Fractons from Vector Gauge Theory

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Motivated by the prediction of fractonic topological defects in a quantum crystal, we utilize a reformulated elasticity duality to derive a description of a fracton phase in terms of coupled *vector* U(1) gauge theories. The fracton order and restricted mobility emerge as a result of an unusual Gauss law where electric field lines of one gauge field act as sources of charge for others. At low energies this vector gauge theory reduces to the previously studied fractonic symmetric tensor gauge theory. We construct the corresponding lattice model and a number of generalizations, which realize fracton phases via a condensation of string-like excitations built out of charged particles, analogous to the p-string condensation mechanism of the gapped X-cube fracton phase.

Introduction. Motivated by continued interest in topological quantum matter and by a search of fault-tolerant quantum memory, recent studies have led to fascinating developments in an exotic class of quantum spin-liquid models[1–7]. These are characterized by many nontrivial properties, the most unusual of which are system-size-dependent ground state degeneracy and the existence of quasi-particles, dubbed "fractons", that exhibit restricted mobility. Namely, there are quasi-particles confined to zero-, one- and/or two-dimensional subspaces of the full three-dimensional space of the model. While such fracton phases were originally discovered in fully gapped phases of commuting projector lattice spin Hamiltonians, it was more recently pointed out [8], that fractonic charges are also realized in gapless phases of U(1) symmetric tensor gauge theories [9].

In a parallel development, it was observed by one of us (L.R.) [10] that such restricted quasi-particle mobility is strongly reminiscent of the immobile disclinations and glide-only dislocations in an ordinary two-dimensional (2D) crystal, described by a symmetric strain tensor field. This conjecture of fracton-elasticity duality was reported and moreover explicitly demonstrated in [11], utilizing a generalization of boson-vortex duality [12, 13]. It was shown that a 2+1D quantum crystal is dual to a symmetric tensor gauge theory, with disclinations and dislocations mapping onto fractonic charges and their dipolar bound states, and with stress tensor σ_{ik} and momentum vector π_k fields respectively corresponding to the electric tensor E_{ik} and magnetic vector B_k fields.

Motivation and results. An important source of insight into fracton physics has been to relate apparently exotic fracton states to more familiar quantum phases of matter. Indeed, the fracton-elasticity duality is an example of such a relationship. Related progress has also been made for certain gapped fracton phases, via a construction of these phases in terms of coupled layers of ordinary 2D topologically ordered states [14, 15]. So far there is a relative paucity of relationships between gapless fracton phases and better understood phases or theories. Most U(1) tensor gauge theories are not dual to elasticity, and even for those that are, developing alternative viewpoints is highly desirable.

Remarkably, the fracton-elasticity duality itself contains the seed of another such point of view. There is a sense in which elasticity, formulated in terms of a symmetrized strain tensor $u_{ik} = \frac{1}{2}(\partial_i u_k + \partial_k u_i)$ is a system of two "spin" flavors of XY models, joined together via "spin-space" coupling, as in systems with spin-orbit interaction. However, in the absence of spin-space coupling, such a system dualizes to two independent flavors of U(1) vector gauge theory, and lacks fractonic charges. It is thus natural to ask whether fractonic tensor gauge theories can be formulated in terms of coupled vector gauge theories and if so, what minimal ingredients are required for such coupling. Some progress has been made along these lines from a different point of view, starting with a vector U(1) gauge theory and gauging certain global symmetries to obtain a fractonic tensor gauge theory [16]. We make contact with this result below in greater detail.

In this Letter, we first utilize a reformulated fracton-elasticity duality to derive a 2+1D U(1) *vector* gauge theory that hosts fractonic charges and is equivalent at low energy to the rank-2 symmetric tensor U(1) gauge theory with scalar charge, dubbed the "scalar-charge theory". We then discuss a lattice version of the same theory, which allows the scalar-charge theory to be understood starting from two decoupled vector U(1) gauge theories, and condensing certain charged loops. Finally, we discuss generalizations of our lattice construction that provide constructions of a new class of fractonic tensor gauge theories as coupled vector gauge theories.

The continuum Hamiltonian density we obtain is given by

$$\begin{aligned}\tilde{\mathcal{H}} &= \frac{1}{2}C|\mathbf{E}_k|^2 + \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}_k)^2 + \frac{1}{2}K|\mathbf{e}|^2 \\ &+\frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{a} + A_a)^2 - \mathbf{A}_k\cdot\mathbf{J}_k - \mathbf{a}\cdot\mathbf{j}\,. \qquad (1)\end{aligned}$$

which involves three U(1) vector gauge fields with electric fields $\mathbf{E}_k$ (with flavors $k = x, y$) and $\mathbf{e}$ and corresponding canonically conjugate vector potentials $\mathbf{A}_k$ and $\mathbf{a}$. We denote the corresponding charge densities by p_k and ρ, and currents by $\mathbf{J}_k$ and $\mathbf{j}$, with p_k and $\mathbf{J}_k$ referred to as the dipole charge and dipole current, respectively. Moreover, $A_a = \epsilon_{ik} A_{ik}$, and we use a short-hand notation where the curl of a 2D vector field is implicitly its scalar z-component, *i.e.*, $\boldsymbol{\nabla} \times \mathbf{a} \equiv \hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times \mathbf{a}$.

The Hamiltonian is supplemented by the Gauss' law constraints:

$$\boldsymbol{\nabla} \cdot \mathbf{E}_k = p_k - e_k, \tag{2}$$
$$\boldsymbol{\nabla} \cdot \mathbf{e} = \rho. \tag{3}$$

Crucially, the components of the electric field e_k appear as additional dipole charge in the Gauss's law (2).

The generalization to d dimensions is straightforward and consists of $d+1$ U(1) gauge fields obeying the same Gauss' laws but with $k = 1, \ldots, d$. The main difference in the Hamiltonian is that the $(\boldsymbol{\nabla}\times\mathbf{a}+A_a)^2$ term is replaced by a sum of the $d(d-1)/2$ terms of the form $(\partial_i a_j - \partial_j a_i + A_{ij} - A_{ji})^2$. The resulting theory is equivalent at low energy to the d-dimensional scalar-charge theory, as we will detail below.

The fractonic nature of the ρ charges can be seen by observing that moving such a charge requires creating or destroying field lines of the $\mathbf{e}$ electric field, but since these field lines themselves carry gauge charge, a single "piece" of field line cannot be locally created or destroyed. The immobility of these charges is also manifest in that gauge invariance requires the current $\mathbf{j}$ to vanish identically, as we elaborate below. We next turn to the derivation of this fractonic coupled *vector* gauge theory and its connection to the previously studied tensor scalar-charge theory.

Derivation. To this end, we pass to Lagrangian formalism and begin with an elastic theory of a 2+1D quantum crystal formulated in terms of the phonon field u_k and its canonically conjugate momentum π_k. For simplicity we take the elastic tensor $C_{ij,kl}$ to be $C_{ij,kl} = C\delta_{ik}\delta_{jl}$. The generalization to an arbitrary $C_{ij,kl}$ is straightforward.

Ref.11 started with such a theory written in terms of the symmetrized strain $u_{ik} = \partial_i u_k + \partial_k u_i$, and showed it is dual to a symmetric tensor gauge theory, where fractonic charges correspond to disclinations and dipoles to dislocations. To get to an equivalent flavored *vector* gauge theory description, we reformulate the elastic theory in terms of "minimally"-coupled quantum XY models, introducing the orientational bond-angle field, θ and its canonically conjugate angular momentum density L. The Lagrangian density is given by

$$\begin{aligned}\mathcal{L} &= \pi_k \partial_t u_k + L\partial_t\theta - \frac{1}{2}\pi_k^2 - \frac{1}{2}C(\partial_i u_k - \theta\epsilon_{ik})^2 \\ &- \frac{1}{2}L^2 - \frac{1}{2}K(\boldsymbol{\nabla}\theta)^2.\end{aligned} \tag{4}$$

Due to the coupling to θ, the anti-symmetric part of the unsymmetrized strain $\partial_i u_k$ is massive, below a scale set by C, similar to the Higgs mechanism for gauge fields. Integrating out θ results in the standard elasticity theory formulated in terms of the symmetrized strain u_{ik}, which is the starting point of Ref.11. To proceed, it is convenient to decouple the elastic and orientational terms in (4) via Hubbard-Stratonovich vector fields, stress $\boldsymbol{\sigma}_k$ and torque $\boldsymbol{\tau}$, resulting in the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \pi_k \partial_t u_k + L\partial_t\theta - \boldsymbol{\sigma}_k \cdot \boldsymbol{\nabla} u_k + \sigma_a\theta - \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\theta \\ &+ \frac{1}{2}C^{-1}\boldsymbol{\sigma}_k^2 - \frac{1}{2}\pi_k^2 - \frac{1}{2}L^2 + \frac{1}{2}K^{-1}\boldsymbol{\tau}^2,\end{aligned} \tag{5}$$

where $\sigma_a \equiv \epsilon_{ik}\sigma_{ik}$.

For a complete description, in addition to the single-valued (smooth) Goldstone mode degrees of freedom, θ^e and u_k^e, we must also include topological defects. A disclination defect is defined by a nonsingle-valued bond angle with winding $\oint d\theta^s = 2\pi s/n$ around the disclination position, or equivalently in a differential form, $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\theta^s = \frac{2\pi s}{n}\delta^2(\mathbf{r}) \equiv \rho(\mathbf{r})$. The integer disclination charge s corresponds to an integer-multiple of $2\pi/n$ missing (added) wedge of atoms for $s > 0$ ($s < 0$) in a C_n symmetric crystal, with most common case of a hexagonal lattice, $n = 6$.

A dislocation is a point vector defect, around which the displacement u_k is not single-valued, with winding $\oint du_k = b_k$, or equivalently in a differential form, $\boldsymbol{\nabla}\times\boldsymbol{\nabla}u_k^s = b_k\delta^2(\mathbf{r}) \equiv b_k(\mathbf{r})$. An elementary dislocation is a dipole of $\pm 2\pi/n$ disclinations and is characterized by a 2D Burgers vector charges, b_k, that takes values in the lattice. An edge dislocation corresponds to a ray of missing or extra lattice sites, with a Burgers vector lying in the 2d plane of the crystal. A nontrivial configuration of dislocations, $\mathbf{b}(\mathbf{r})$ can also contribute to a disclination density, given by $s_b(\mathbf{r}) = \hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\mathbf{b}(\mathbf{r})$, with a single disclination corresponding to an end point of a ray of dislocations.

Expressing the phonon and bond-angle fields in terms of corresponding singular (s) and elastic (e) parts, $u_k = u_k^e + u_k^s$, $\theta = \theta^e + \theta^s$, and integrating over the elastic parts, gives the conservation of linear and angular momentum, $\partial_t\pi_k - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_k = 0$ and $\partial_t L - \boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \sigma_a$.

Expressing linear momentum conservation constraint in terms of dual magnetic and electric fields, $\pi_k = \epsilon_{kj}B_j$, $\sigma_{ik} = -\epsilon_{ij}\epsilon_{k\ell}E_{j\ell}$, leads to the k-flavored Faraday equations, $\partial_t B_k + \boldsymbol{\nabla}\times\mathbf{E}_k = 0$. As in standard electrodynamics, the Faraday law is solved by k-flavored vector $\mathbf{A}_k$ and scalar A_{0k} gauge potentials, $B_k = \boldsymbol{\nabla}\times\mathbf{A}_k$, $\mathbf{E}_k = -\partial_t\mathbf{A}_k - \boldsymbol{\nabla}A_{0k}$. We emphasize that, in contrast to the *symmetric tensor* approach[11, 17], here, the $k = (x, y)$-flavored *vector* gauge field $\mathbf{A}_k$ has components A_{ik} that form an unsymmetrized tensor field.

Using these definitions reduces the conservation of angular momentum to $\partial_t(L - A_a) - \boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - \hat{\mathbf{z}}\times\mathbf{A}_0) = 0$, which is then solved by introducing another set of vector $\mathbf{a}$ and scalar a_0 gauge fields, giving

$$L = \boldsymbol{\nabla}\times\mathbf{a} + A_a, \quad \tau_k = \epsilon_{kj}(\partial_t a_j + \partial_j a_0 - A_{0j}). \tag{6}$$

Using these gauge fields to eliminate $\boldsymbol{\sigma}_k$, π_k, $\boldsymbol{\tau}$ and L, gives an effective Lagrangian density

$$\begin{aligned}\tilde{\mathcal{L}} &= \frac{1}{2}C^{-1}(\partial_t \mathbf{A}_k + \boldsymbol{\nabla} A_{0k})^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}_k)^2 \\ &+ \frac{1}{2}K^{-1}(\partial_t a_k + \partial_k a_0 - A_{0k})^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{a} + A_a)^2 \\ &+ \mathbf{A}_k\cdot\mathbf{J}_k - A_{0k}p_k + \mathbf{a}\cdot\mathbf{j} - a_0\rho.\end{aligned} \tag{7}$$

Here, the dipole charge p_k is given by the dislocation density $p_k = \epsilon_{lk}b_l = (\hat{\mathbf{z}}\times\mathbf{b})_k$, the fracton charge ρ is the disclination density, and the corresponding currents are given by $\mathbf{J}_k = \epsilon_{lk}\hat{\mathbf{z}}\times(\partial_t\boldsymbol{\nabla}u_l - \boldsymbol{\nabla}\partial_t u_l)$ and $\mathbf{j} = \hat{\mathbf{z}}\times(\partial_t\boldsymbol{\nabla}\theta - \boldsymbol{\nabla}\partial_t\theta)$. Finally, using Hubbard-Stratonovich transformations to introduce electric fields canonically conjugate to each vector potential, we obtain (1) in Lagrangian form.

The unusual Gauss's law (2) couples three vector gauge theories, with k-th component of the $\mathbf{e}$ field acting as an additional source of charge in the Gauss's law for $\mathbf{E}_k$. We also note that taking the divergence on the second index k of the Gauss's law for $\mathbf{E}_k$, (2) and using the second law for $\mathbf{e}$ to eliminate $\boldsymbol{\nabla}\cdot\mathbf{e}$ from the resulting right hand side gives,

$$\partial_i\partial_k E_{ik} = \tilde{\rho}. \tag{8}$$

This thereby recovers the generalized Gauss's law of scalar-charge tensor gauge theory, with $\tilde{\rho} \equiv -\rho + \boldsymbol{\nabla}\cdot\mathbf{p}$ the total charge contribution,[11] that encodes the additional dipole conservation responsible for immobility of fractonic charges.[8] We note that, in contrast to the scalar-charge theory, E_{ik} is not a symmetric tensor, but effectively becomes symmetric at low energy as we demonstrate below.

We note that the Lagrangian (7) is invariant under a deformed gauge transformation,

$$\mathbf{A}_k \to \mathbf{A}_k + \boldsymbol{\nabla}\chi_k, \quad A_{0k} \to A_{0k} - \partial_t\chi_k, \tag{9}$$

$$a_k \to a_k + \partial_k\phi - \chi_k, \quad a_0 \to a_0 - \partial_t\phi, \tag{10}$$

with (10) ensuring that $\boldsymbol{\nabla}\times\mathbf{a} + A_a$ is gauge invariant. Under the χ_k gauge transformation, the current source terms in (7) shift by $-\chi_k j_k + \partial_t\chi_k\tilde{p}_k + \boldsymbol{\nabla}\chi_k\cdot\mathbf{J}_k$, where $\tilde{p}_k = p_k - e_k$ is the effective dipole density, that is a combination of microscopic dipoles and electric field generated by pairs of fracton charges. Requiring gauge invariance then leads to the dipole continuity equation $\partial_t\tilde{p}_k + \boldsymbol{\nabla}\cdot\mathbf{J}_k = -j_k$, where dipole conservation is violated by a nonzero fracton current $\mathbf{j}$. It follows that in the absence of gapped dipoles, $\mathbf{j} = 0$ for on-shell processes, *i.e.* isolated disclinations are immobile fractonic charges.

The harmonic [21] elasticity theory (4) enjoys the symmetries

$$\begin{aligned} u_k &\to u_k + \alpha_k + \beta\epsilon_{kj}r_j, \\ \theta &\to \theta - \beta.\end{aligned} \tag{11}$$

The constant shift of u_k by α_k can be interpreted as continuous translational symmetry. The terms proportional to β are a small-angle rotation, where the displacements u_k in the initial configuration (before symmetry transformation) are also small. By introducing background gauge fields for these symmetries (see Supplementary Material) and carrying out the duality in the presence of the background fields, we identify corresponding conserved currents on the gauge theory side. Associated with the α_k translational symmetry, we have the linear momentum current $J^m_{\mu k}$, where the conserved density $J^m_{0k} = \epsilon_{kj}B_j$ is the magnetic flux, and $J^m_{ik} = \epsilon_{kj}\epsilon_{i\ell}E_{\ell j}$. Operators transforming under α_k are thus monopole operators of the $\mathbf{A}_k$ gauge fields. Associated with the β rotation symmetry and conservation of angular momentum, we have the magnetic flux current of the $\mathbf{a}$ gauge field, j^m_μ. Due to the coupling between the vector gauge fields, the naive magnetic flux $\nabla\times\mathbf{a}$ is not gauge-invariant, requiring modified expressions $j^m_0 = \epsilon_{ij}(\partial_i a_j + A_{ij}) - r_i\epsilon_{ij}J^m_{0j}$ and $j^m_j = \epsilon_{ij}(\partial_i a_0 + \partial_t a_i - A_{0i}) - r_i\epsilon_{ik}J^m_{jk}$. We note that the current j^m_μ is explicitly position-dependent, similar to the symmetries discussed in [18]. If the position-dependent terms are dropped, j^m_μ remains gauge-invariant but is no longer conserved. This identification of magnetic flux currents in the coupled vector gauge theory with symmetries of the elasticity theory is useful in our discussion of lattice models below.

We conclude by demonstrating that in fact this dual coupled *vector* U(1) gauge theory, at low energies is indeed equivalent to the *symmetric tensor* gauge theory. To this end, we observe that the enlarged gauge redundancy allows us to completely eliminate a_k from the Lagrangian (7), by choosing $\chi_k = a_k$. The term $\frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{a} + A_a)^2$ reduces to $\frac{1}{2}A_a^2$, thereby gapping out the antisymmetric component $A_a = \epsilon_{ij}A_{ik}$. Thus, at energies well below this gap, $\epsilon_{ik}A_{ik} \approx 0$, and only the symmetric components of A_{ik} remain as active degrees of freedom. Furthermore, the electric field term reduces to $\frac{1}{2}K^{-1}(\partial_t\mathbf{a} + \boldsymbol{\nabla}a_0 - A_{0k})^2 \to \frac{1}{2}K^{-1}(\boldsymbol{\nabla}a_0 - A_{0k})^2$, enforcing $A_{0k} = \partial_k\alpha_0$. Thus at low energies, this reduces the Lagrangian exactly to that of the symmetric tensor gauge theory, with the Gauss' law (8).

Lattice model and charged loop condensation. We now consider a lattice version of the Hamiltonian (1). For simplicity of presentation we first work in 2D, then discuss the generalization to arbitrary dimension. The 3D version of the model appeared previously in [16]. We use the resulting model to obtain a physical picture of the scalar-charge theory in terms of condensation of certain charged loops. This differs in perspective from the results of [16], where the fracton phase arises upon gauging certain global symmetries. There are subtleties particular to the 2D case that we discuss, having to do with the role of compact gauge fields and associated symmetries.

The lattice geometry consists of three interpenetrating 2D square lattices as shown in Fig. 1. One of these we refer to as the "underlying lattice," and the k-lattice ($k = x, y$) is a square lattice with vertices the k-directed links of the underlying lattice. We place the electric field $\mathbf{e}$ and vector potential $\mathbf{a}$ on the links of the underlying lattice, with $\mathbf{E}_k, \mathbf{A}_k$ placed on the links of the k-lattice. The $\mathbf{e}, \mathbf{a}$ gauge field is taken to be compact, thereby allowing a loop-condensation phase transition into the fracton phase, with $\mathbf{e}$ taking integer eigenvalues and $\mathbf{a}$ a 2π-periodic phase. For simplicity (apart from 2D), $\mathbf{E}_k$ and $\mathbf{A}_k$ are taken non-compact, with real eigenvalues. The Gauss's laws are $\nabla \cdot \boldsymbol{E}_k = e_k$ and $\nabla \cdot \boldsymbol{e} = 0$, where the derivatives denote lattice finite differences. For simplicity, we do not include any additional charged matter; including it does not affect the following discussion.

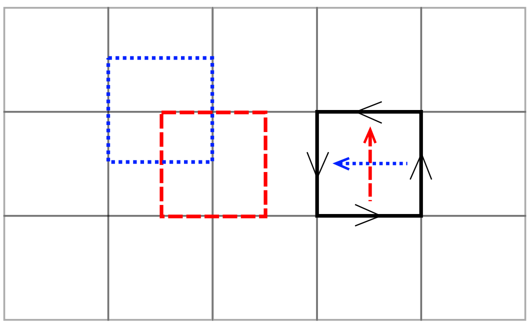

FIG. 1: Geometry of the lattice coupled vector gauge theory Hamiltonian (12). Gray solid lines represent the underlying square lattice. On the left, a single plaquette is shown for the x-lattice (dashed lines; red online) and the y-lattice (dotted lines; blue online). On the right, the thick solid line represents a loop of $\mathbf{e}$ electric field on a plaquette of the underlying lattice, with the arrows indicating the direction of $\mathbf{e}$. Because the e_k electric field lines carry gauge charge of the $\mathbf{E}_k$ electric field as expressed by the Gauss law (2), there are also necessarily lines of $\mathbf{E}_x$ (dotted line with arrow; blue online) and $\mathbf{E}_y$ (dashed line with arrow; red online) electric field.

We consider the following lattice Hamiltonian counterpart of (1):

$$\begin{aligned} H &= \frac{U_E}{2} \sum_{\ell \in L_k} E_{k\ell}^2 + \frac{U_e}{2} \sum_{\ell \in L} e_\ell^2 + \frac{K_E}{2} \sum_{\square_k} (\nabla \times \mathbf{A}_k)^2 \\ &- K_e \sum_{\square} \cos\left[(\nabla \times \mathbf{a})_\square + A_{xy}(\ell_x) - A_{yx}(\ell_y) \right]. \quad (12) \end{aligned}$$

Here, $k = x, y$ is summed over in those terms where it appears, and L and L_k are the sets of links in the underlying lattice and k-lattice, respectively. Similarly $\square$ and $\square_k$ denote plaquettes of the underlying lattice and the k-lattice. The expression $(\nabla \times \mathbf{a})_\square$ is the lattice line integral of $\mathbf{a}$ taken counterclockwise around the perimeter of the plaquette $\square$. To understand the last term, note that a single x-lattice link ℓ_x and single y-lattice link ℓ_y pass through the center of each plaquette $\square$, as shown on the left of Fig. 1. This term is thus a gauge-invariant operator that creates electric field configurations like that shown on the right of Fig. 1.

For the generalization to d dimensions, we start with an underlying hypercubic lattice, and d hypercubic k-lattices, with $k = 1, \ldots, d$, whose vertices are centered on k-directed links of the underlying lattice. The K_e operator is replaced with a term proportional to $\cos\left[(\nabla \times \mathbf{a})_\square + A_{ij} - A_{ji} \right]$, where $\square$ is one of the $d(d-1)/2$ types of plaquettes in the underlying lattice, bisected by the two perpendicular links ℓ_i and ℓ_j.

Considering the 3D case, Ref. [16] showed that one obtains the Hilbert space of the scalar-charge theory upon taking the limit $K_e \to \infty$. We take a different point of view, considering the phases of the Hamiltonian (12). When $U_e \gg K_e$, we put the $\mathbf{e}$ gauge field into its confining phase, with $\mathbf{e} \approx 0$, and electric field loops costing an energy proportional to their length. In this limit $\mathbf{e}$ can be integrated out and we wind up with d decoupled $\mathbf{E}_k, \mathbf{A}_k$ non-compact U(1) vector gauge theories.

Starting from this phase, we increase K_e, thus increasing the fluctuations of the $\mathbf{e}$ loops. After K_e is raised above a critical value, the $\mathbf{e}$ loops proliferate and condense. This is analogous to the p-string condensation in the coupled layer construction of the X-cube fracton model [14, 15], because each $\mathbf{e}$ loop is built from point-like charged particles of the $\mathbf{E}_k$ gauge fields. We can access the condensed phase by taking K_e large and expanding the cosine in the last term. The resulting Gaussian model is simply a lattice regularization of the continuum Hamiltonian (1), and is identical at low energy to the scalar-charge tensor gauge theory.

In $d = 2$ this picture breaks down, and – as for a single compact U(1) vector gauge theory [19] – there is only one phase. This occurs because when $\mathbf{e}$ is compact, we only have the two U(1) symmetries associated with conservation of $\nabla \times \mathbf{A}_k$ magnetic flux, with currents $J^m_{\mu k}$ discussed above. When $U_e \gg K_e$, we have two decoupled non-compact U(1) gauge theories in their deconfined phase, which is of course dual to the ordered phase of two decoupled XY models, where the two XY currents are $J^m_{\mu k}$. When $U_e \ll K_e$, naively one may expect to be able to expand the cosine to obtain the Hamiltonian (1). But, as we can see from the dual elasticity description, the resulting theory is only distinct from two decoupled XY models in the presence of the *third* j^m_μ U(1) symmetry, which is absent for compact $\mathbf{a}$ for any finite K_e/U_e. The absence of a phase transition is obvious on the elasticity side, corresponding to an incommensurate substrate that breaks rotational symmetry and thereby gaps out θ in (4), reducing to two XY models for u_x and u_y. Therefore there is only one phase in the $d = 2$ case. Alternatively we could have taken $\mathbf{e}$ to be non-compact, in which case we do have all three U(1) symmetries, but there is still only one phase.

These considerations have an interesting consequence for the $d = 2$ scalar-charge theory. Namely, if only the two $J^m_{\mu k}$ symmetries are present, there is no difference between this theory and two non-compact vector U(1) gauge theories. We note that these conclusions can be obtained working entirely on the lattice; that is, we can follow standard techniques to obtain a dual lattice theory of (12), which at the Gaussian level and in the continuum limit is identical to elasticity.

Generalization to vector-charge theory. We briefly describe a generalization of the above construction that reduces to a different fractonic symmetric-tensor gauge theory at low energy, the *vector*-charge theory [8]. In the vector-charge theory, as with the scalar-charge theory, the electric field ε_{ij} and gauge potential α_{ij} are symmetric tensors, but the Gauss' law constraint is different, given by $\partial_i \varepsilon_{ij} = \rho_j$, with the gauge charge ρ_j now carrying a vector index.

Focusing on 3D and working in the continuum for simplicity, we introduce a theory of coupled vector gauge theories that reduces to the vector-charge tensor gauge theory at low energy. We introduce *six* U(1) gauge fields, with electric fields $\mathbf{e}_k$ and $\mathbf{E}_k$ $(k = x, y, z)$, and corresponding vector potentials $\mathbf{a}_k$ and $\mathbf{A}_k$. We take the Gauss' law constraints to be

$$\partial_i e_{ik} = 0 \tag{13}$$
$$\partial_i E_{ik} = \epsilon_{kij} e_{ij}, \tag{14}$$

which express that the anti-symmetric components of e_{ij} act as sources of gauge charge for the $\mathbf{E}_k$ electric fields. These constraints are encoded in the Lagrangian density

$$\begin{aligned} \mathcal{L} &= -e_{ik}(\partial_t a_{ik} + \partial_i a_{0k} + \epsilon_{lik} A_{0l}) - E_{ik}(\partial_t A_{ik} + \partial_i A_{0k}) \\ &- \frac{C}{2}\mathbf{E}_k^2 - \frac{1}{2}(\nabla \times \mathbf{A}_k)^2 - \frac{K}{2}\mathbf{e}_k^2 - \frac{1}{2}\sum_{ij} b_{ij}^2, \end{aligned} \tag{15}$$

where $b_{ij} = \epsilon_{ikl}\partial_k a_{lj} + A_{ji} - \delta_{ij} A_{kk}$. In the Supplementary Material, we show that this theory reduces at low energy to the vector-charge theory, and discuss how to carry out the construction on the lattice. We also briefly remark on a generalization to the 2D vector charge theory. The Supplementary Material also discusses some further generalizations of the lattice coupled vector gauge theory construction of the scalar-charge theory, that reduce at low energy to the (m, n) theories of [20], and the version of the scalar-charge theory where the electric field is a traceless, symmetric tensor [8].

In summary, motivated by the fracton-elasticity duality[10, 11, 17], we utilized its reformulation to derive a fractonic coupled U(1) *vector* gauge theory representation in terms of $d + 1$-coupled gauge fields, where components of one type of electric field act as charges for the remaining d gauge theories. At low energies this vector description is identical to fractonic scalar-charge tensor gauge theory. We used a lattice version of this model to discuss fracton order in terms of proliferation of electric field loops. We also proposed a number of generalizations of this construction, making contact with fractonic tensor gauge theories that are not dual to elasticity.

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[21] In a fully nonlinear elasticity, the Hamiltonian is an expansion in the *nonlinear* strain tensor, $u_{ij} = \frac{1}{2}(\partial_i \mathbf{R} \cdot \partial_j \mathbf{R} - \delta_{ij}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i \mathbf{u} \cdot \partial_j \mathbf{u})$, that, by construction is a target-space $\mathbf{R}(r_i)$ scalar, i.e., invariant under rotation of $\mathbf{R} \to \mathbf{R}' = O \cdot \mathbf{R}$. This corresponds to $u'_k = (O_{kj} - \delta_{kj})x_j + O_{kj}u_j$, that, in a linear approximation (small rotation angle β) reduces to (11).