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# Emergent Periodic and Quasiperiodic Lattices on Surfaces of Synthetic Hall Tori and Synthetic Hall Cylinders

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Phys. Rev. Lett. **123**, 260405 — Published 30 December 2019

DOI: [10.1103/PhysRevLett.123.260405](https://doi.org/10.1103/PhysRevLett.123.260405)

# Emergent periodic and quasiperiodic lattices on surfaces of synthetic Hall tori and synthetic Hall cylinders

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(Dated: October 24, 2019)

Synthetic spaces allow physicists to bypass constraints imposed by certain physical laws in experiments. Here, we show that a synthetic torus, which consists of a ring trap in the real space and internal states of ultracold atoms cyclically coupled by Laguerre-Gaussian Raman beams, could be threaded by a net effective magnetic flux through its surface—an impossible mission in the real space. Such synthetic Hall torus gives rise to a periodic lattice in the real dimension, in which the periodicity of density modulation of atoms fractionalizes that of the Hamiltonian. Correspondingly, the energy spectrum is featured by multiple bands grouping into clusters with nonsymmorphic-symmetry-protected band crossings in each cluster, leading to swaps of wavepackets in Bloch oscillations. Our scheme allows physicists to glue two synthetic Hall tori such that localization may emerge in a quasicrystalline lattice. If the Laguerre-Gaussian Raman beams and ring traps were replaced by linear Raman beams and ordinary traps, a synthetic Hall cylinder could be realized and deliver many of the aforementioned phenomena.

Spaces with nontrivial topologies provide quantum systems unprecedented properties [1–7]. As a prototypical space of a finite genus, the importance of a torus in modern physics is more far-reaching than applying periodic boundary conditions (PBC) in theoretical calculations. It plays a crucial role in quantum Hall physics. The ground state of a fractional quantum Hall state becomes degenerate on a torus or any surface with a finite genus [8]. Such degeneracy, which is unavailable on a cylinder or a flat space and defines the concept of the topological order lays the foundation of topological quantum computation [9]. However, due to the absence of magnetic monopoles in nature, it is impossible to generate a net magnetic flux through a closed surface in the real space. The study of quantum Hall states on a torus has eluded experiments so far.

Ultracold atoms provide physicists a unique platform to engineer Hamiltonians and allow physicists to achieve many quantum Hall states unattainable in electronic systems [10], such as quantum Hall states of bosons and quantum Hall states with high spins. Other than the typical harmonic potentials, ring traps have been implemented in an annular geometry [11, 12]. Linear and Laguerre-Gaussian (LG) Raman beams have been used to create spin-momentum coupling and spin-angular momentum coupling, respectively. [13–17]. If one considers the internal degree of freedom as a synthetic dimension, the spin-momentum coupling gives rise to a synthetic magnetic field in a two-dimensional plane [18, 19]. Whereas experiments have been focusing on open boundary conditions in the synthetic dimension [20, 21], there have been theoretical proposals on creating a periodic or twisted boundary condition [22–27]. However, few experiments have fulfilled the requirements of these proposals [28, 29].

We propose a simple scheme to realize a synthetic torus

penetrated by a net effective magnetic flux. Ultracold atoms confined in a ring trap in the real space are subjected to spin-angular momentum coupling induced by LG Raman beams. Either hyperfine spins or nuclear spins could be used to enable a cyclic coupling and form a loop in the discrete synthetic dimension. Cyclic couplings have been studied for different purposes, including realizing two-dimensional spin-orbit coupling and creating Yang monopoles [30–32]. Here, we use spin-angular momentum coupling to synthesize internal states and the real dimension into a synthetic Hall torus. PBCs in both the synthetic and the real dimension deliver a torus. Spin-angular momentum coupling produces finite effective magnetic fluxes penetrating its toroidal surface, signifying the rise of a synthetic Hall torus. Replacing Laguerre-Gaussian Raman beams by linear ones, our scheme applies to ordinary traps with open boundary conditions for creating synthetic Hall cylinders, which have been realized by experiments recently [28, 29].

We further unfold unique properties of synthetic Hall tori and cylinders. Unlike previous works including optical lattices in the real dimension [20, 21, 24, 26–28], we consider a continuous real space trap. Interestingly, periodic or quasiperiodic lattices emerge in the continuous real dimension, as a result of PBC in the synthetic dimension. The periodic lattice modulates the density of atoms with a fractionalized periodicity of the Hamiltonian and a unique band structure shows up. Energy bands form clusters with nonsymmorphic-symmetry-protected band crossings in each cluster. Wavepackets in each cluster swap with each other in Bloch oscillations. Though each single synthetic Hall torus or cylinder supports only extend states, once two of them are glued together, quasiperiodic lattices may emerge and lead to localized states in the real space. Such “localization from gluing” demonstrates the power of synthetic Hall tori or cylinders

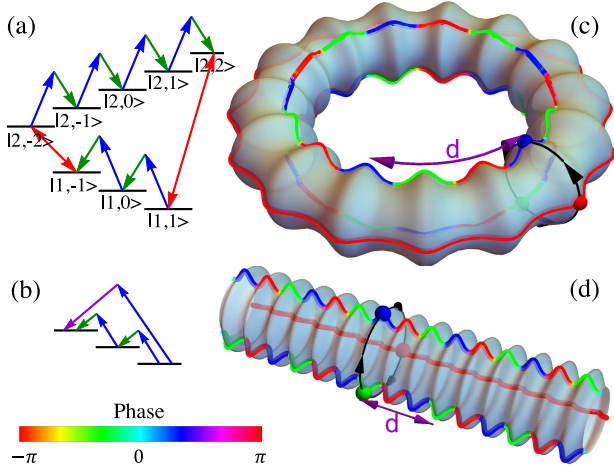


FIG. 1. (Color online) (a) Energy diagram for the hyperfine states and the laser coupling scheme. Blue and green arrows represent the Raman coupling. Bidirectional arrows represent microwave couplings. (b) Simplified coupling diagram for three internal states coupled by Raman beams. (c-d) Torus (Cylinder) formed by cyclically coupled three internal states in a real space ring trap. The density oscillation is depicted as the fluctuation of the radius of the torus or cylinder. Colored curves represent phases of the wavefunction of each spin component.

in accessing even more complex synthetic spaces and intriguing quantum phenomena there.

*Proposed scheme and Hamiltonian.* We consider  $M$  internal states in a real space ring trap. For alkali atoms, these  $M$  spins involves both  $F = 1$  and  $F = 2$ , as shown in Fig. 1(a). At weak magnetic fields, linear Zeeman splitting dominates, thus a single pair of LG Raman beams simultaneously couples every consecutive states within each manifold; Microwave fields couple  $|1,1\rangle$  ( $|1,-1\rangle$ ) and  $|2,2\rangle$  ( $|2,-2\rangle$ ). These eight states form a circle in the synthetic dimension. Due to the opposite  $g$  factors between  $F = 1$  and  $F = 2$ , a finite angular momentum transfer occurs once an atom finishes the loop in the synthetic dimension. A net effective magnetic flux emerges on the torus. Each hyperfine spin state has multiple angular momenta.

The number of internal states are controllable. On the one hand, at large magnetic fields, quadratic Zeeman splittings become important, and fewer spins can be separated out from the rest to form a smaller circle [26, 28–30, 32]. Note that shrinking the synthetic dimension does not change any results qualitatively, as fractional quantum Hall states can be adiabatically connected to 1D charge-density waves [33]. Chiral edge currents in the quantum Hall strips have also been observed using only three internal states [20, 21]. On the other hand, using the  $^1S_0$  and  $^3P_0$  states of Sr87 [34], one could cyclically couple up to 20 internal states.

Here, we consider nearest neighbors in the synthetic dimension coupled by LG Raman beams. The spin flip

from the  $j$ th spin state to the  $j + 1$ th one is thus accompanied by an angular momentum increase  $m_{j,j+1}$ , which is the difference between the angular momenta carried by the two LG beams. A microwave coupling then corresponds to  $m_{j,j+1} = 0$ .

We define the position  $x = \phi L / (2\pi)$  and the momentum  $p = 2\pi m / L$ , where  $\phi$  is the azimuthal angle and  $L$  is the circumference of the real space ring. We also define  $q_{j,j+1} = 2\pi m_{j,j+1} / L$  as the “momentum” transfer along the azimuthal direction. The advantage of the notation is that all results directly apply to a cylinder. In both the cylinder and the torus,  $x$  represents the direction in the real dimension. The Hamiltonian reads

$$H = \sum_{j=1}^M |\psi^j(x)\rangle \left( -\frac{\hbar^2}{2m_0} \partial_x^2 + \epsilon_j \right) \langle \psi^j(x)| + \sum_{j=1}^M (\Omega_{j,j+1} e^{iq_{j,j+1}x} |\psi^{j+1}(x)\rangle \langle \psi^j(x)| + h.c.), \quad (1)$$

where  $\psi^j(x)$  denote the spacial wave function for the  $j$ th spin,  $\epsilon_j$  the one (two) photon detuning in the microwave (Raman) transition,  $\Omega_{j,j+1}$  the coupling strength between the  $j$ th and the  $j + 1$ th spin state, and  $\psi^{M+1}(x) = \psi^1(x)$ . Whereas our results are very general and do not require tuning every single parameter arbitrarily, quadratic Zeeman splitting could create uneven energy separations and each pair of hyperfine spin states could be coupled by different lasers. Thus,  $\Omega_{j,j+1}$  can be, in principle, tuned independently. The total phase accumulated after an atom finishes a circle  $j \rightarrow j+1 \rightarrow j+2 \dots \rightarrow j-1 \rightarrow j$ ,  $\varphi(x) \equiv e^{iQx}$  is finite and spatially dependent, where  $Q \equiv \sum_{j=1}^M q_{j,j+1}$ . The total synthetic magnetic flux on the surface of the torus per unit length in the physical dimension is then propotional to  $Q$ .

*Nonsymmorphic symmetry and band structures.* We start from commensurate momentum transfers, i.e.,  $q_{j,j+1} = n_j q_L$ , where  $n_j$  are integers. For any coupling strengths  $\Omega_{j,j+1}$ , the reciprocal lattice vector  $q_L$  determines the periodicity of  $H(x)$ ,  $H(x) = H(x + \frac{2\pi}{q_L})$ . If one of these couplings vanishes, the phases,  $e^{iq_{j,j+1}x}$ , can be absorbed to  $|\psi^{j+1}(x)\rangle$ , and any spin component in an eigenstate contains a single plane wave. In contrast, when all  $\Omega_{j,j+1} \neq 0$ , the phases cannot be gauged away under such PBC. A single spin component in any eigenstate contains multiple plane waves and the densities form standing waves. The lattice in the real space is therefore an emergent one from the PBC in the synthetic dimension.

The Bloch wavefunctions  $\vec{\psi}_k(x)$  are simultaneous eigenstates of  $H$  and  $\hat{T}(d)$  [ $\hat{T}(d) \vec{\psi}_k(x) = e^{ikd} \vec{\psi}_k(x)$ ], where  $\vec{\psi}(x)$  is a  $M$ -component wavefunction,  $k$  the quasi-momentum, and  $d \equiv \frac{2\pi}{q_L}$  the lattice spacing,  $T(d)$  the translation operator of distance  $d$ . We define the nonsymmorphic symmetry operator  $\hat{G}$  as a combination of

a translation for a fraction of the lattice space  $T(2\pi/Q)$  in the real dimension and a unitary transformation  $U_s$  in the synthetic direction,

$$x \rightarrow x + \frac{2\pi}{Q}, \quad |\psi^{j>1}\rangle \rightarrow e^{-i\frac{2\pi}{Q} \sum_{j'=1}^{j-1} q_{j',j'+1}} |\psi^j\rangle. \quad (2)$$

Again, it is understood that  $M+1$  is equivalent to 1. A simple example of the nonsymmorphic symmetry is the glide-reflection symmetry, a translation for half of the lattice spacing combined with a reflection in the perpendicular direction, which has played an important role in topological quantum matters [35–37]. Consider a special case,  $\epsilon_j = 0$ ,  $\Omega_{j,j+1} = \bar{\Omega}$  and  $n_j = \bar{n}$ , the synthetic dimension becomes translational invariant in the real dimension.  $\hat{G}$  and its multiples, together with the translation in the synthetic dimension, then form the conventional magnetic translation group [38]. In generic cases where the synthetic dimension does not have translation invariance, i.e., nonuniform  $\epsilon_j$ ,  $\Omega_{j,j+1}$  or  $n_j$ .  $[H, \hat{G}] = 0$  is still satisfied and signifies a nonsymmorphic symmetry.

We define  $n \equiv Q/q_L = \sum_{j=1}^M n_j$ . Many physical quantities depend on  $n$ , i.e., properties of the system crucially rely on how the synthetic magnetic field is distributed on the surface of the torus, not just the total flux. Applying  $\hat{G}$  for  $n$  times is equivalent to a translation in the physical dimension for one lattice spacing,  $\hat{G}^n \psi_k(x) = e^{ikd} \psi_k(x)$ . Thus,

$$\hat{G} \vec{\psi}_k(x) = c_s \vec{\psi}_k(x), \quad c_s = e^{i(\frac{kd}{n} + \frac{2s\pi}{n})}, \quad (3)$$

where  $s = 1, 2, \dots, n-1, n$ . Equation (3) shows that, as the quasimomentum  $k$  changes by a reciprocal lattice vector  $q_L$ , the eigenvalue of  $\hat{G}$  changes by  $e^{2i\pi/n}$ , i.e., the  $s$ th eigenvalue becomes the  $s+1$  one. Meanwhile,  $\vec{\psi}_k(x) = \vec{\psi}_{k+q}(x)$  is satisfied. Thus, we conclude that bands must form clusters, each of which contains  $n$  bands. These  $n$  bands are the  $n$  eigenstates of the operator  $\hat{G}$  with the  $s$ th eigenvalue  $c_s$ , and intersect with each other within the Brillouin zone (BZ).

We solve  $H$  in Eq. (1) using plane-wave expansions. The band structure fully agrees with the prediction from the above symmetry considerations. Fig. 2(a) shows the energy bands when  $M = 3$ ,  $q_{1,2} = q_{2,3} = q_{3,1} = q$ , and coupling strength  $\Omega_{1,2} = 1.2E_R$ ,  $\Omega_{2,3} = 1.8E_R$ , and  $\Omega_{3,1} = 1.5E_R$ , where  $E_R = \hbar^2 Q^2 / 2m_0$  is the recoil energy defined by  $Q$ . Here, the reciprocal lattice vector  $q_L = q$  and  $n = Q/q_L = 3$ . The cylindrical or toroidal surface is penetrated by a uniform flux. Thus, three bands exist in each cluster. The eigenstate of the  $s$ th band is

$$\begin{aligned} \vec{\psi}_k(x) &= e^{i(k+sq)x} (u_k^1(x), u_k^2(x), u_k^3(x))^T, \\ u_k^j(x) &= e^{i(j-1)qx} \sum_{l=-\infty}^{\infty} c_l^j(k) e^{ilQx}, \end{aligned} \quad (4)$$

where  $u_k^j(x)$  is the periodic Bloch wavefunction of the  $j$ th spin state,  $l$  an integer, and  $c_l^j(k)$  determined by Eq. (1).

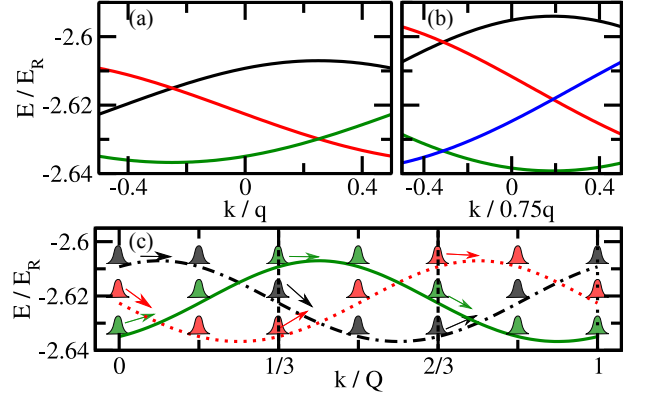


FIG. 2. Band structures when  $q_{1,2} : q_{2,3} : q_{3,1} = 1 : 1 : 1$  (a) and  $q_{1,2} : q_{2,3} : q_{3,1} = 1 : 1 : 2$  (b). (c): Bloch oscillation for (a). Vertical lines represent the boundaries of BZ.

The density of the  $j$ th spin state,  $\rho_k^j(x) \equiv |u_k^j(x)|^2$  satisfies

$$\rho_k^j(x) = \rho_k^j(x + \frac{2\pi}{Q}) = \rho_k^j(x + \frac{d}{3}). \quad (5)$$

The total density  $\rho(x) = \sum_j \rho_k^j(x)$ , by definition, also satisfy Eq. (5) [39]. Despite the continuous real dimension, the density of atoms oscillates with a period only  $1/3$  of that of the Hamiltonian, as shown in Fig. 1(c-d). In contrast, the relative phase between  $u_k^2(x)$  [ $u_k^3(x)$ ] and  $u_k^1(x)$  has a periodicity of  $d$ , as shown by the colored curves in Fig. 1(c-d). As previously discussed, such results crucially depend on the PBC. Both periodic density and phase oscillations vanish once the synthetic dimension has an open boundary condition.

*Swapping wave packets in Bloch oscillations.* When a constant force is applied, a wavepacket in the momentum space experiences a Bloch oscillation, which has exactly the same period of the Hamiltonian in an ordinary band structure. In contrast, the period of the Bloch oscillation here is given by  $3q$ , tripling the reciprocal lattice vector. Due to the presence of band crossings, a wavepacket does not return to the original band after the momentum changes by  $q$ . Instead, it swaps with another wavepacket from a different band.

The significance of such Bloch oscillations here is that it traces the Wilson lines, a path-ordered integral of non-abelian Berry connections in the momentum space [40, 41]. Due to the nonsymmorphic-symmetry-protected band crossings, abelian Berry connections no longer applies when studying topological properties of the band structure. Non-abelian Berry connections and the Wilson lines characterize how a quantum state changes to a different one while the Hamiltonian returns to the original one [42, 43], a prototypical non-abelian operation. This is precisely what we see from Fig. 2(c). If we label the bands as 1, 2, 3 from bottom to top based on the energies, the green wave packet initially at band-1 moves to

band-3, meanwhile the red (black) one initially at band-2 (band-3) moves to band-1 (band-2) when  $\Delta k = q$  [44]. When  $\Delta k = Q = 3q$ , these three wave packets swap with each other for six times, as shown in Fig. 2(c).

The above discussions can be directly generalized to other choices of  $\{q_{j,j+1}\}$ . If  $q_{1,2} = q_{2,3} = \frac{3}{4}q$ ,  $q_{3,1} = \frac{3}{2}q$ , though the total momentum transferred,  $Q$ , is still  $3q$ , the same as the previously discussed case,  $q_L$  becomes  $3q/4$  and  $n = 4$ . One thirds of the surface has a larger magnetic flux than the remaining region [44], and a cluster consists of four bands [Fig. 2(b)]. Changing the value of some of the wavevectors is equivalent to redistributing the magnetic flux on the surface, and leads to distinct band structures. We emphasize that the total number of states of the system remains unchanged. The change of the number of bands is associated with the change of BZ. In this example, the reciprocal lattice vector  $q'_L$  become  $\frac{3}{4}q$ . Shrinking the size of BZ then leads to an increase of the number of bands in a cluster.

*Quasiperiodic lattices.* If  $q_{j,j+1}$  are incommensurate, e.g.,  $q_{1,2} : q_{2,3} : \dots : q_{M,1} = 1 : 1 : \dots : \gamma$ , where  $\gamma = \frac{\sqrt{5}-1}{2}$ , a peculiar quasiperiodic lattice arises: certain quantities have well defined periodicities but others do not. For example, Eq. (4-4) holds for a generic  $\{q_{j,j+1}\}$  when  $M = 3$ . Though  $H$  in Eq. (1) is aperiodic, the density of each spin still satisfies  $\rho_k^j(x) = \rho_k^j(x + \frac{2\pi}{Q})$ . The wavefunction of each spin component is still extended, as its plane wave expansion only includes multiples of  $Q$ . In contrast, the relative phases between different spin component are spatially variant and are not commensurate. Thus, the wavefunction  $\psi_k(x)$  is aperiodic in the real dimension. Defining a pseudospin-1,  $S_\mu = \sum_{j,j'} u_k^{j*}(x) \mathcal{F}_\mu^{j,j'} u_k^j(x)$ , where  $j, j' = 1, 2, 3$ ,  $\mathcal{F}_\mu^{j,j'}$  are the spin-1 Pauli matrices, and  $\mu = x, y, z$ .  $S_z(x)$  is periodic,  $S_z(x) = S_z(x + 2\pi/Q)$ , but  $S_x(x)$  and  $S_y(x)$  do not have well defined periods, as shown in Fig. 3(a).

On a cylinder, there is no restriction on the choice of  $q_{j,j+1}$ . In contrast, an irrational ratio  $q_{j,j+1}/q_{j',j'+1}$  is not allowed on a torus, as the PBC in the real dimension require that all momentum scales are multiples of  $2\pi/L$ . Nevertheless, any irrational number can be approached by the ratio of two integers with increasing the integers' values. For instance,  $\gamma$  can be approximated by  $\gamma_\alpha = a_{\alpha-1}/a_\alpha$ , where  $\{a_\alpha\}$  is the Fibonacci series 1, 1, 2, 3, 5, ..., with increased accuracy. When the approximation order  $\alpha$  increases, the periodicity of  $S_{x,y}(x)$  increases.  $\gamma_\alpha$  with a small  $\alpha$  well reproduces the result for a small  $x$  and a large  $\gamma$ .

*Localization by gluing.* Our scheme can be implemented to access more complex synthetic spaces. For instance, adding extra couplings to the synthetic dimension is equivalent to gluing multiple tori or cylinders. Figure 3(b) shows that two tori or cylinders with  $M = 3$  can be glued to a single one with  $M = 4$ . In the real space, it is difficult to realize such gluing, as it is required

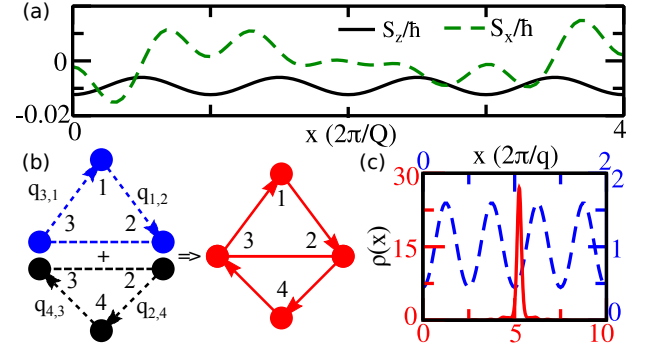


FIG. 3. (a) Spin polarization along the  $x$  and  $z$  direction as a function of  $x$ . (b) Schematic of the cross sections of two tori or cylinders when they are glued together. (c) Total density of the ground eigenstate at zero quasimomentum before (dashed line, only the top blue torus or cylinder is shown) and after (solid line) the gluing.  $q_{1,2} = q_{3,1} = q$ ,  $q_{2,3} = 0$ ,  $q_{2,4} = q_{4,3} = \gamma q$ , and all couplings  $\Omega_{j,j'}$  are  $2E_r$ , where  $E_r = \hbar^2 q^2 / 2m_0$  is the recoil energy defined by  $q$ .  $\gamma_8 = 13/21$  has been used as an approximation of  $\gamma = (\sqrt{5} - 1)/2$ .

to identify certain parts of two different objects. Using the synthetic dimension, adding an additional tunneling through the interior of single tori or cylinder with  $M = 4$ ,  $\Omega' = \Omega_{2,3}$ , immediately realizes this gluing and delivers a system with a different topology. Though each single torus or cylinder supports only extended eigenstates, after gluing them together, eigenstates at low energies could become localized, as shown in Fig. 3(c) [45]. The wavefunction of a single spin component now includes multiple momentum scales,  $q_{1,2}$ , and  $q_{4,3}$ , unlike a single torus or cylinder case where only  $Q$  is relevant. The interference of plane waves with incommensurate wave vectors could thus potentially localize the wavefunction.

To quantitatively characterizes the localization, we compute the width of the lowest band as a function of  $\Omega'$ . It has been shown that the ground band width scales with  $a_\alpha^{-2}$  for extended states, and decays much faster for localized states [46, 47]. Here, the scaled band width almost vanish at an intermediate value of  $\Omega'$ , where eigenstates are localized, as shown in Fig. 4. When  $\Omega'$  is very small, the wavefunction is still extended, similar to a single torus without  $\Omega'$ . For large  $\Omega'$ , dominating contributions to the wavefunction come from only two hyperfine spin states [2 and 3 in Fig. 3(b)] such that the incommensurate wave vectors are no longer relevant and the wavefunction is still extended.

The localization can also be characterized by the expansion of an initially localized wavepacket with a width  $\sigma_0$  in the real space. We consider a Gaussian wave packet as the initial state. For small or large  $\Omega'$ , where the eigenstates at low energies are delocalized, the width of the wave packet,  $\sigma$ , increases quickly. In contrast,  $\sigma$  grows much slower in the localized regime. To further consider interaction effects, we numerically solve a time-

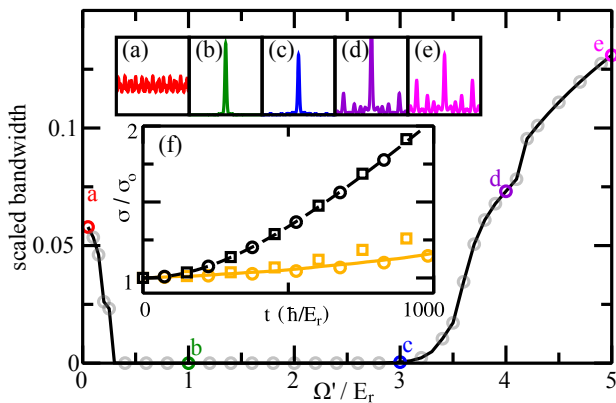


FIG. 4. Bandwidth multiplied by  $(a_\alpha/E_r)^2$  as a function of  $\Omega'$  for  $\alpha = 12$ . The inset (a), ..., (e) show the corresponding zero quasi-momentum ground state at  $\Omega'/E_r = 0.05, 1, 3, 4, 5$ , respectively with  $\alpha = 8$ . (f) The width of the Gaussian wave packet as a function of time. The solid orange (black dashed) curve represents non-interacting results for  $\Omega' = 2E_r$  ( $\Omega' = 0$ ). Squares and circles represent results for  $\int g\rho_i^2 dx = \pm 0.003E_r$ , respectively, where  $\rho_i$  is the density for the initial state.

dependent Gross-Pitaevskii equation,

$$i\hbar \frac{\partial \vec{\psi}(x)}{\partial t} = (\hat{H} + g\rho)\vec{\psi}(x), \quad (6)$$

where  $g$  is the interaction strength. We find that a weak repulsive (attractive) interaction slightly enhance (suppress) the dynamics. In addition, we have considered the ground state of interacting bosons in the trap. The main conclusions remain unchanged [44].

We have shown that atom-laser interactions allow physicists to synthesize Hall tori and cylinders hosting intriguing quantum phenomena in the emergent periodic and quasiperiodic lattices. We hope that our results will stimulate more works on synthetic spaces so as to explore physics that are not easy to access in conventional traps.

This work is supported by startup funds from Purdue University.

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