of the *eigenvalue rigidity criterion* in Ref. [40], adapted for our analysis. Let $\varepsilon \geq 0$ and $n^{-1/3} \leq p \leq 1 - n^{-1/3}$. Then it can be proved that the eigenvalues of $\bar{A}_{G(n,p)}$ satisfy

$$|\lambda_i - \gamma_i| \leq \frac{n^{\varepsilon}(n^{-2/3}\alpha_i^{-1/3} + n^{-2\phi})}{(pn)^{1/2}} \tag{8}$$

with probability $1 - o(1)$, where $\phi := \frac{\log pn}{2\log n}$ and $\alpha_i := \max\{i, n-i\}$. Note that eigenvalue rigidity does not provide any information about the smallest eigenvalue gaps. However as we examine gaps $\lambda_{i+r} - \lambda_i$ for large enough r, rigidity provides a better estimate on the gap size than the tail bounds from Ref. [43]. Combining both estimates at the different scales of r yields an improved estimate for Σ_r. We prove that for $n^{-1/3} \leq p \leq 1 - n^{-1/3}$ and $\phi = \frac{\log pn}{2\log n}$, the eigenvalues of $\bar{A}_{G(n,p)}$ satisfy

$$\Sigma = \sum_{i=1}^{n-1}\sum_{r=1}^{n-i} \frac{1}{|\lambda_{i+r} - \lambda_i|} \leq n^{1-2\phi+o(1)}n^{5/2}\sqrt{p}, \tag{9}$$

with probability $1-o(1)$. Here, we provide an intuition of the proof of this bound, with the rigorous details being relegated to the Supplemental Material [38]. As λ_n is isolated from the bulk of the spectrum, we have that $\sum_{i=1}^{n-1} |\lambda_n - \lambda_i|^{-1} = \mathcal{O}(n)$, almost surely.

For the remaining terms of Σ, we can exploit *eigenvalue rigidity* once r is large enough to guarantee that $\gamma_{i+r}-\gamma_i$ from Eq. (7) is larger than the error $|\lambda_i-\gamma_i|+|\lambda_{i+r}-\gamma_{i+r}|$ from Eq. (8). This motivates the following definition: Let $c_\star(i) = n^\varepsilon \max\{1, n^{2/3}\alpha_i^{1/3} n^{-2\phi}\} \leq n^{1+\varepsilon-2\phi}$ such that Σ is split into the following parts:

$$\Sigma = \sum_{i=1}^{n-1}\sum_{r=1}^{c_\star(i)} \frac{1}{|\lambda_{i+r}-\lambda_i|} + \sum_{i=1}^{n-1}\sum_{r=c_\star(i)+1}^{n-i} \frac{1}{|\lambda_{i+r}-\lambda_i|} + \mathcal{O}(n). \tag{10}$$

For the first double sum, we use the bound from Eq. (6) to obtain

$$\sum_{i=1}^{n-1}\sum_{r=1}^{c_\star(i)} \frac{1}{|\lambda_{i+r}-\lambda_i|} \leq n^{1+\varepsilon-2\phi}\Sigma_1 \leq n^{1+\varepsilon-2\phi} n^{5/2+o(1)}\sqrt{p}, \tag{11}$$

almost surely. On the other hand, for the second double sum, r is large enough for rigidity to kick in and one obtains that

$$\sum_{i=1}^{n-1}\sum_{r=c_\star(i)+1}^{n-i} \frac{1}{|\lambda_{i+r}-\lambda_i|} \leq \mathcal{O}\Big(n^{5/2+o(1)}\sqrt{p}\Big), \tag{12}$$

almost surely. Combining these two bounds, coupled with the observation that a choice of $\varepsilon = o(1)$ suffices, leads to the required upper bound for Σ in Eq. (9).

As $\Sigma \geq \Delta_{\min}^{-1}$, our upper bound for Σ is close to the best possible upper bound for this quantity for *dense random networks*. Intuitively, for such networks, semicircle law provides an excellent approximation for each λ_i. As such, rigidity kicks in soon: $|\lambda_{i+r} - \lambda_i| \approx |\gamma_{i+r} - \gamma_i|$, for all $r \geq \log n$. This is no longer the case for *sparse random networks*.

In order to obtain the mixing time $T_{\text{mix}}^{G(n,p)}$, we also require bounds on the overlap factors in the numerator of Eq. (4). We show in the Supplemental Material [38] that the eigenstate corresponding to the highest eigenvalue of $\bar{A}_{G(n,p)}$ has a significant overlap with the state that is an equal superposition of all nodes of the network, i.e. $|s\rangle = 1/\sqrt{n}\sum_{i=1}^{n}|i\rangle$. In particular, we show that $|\langle v_n|s\rangle| \geq 1 - 2/\sqrt{np}$, almost surely. It has also been established that with probability $1-o(1/n)$, all other eigenstates of $\bar{A}_{G(n,p)}$ are also completely delocalized as long as $p \geq \log(n)/n$ [40, 45, 46]. That is, for $j \in \{1, \cdots, n\}$, $\|\|v_j\rangle\|_\infty \leq n^{-1/2+o(1)}$, implying that $|\langle v_j|l\rangle| \leq n^{-1/2+o(1)}$, for all states $|l\rangle$, localized at any of the nodes of $G(n,p)$. So from Eq. (1), we obtain that $P_f(T\to\infty) \leq \widetilde{\mathcal{O}}(1/n)$, i.e. the limiting distribution of quantum dynamics on random networks is close to an uniform distribution, almost surely, independently of the initial state $|\psi_0\rangle$.

The delocalization of eigenvectors, along with the bound on Σ allows us to conclude that as long as $n^{-1/3} \leq p \leq 1 - n^{-1/3}$,

$$T_{\text{mix}}^{G(n,p)} = \widetilde{\mathcal{O}}\Big(n^{5/2-2\phi}\sqrt{p}/\epsilon\Big). \tag{13}$$

Interestingly, on decreasing p, our bound for the quantum mixing time actually increases. Clearly, when $p = 1/2$, i.e. when a network is picked up uniformly at random from the set of all networks, its quantum mixing time is almost surely $T_{\text{mix}}^{G(n,1/2)} = \widetilde{\mathcal{O}}(n^{3/2}/\epsilon)$.

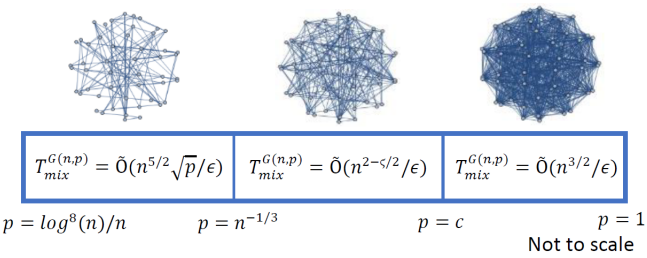

FIG. 1: Summary of the quantum mixing time (time after which the instantaneous distribution of a quantum walk is ϵ-close to its limiting distribution) for $G(n,p)$ for different regimes of p: For dense random networks (when p is some constant c), the quantum mixing time is in $\widetilde{\mathcal{O}}(n^{3/2}/\epsilon)$. For sparse random networks, for $p \geq n^{-1+\zeta}$, where $\zeta \geq 2/3$, the quantum mixing time is in $\mathcal{O}(n^{2-\zeta/2}/\epsilon)$. Finally for sparser random networks, i.e. when $\log^8(n)/n \leq p < n^{-1/3}$, the quantum mixing time is in $\widetilde{\mathcal{O}}(n^{5/2}\sqrt{p}/\epsilon)$.

Unfortunately for sparser random networks, i.e. for $p = \log^D(n)/n$, such that $D \geq 8$, we cannot make use of eigenvalue rigidity. However simply using Eq. (6) along with the observation that $\Sigma_r \leq \Sigma_1$, for $2 \leq r \leq n-1$, gives us a weaker upper bound for the quantum mixing time in such regimes of sparsity. We obtain that $T_{\text{mix}}^{G(n,p)} = \widetilde{\mathcal{O}}\big(n^{5/2}\sqrt{p}/\epsilon\big)$. Our bounds for the quantum mixing time for $G(n,p)$ for different regimes of p are summarized in Fig. 1.

In fact, the breakdown of rigidity estimates in Ref. [40] is not an artifact of the proof. For extremely sparse networks, the optimal rigidity estimates that hold in dense networks are known to break down [47]. Finally, the dependence on precision for the quantum mixing time can be exponentially improved by the amplification techniques in Refs. [10, 17].

Discussion: Prior to this work, the fundamental problem of quantum walk mixing was explored for only a handful of specific graphs and subsequently the quantum mixing time for general classes of graphs was unknown. We have proved an upper bound on the mixing time of quantum walks for *almost all* networks. We do so by proving that the mixing time for quantum walks on Erdös-Renyi random networks is in $\widetilde{\mathcal{O}}\big(n^{5/2-2\phi}\sqrt{p}\big)$, for $p \geq n^{-1/3}$ and

$\phi = \frac{\log pn}{2 \log n}$. Our bound for the quantum mixing time is slower than its classical counterpart which hints at fundamental differences between classical and quantum dynamics on random networks.

In the process of obtaining our general results for quantum mixing, we have also derived several results in random matrix theory, which could be of interest to areas beyond quantum information. Random matrices are ubiquitous in several areas in physics ranging from condensed matter physics [48], nuclear physics [49] and high energy physics [50]. The techniques used to derive bounds on Σ for $A_{G(n,p)}$ can also be applied to random matrices in very general scenarios, namely for random symmetric matrices with each entry having mean μ and standard deviation σ, known as Wigner matrices [51]. Moreover, they can be extended to hold for the so-called *Band Wigner Matrices*: symmetric $n \times n$ random matrices H with random entries such that any entry $H_{ij} = 0$, if $|i-j| > W$, where $W \leq n/2$ is the *band-width*. These matrices also find many applications in several areas of physics and mathematics [52].

For example, consider the problem of equilibration of isolated quantum systems, widely studied in quantum statistical mechanics [53]. Therein one is interested in the limiting distribution of the expectation value of an observable defined in a time-averaged sense and also in the time after which the expectation value remains close to this limiting distribution [54, 55]. A crucial quantity in determining the equilibration time is the distribution of gaps of the underlying Hamiltonian [56]. Thus, the techniques used in this letter to obtain bounds on Σ can be applied to obtain better upper bounds for the equilibration times of isolated quantum systems defined by random Hamiltonians. Besides this, we believe that our results could be used in other areas of physics such as in Eigenstate Thermalization Hypothesis (ETH) [57]. They could also lead to generic analytical results in the field of quantum chaos [58] and in the analysis of scrambling of information in black holes [59].

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[1] Andrew M Childs. Universal computation by quantum walk. *Physical review letters*, 102(18):180501, 2009.

[2] Andrew M Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, and Daniel A Spielman. Exponential algorithmic speedup by a quantum walk. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 59–68. ACM, 2003.

[3] Andris Ambainis. Quantum walk algorithm for element distinctness. *SIAM Journal on Computing*, 37(1):210–239, 2007.

[4] Andrew M Childs and Jeffrey Goldstone. Spatial search by quantum walk. *Physical Review A*, 70(2):022314, 2004.

[5] Frédéric Magniez, Ashwin Nayak, Jérémie Roland, and Miklos Santha. Search via quantum walk. *SIAM Journal on Computing*, 40(1):142–164, 2011.

[6] Hari Krovi, Frédéric Magniez, Maris Ozols, and Jérémie Roland. Quantum walks can find a marked element on any graph. *Algorithmica*, 74(2):851–907, 2016.

[7] Masoud Mohseni, Patrick Rebentrost, Seth Lloyd, and Alan Aspuru-Guzik. Environment-assisted quantum walks in photosynthetic energy transfer. *The Journal of chemical physics*, 129(17):11B603, 2008.

[8] Patrick Rebentrost, Masoud Mohseni, Ivan Kassal, Seth Lloyd, and Alán Aspuru-Guzik. Environment-assisted quantum transport. *New Journal of Physics*, 11(3):033003, 2009.

[9] Alex W Chin, Animesh Datta, Filippo Caruso, Susana F Huelga, and Martin B Plenio. Noise-assisted energy transfer in quantum networks and light-harvesting complexes. *New Journal of Physics*, 12(6):065002, 2010.

[10] Dorit Aharonov, Andris Ambainis, Julia Kempe, and Umesh Vazirani. Quantum walks on graphs. In *Proceedings of the thirty-third annual ACM symposium on Theory of computing*, pages 50–59. ACM, 2001.

[11] Shantanav Chakraborty, Leonardo Novo, and Jérémie Roland. Finding a marked node on any graph by continuous time quantum walk. *arXiv preprint arXiv:1807.05957*, 2018.

[12] Dorit Aharonov, Wim Van Dam, Julia Kempe, Zeph Landau, Seth Lloyd, and Oded Regev. Adiabatic quantum computation is equivalent to standard quantum computation. *SIAM review*, 50(4):755–787, 2008.

[13] Libor Caha, Zeph Landau, and Daniel Nagaj. Clocks in feynman's computer and kitaev's local hamiltonian: Bias, gaps, idling, and pulse tuning. *Physical Review A*, 97(6):062306, 2018.

[14] Amir Ahmadi, Ryan Belk, Christino Tamon, and Carolyn Wendler. On mixing in continuous-time quantum walks on some circulant graphs. *Quantum Information & Computation*, 3(6):611–618, 2003.

[15] Viv Kendon and Ben Tregenna. Decoherence can be useful in quantum walks. *Physical Review A*, 67(4):042315, 2003.

[16] Leonid Fedichkin, Dmitry Solenov, and Christino Tamon. Mixing and decoherence in continuous-time quantum walks on cycles. *Quantum Information & Computation*, 6(3):263–276, 2006.

[17] Peter C Richter. Almost uniform sampling via quantum walks. *New Journal of Physics*, 9(3):72, 2007.

[18] Peter C Richter. Quantum speedup of classical mixing processes. *Physical Review A*, 76(4):042306, 2007.

[19] FL Marquezino, R Portugal, G Abal, and R Donangelo. Mixing times in quantum walks on the hypercube. *Physical Review A*, 77(4):042312, 2008.

[20] Franklin L Marquezino, Renato Portugal, and Gonzalo Abal. Mixing times in quantum walks on two-dimensional grids. *Physical Review A*, 82(4):042341, 2010.

[21] Maria Kieferova and Daniel Nagaj. Quantum walks on

necklaces and mixing. *International Journal of Quantum Information*, 10(02):1250025, 2012.

[22] P. Erdős and A. Rényi. On random graphs. I. *Publ. Math. Debrecen*, 6:290–297, 1959.

[23] P. Erdős and A. Rényi. On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.

[24] Pawel Wocjan and Anura Abeyesinghe. Speedup via quantum sampling. *Physical Review A*, 78(4):042336, 2008.

[25] Vedran Dunjko and Hans J Briegel. Quantum mixing of markov chains for special distributions. *New Journal of Physics*, 17(7):073004, 2015.

[26] Davide Orsucci, Hans J Briegel, and Vedran Dunjko. Faster quantum mixing for slowly evolving sequences of markov chains. *Quantum*, 2:105, 2018.

[27] Shantanav Chakraborty, Leonardo Novo, Andris Ambainis, and Yasser Omar. Spatial search by quantum walk is optimal for almost all graphs. *Physical review letters*, 116(10):100501, 2016.

[28] Shantanav Chakraborty, Leonardo Novo, Serena Di Giorgio, and Yasser Omar. Optimal quantum spatial search on random temporal networks. *Physical review letters*, 119(22):220503, 2017.

[29] Adam Glos, Aleksandra Krawiec, Ryszard Kukulski, and Zbigniew Puchała. Vertices cannot be hidden from quantum spatial search for almost all random graphs. *Quantum Information Processing*, 17(4):81, 2018.

[30] Bálint Kollár, Tamás Kiss, Jaroslav Novotnỳ, and Igor Jex. Asymptotic dynamics of coined quantum walks on percolation graphs. *Physical review letters*, 108(23):230505, 2012.

[31] Oliver Mülken and Alexander Blumen. Continuous-time quantum walks: Models for coherent transport on complex networks. *Physics Reports*, 502(2-3):37–87, 2011.

[32] Mauro Faccin, Tomi Johnson, Jacob Biamonte, Sabre Kais, and Piotr Migdał. Degree distribution in quantum walks on complex networks. *Physical Review X*, 3(4):041007, 2013.

[33] Mauro Faccin, Piotr Migdał, Tomi H Johnson, Ville Bergholm, and Jacob D Biamonte. Community detection in quantum complex networks. *Physical Review X*, 4(4):041012, 2014.

[34] Throughout the letter, $\tilde{O}(f(n))$ denotes $\mathcal{O}(f(n)\text{polylog}(f(n)))$.

[35] László Lovász. Random walks on graphs. *Combinatorics, Paul erdos is eighty*, 2(1-46):4, 1993.

[36] Mario Szegedy. Quantum speed-up of markov chain based algorithms. In *Proceedings. 45th Annual IEEE Symposium on Foundations of Computer Science, 2004.*, pages 32–41. IEEE, 2004.

[37] Terence Tao and Van Vu. Random matrices have simple spectrum. *arXiv preprint arXiv:1412.1438*, 2014.

[38] See supplemental material at [url], which includes the detailed derivations of the results of the letter.

[39] Zoltán Füredi and János Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[40] László Erdös, Antti Knowles, Horng-Tzer Yau, Jun Yin, et al. Spectral statistics of erdős–rényi graphs i: local semicircle law. *The Annals of Probability*, 41(3B):2279–2375, 2013.

[41] Van H. Vu. Spectral norm of random matrices. *Combinatorica*, 27(6):721–736, 2007.

[42] Kyle Luh and Van Vu. Sparse random matrices have simple spectrum. *arXiv preprint arXiv:1802.03662*, 2018.

[43] Patrick Lopatto and Kyle Luh. Tail bounds for gaps between eigenvalues of sparse random matrices. *arXiv preprint arXiv:1901.05948*, 2019.

[44] Hoi Nguyen, Terence Tao, and Van Vu. Random matrices: tail bounds for gaps between eigenvalues. *Probability Theory and Related Fields*, pages 1–40, 2015.

[45] Yael Dekel, James R Lee, and Nathan Linial. Eigenvectors of random graphs: Nodal domains. *Random Structures & Algorithms*, 39(1):39–58, 2011.

[46] Yukun He, Antti Knowles, and Matteo Marcozzi. Local law and complete eigenvector delocalization for supercritical erdös-rényi graphs. *arXiv preprint arXiv:1808.09437*, 2018.

[47] Jiaoyang Huang, Benjamin Landon, and Horng-Tzer Yau. Transition from tracy-widom to gaussian fluctuations of extremal eigenvalues of sparse erd\h {o} sr\'enyi graphs. *arXiv preprint arXiv:1712.03936*, 2017.

[48] Carlo WJ Beenakker. Random-matrix theory of quantum transport. *Reviews of modern physics*, 69(3):731, 1997.

[49] Gary E Mitchell, Achim Richter, and Hans A Weidenmüller. Random matrices and chaos in nuclear physics: Nuclear reactions. *Reviews of Modern Physics*, 82(4):2845, 2010.

[50] Jordan S Cotler, Guy Gur-Ari, Masanori Hanada, Joseph Polchinski, Phil Saad, Stephen H Shenker, Douglas Stanford, Alexandre Streicher, and Masaki Tezuka. Black holes and random matrices. *Journal of High Energy Physics*, 2017(5):118, 2017.

[51] Terence Tao. *Topics in random matrix theory*, volume 132. American Mathematical Soc., 2012.

[52] Paul Bourgade, Horng-Tzer Yau, and Jun Yin. Random band matrices in the delocalized phase, i: Quantum unique ergodicity and universality. *arXiv preprint arXiv:1807.01559*, 2018.

[53] Anatoli Polkovnikov, Krishnendu Sengupta, Alessandro Silva, and Mukund Vengalattore. Colloquium: Nonequilibrium dynamics of closed interacting quantum systems. *Reviews of Modern Physics*, 83(3):863, 2011.

[54] Noah Linden, Sandu Popescu, Anthony J Short, and Andreas Winter. Quantum mechanical evolution towards thermal equilibrium. *Physical Review E*, 79(6):061103, 2009.

[55] Anthony J Short and Terence C Farrelly. Quantum equilibration in finite time. *New Journal of Physics*, 14(1):013063, 2012.

[56] Henrik Wilming, Thiago R de Oliveira, Anthony J Short, and Jens Eisert. Equilibration times in closed quantum many-body systems. In *Thermodynamics in the Quantum Regime*, pages 435–455. Springer, 2018.

[57] Marcos Rigol, Vanja Dunjko, and Maxim Olshanii. Thermalization and its mechanism for generic isolated quantum systems. *Nature*, 452(7189):854, 2008.

[58] Mark Srednicki. Chaos and quantum thermalization. *Physical Review E*, 50(2):888, 1994.

[59] Nima Lashkari, Douglas Stanford, Matthew Hastings, Tobias Osborne, and Patrick Hayden. Towards the fast scrambling conjecture. *Journal of High Energy Physics*, 2013(4):22, 2013.