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All-Gaussian universality and fault tolerance with the Gottesman-Kitaev-Preskill code

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The Gottesman-Kitaev-Preskill (GKP) encoding of a qubit within an oscillator is particularly appealing for fault-tolerant quantum computing with bosons because Gaussian operations on encoded Pauli eigenstates enable Clifford quantum computing with error correction. We show that applying GKP error correction to Gaussian input states, such as vacuum, produces distillable magic states, achieving universality without additional non-Gaussian elements. Fault tolerance is possible with sufficient squeezing and low enough external noise. Thus, Gaussian operations are sufficient for fault-tolerant, universal quantum computing given a supply of GKP-encoded Pauli eigenstates.

Introduction—The promise of a quantum computer lies in its ability to dramatically outpace classical computers for certain tasks [1]. Computation using operations restricted to Pauli-eigenstate preparation, Clifford transformations, and Pauli measurements—henceforth referred to as *Clifford quantum computing (QC)*—cannot outperform classical computation since it is efficiently simulable on a classical computer [2]. Universal quantum computation requires supplementing Clifford QC by a non-Clifford resource—that is, a preparation, gate, or measurement that is not an element of Clifford QC.

In the presence of noise, universality is not enough. The celebrated Threshold Theorem [3] proves that given low enough physical noise, quantum error correction can be used to reduce logical noise to arbitrarily low levels—a property called *fault tolerance* [4]. Fortunately, Clifford QC provides all the necessary tools for quantum error correction. The question then is how to augment Clifford QC such that the result is both universal and fault tolerant. One way is to use a non-Pauli eigenstate, referred to as a *magic state* [5].

The continuous-variable (CV) analog of Clifford QC is *Gaussian QC*, which includes Gaussian state preparation, Gaussian operations (i.e., Hamiltonians quadratic in $\hat{a}, \hat{a}^\dagger$), and homodyne detection. CV systems arise naturally in many quantum architectures, including optical modes [6–9], microwave-cavity modes [10–13], and vibrational modes of trapped ions [14]. Gaussian QC lends itself to optics because the nonlinearities required are limited and of low order and because homodyne detection is very high efficiency. However, Gaussian QC is efficiently simulable by a classical computer [15] and requires any single non-Gaussian resource (preparation, gate, or measurement) for universal QC [16–18]. Further, Gaussian QC alone is insufficient to correct Gaussian noise [19].

Fault tolerance requires discrete quantum information. Bosonic quantum error-correcting codes (*bosonic codes* for short) embed discrete quantum information into CV systems in a way that maps CV noise into effective logical noise acting on the encoded qubits [20–23]. Such codes are promising for fault-tolerant computation [24, 25] due to the built-in redundancy afforded by their infinite-dimensional Hilbert space. High precision controllability of optical-cavity [10, 12, 13] and vibrational [14] modes further enhances their appeal. With a bosonic code, one may define *logical-Clifford QC*, comprising encoded Pauli eigenstates and logical-Clifford operations—allowing error correction at the encoded-qubit level. This, too, is efficiently simulable and thus requires additional logical-non-Clifford resources for fault-tolerant universality.

The Gottesman-Kitaev-Preskill (GKP) encoding of a qubit into an oscillator [21] is currently experiencing significant theoretical [26–29] and experimental [14, 30] interest due to its favorable error-correction properties [31], integration into scalable CV cluster states for measurement-based QC [32–34], and all-Gaussian Clifford gates and measurements. That is, the GKP encoding is the only known bosonic code for which logical-Clifford QC and error correction require only Gaussian QC along with a supply of logical-Pauli eigenstates, which are non-Gaussian [58]. Until now, fault-tolerant universal QC with the GKP code has required an additional non-Gaussian element—cubic phase gate, cubic phase state, or logical magic state [21, 35, 36]. In this Letter we show that no such additional non-Gaussian element is required.

Specifically, we show that high-quality magic states for both square- and hexagonal-lattice GKP codes [22] can be produced by applying GKP error correction to vacuum or low-temperature thermal states. The result is that Gaussian QC and just one type of non-Gaussian resource—a high-quality GKP Pauli eigenstate—suffice for both universality and fault tolerance.

Notation and conventions—Here we define notation and conventions to be used throughout this Letter. We define position $\hat{q} \coloneqq \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ and momentum $\hat{p} \coloneqq \frac{-i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$ for any mode $\hat{a}$. This means

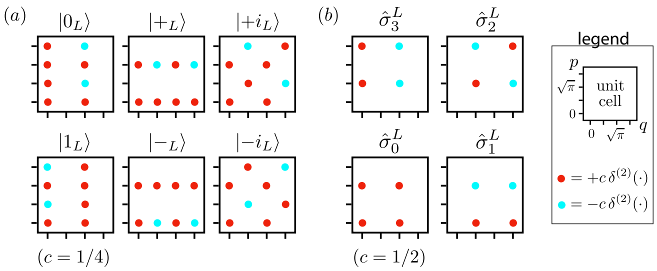


FIG. 1: Wigner-function representations of the square-lattice GKP (a) Pauli eigenstates and (b) logical Pauli operators in a single unit cell of phase space with dimensions $(2\sqrt{\pi}) \times (2\sqrt{\pi})$. The states are normalized to 1 over one unit cell, which determines the coefficients c.

$[\hat{q}, \hat{p}] = i$, with a vacuum variance of $\frac{1}{2}$ in each quadrature and $\hbar = 1$.

The Weyl-Heisenberg displacement operators $\hat{X}(s) \coloneqq e^{-is\hat{p}}$ and $\hat{Z}(s) \coloneqq e^{is\hat{q}}$ displace a state by $+s$ in position and momentum, respectively. For brevity, we also define a joint displacement $\hat{V}(\mathbf{s}) \coloneqq \hat{Z}(s_p)\hat{X}(s_q)$, where $\mathbf{s} = (s_q, s_p)^{\mathrm{T}}$.

The functions $\psi(s) \coloneqq {}_q\langle s|\psi\rangle$ and $\tilde{\psi}(s) \coloneqq {}_p\langle s|\psi\rangle$ denote position- and momentum-space wave functions for a state $|\psi\rangle$, respectively (tilde indicates momentum space). Any function, including wave functions, can be evaluated with respect to position, $\varphi(\hat{q}) \coloneqq \int \mathrm{d}s\, \varphi(s) |s\rangle_{q\,q}\langle s|$, to produce an operator diagonal in the position basis—and similarly for momentum. Finally, we define $Ш_T(x) \coloneqq \sum_{n\in\mathbb{Z}} \delta(x - nT)$ as a Dirac comb with spacing T.

The GKP encoding—In the original square-lattice GKP encoding [21], the wave functions for the logical basis states $\{|0_L\rangle, |1_L\rangle\}$ are Dirac combs in position space with state-dependent offset: $\psi_{j,L}(s) = Ш_{2\sqrt{\pi}}(s - j\sqrt{\pi})$ for $j \in \{0, 1\}$. Their momentum-space wave functions are also Dirac combs but with no offset, different spacing, and a relative phase between the spikes: $\tilde{\psi}_{j,L}(s) = \frac{1}{\sqrt{2}}(-1)^{js/\sqrt{\pi}} Ш_{\sqrt{\pi}}(s)$. Note that the momentum-space spikes for $|1_L\rangle$ alternate sign, and those for $|0_L\rangle$ are uniform.

GKP logical operators $\hat{X}_L$ and $\hat{Z}_L$ are implemented by displacements $\hat{X}(\sqrt{\pi})$ and $\hat{Z}(\sqrt{\pi})$, respectively, while displacements by integer multiples of $2\sqrt{\pi}$ in either quadrature leave the GKP logical subspace invariant. For later use, we define the four GKP-encoded logical Paulis

$$\hat{\sigma}_L^\mu \coloneqq \sum_{jk} \sigma_{jk}^\mu \, |j_L\rangle\langle k_L| \,, \tag{1}$$

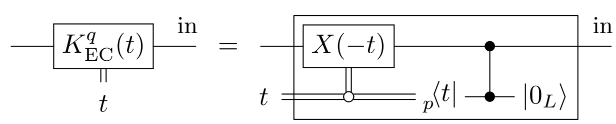
where σ_{jk}^μ is the jk'th element of Pauli matrix $\boldsymbol{\sigma}^\mu$ (with $\boldsymbol{\sigma}^0 = \mathbf{I}$). Note that $\hat{\sigma}_L^\mu$ have support only on the GKP logical subspace, while $\hat{X}_L$ and $\hat{Z}_L$ have full support and act both within and outside of the GKP subspace. We denote the (rank-two) projector onto the square-lattice GKP logical subspace [21, 37]

$$\hat{\Pi}_{\mathrm{GKP}} \coloneqq \hat{\sigma}_L^0 = \tilde{\psi}_{0,L}(\hat{q})\tilde{\psi}_{0,L}(\hat{p}) = \tilde{\psi}_{0,L}(\hat{p})\tilde{\psi}_{0,L}(\hat{q}). \tag{2}$$

We assume that the physical GKP Pauli eigenstates used in the following analysis are high quality enough to enable fault-tolerant GKP Clifford QC. This allows us to approximate them as ideal states with noiseless Cliffords for the purpose of magic-state preparation [5, 38]. We justify this in the penultimate section.

Kraus operator for GKP error correction—In its original formulation [21], *GKP error correction* is a quantum operation designed to correct an encoded qubit that has acquired some noise (leakage of its state outside of the logical subspace) by projecting it back into the GKP logical subspace, possibly at the expense of an unintended logical operation. Standard implementations of error correction strive to avoid these unintended logical operations (residual errors). In what follows, we apply the machinery of GKP error correction to a known Gaussian state, which means the outcome-dependent final state is known perfectly.

GKP error correction [21, 39] proceeds in two steps: First, one quadrature is corrected, then the conjugate quadrature. We define the Kraus operator that corrects just the q quadrature $\hat{K}_{\mathrm{EC}}^q(t)$ via the circuit (read right to left):



where the controlled operation is $\hat{\mathrm{C}}_Z = e^{i\hat{q}\otimes\hat{q}}$, and $t \in \mathbb{R}$ is the measurement outcome. This circuit differs from the original [21] in that the correction here is a negative displacement by t rather than by t rounded to the nearest integer multiple of $\sqrt{\pi}$. The outputs may differ by a logical operation $\hat{X}(\pm\sqrt{\pi})$, but this is unimportant because the input state is known.

Direct evaluation shows $\hat{K}_{\mathrm{EC}}^q(t) = \tilde{\psi}_{0,L}(\hat{q})\hat{X}(-t)$. A similar calculation shows that the Kraus operator for correcting the p quadrature is $\hat{K}_{\mathrm{EC}}^p(t) = \tilde{\psi}_{0,L}(\hat{p})\hat{Z}(-t)$. Applying both corrections (in either order since they commute up to a phase) performs full GKP error correction:

$$\hat{K}_{\mathrm{EC}}(\mathbf{t}) = \hat{K}_{\mathrm{EC}}^p(t_p)\hat{K}_{\mathrm{EC}}^q(t_q) = \hat{\Pi}_{\mathrm{GKP}}\hat{V}(-\mathbf{t}), \tag{3}$$

with measurement outcomes $\mathbf{t} = (t_q, t_p)^{\mathrm{T}}$. This Kraus operator (i) displaces the state by an outcome-dependent amount, $\hat{V}(-\mathbf{t})$, and then (ii) projects it back into the GKP logical subspace with $\hat{\Pi}_{\mathrm{GKP}}$ [40].

Applying $\hat{K}_{\mathrm{EC}}(\mathbf{t})$ to an input state $\hat{\rho}_{\mathrm{in}}$ produces the unnormalized state $\hat{\bar{\rho}}(\mathbf{t}) = \hat{K}_{\mathrm{EC}}(\mathbf{t})\hat{\rho}_{\mathrm{in}}\hat{K}_{\mathrm{EC}}^\dagger(\mathbf{t})$, where the bar indicates lack of normalization. The joint probability density function (pdf) for the outcomes, $\mathrm{pdf}(\mathbf{t}) = \mathrm{Tr}[\hat{\bar{\rho}}(\mathbf{t})]$, normalizes the output state: $\hat{\rho}(\mathbf{t}) = \hat{\bar{\rho}}(\mathbf{t}) / \mathrm{pdf}(\mathbf{t})$.

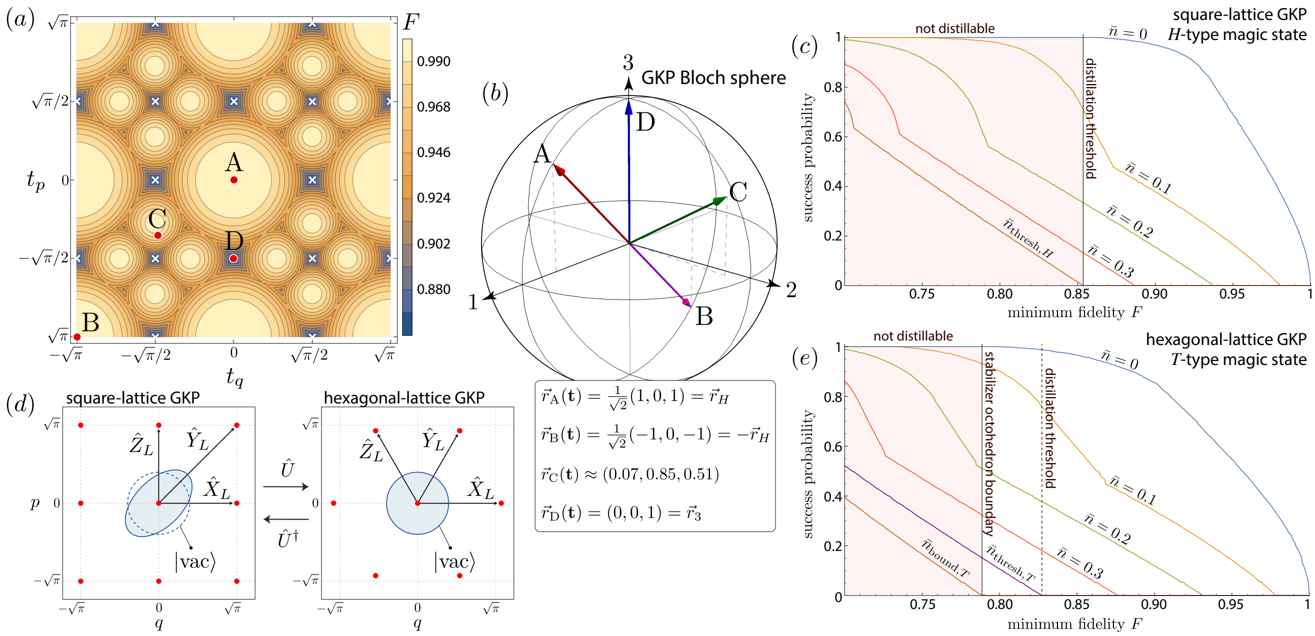


FIG. 2: (a) GKP error correction of the vacuum: outcome-dependent fidelity F with the nearest H-type magic state. The outcomes that do not yield a distillable magic state are marked with a white "x" (these yield GKP Pauli eigenstates). Bloch vectors for the representative outcomes (A–D) are shown on the GKP Bloch sphere in (b). (c) Probability of producing an H-type resource state of at least fidelity F using a thermal state of mean occupation $\bar{n}$. Resource states with fidelity higher than the distillation threshold, $F > 0.853$—*i.e.* outside the stabilizer octohedron—can be distilled into higher-quality $|{+H_L}\rangle$ states [41, 42]. Distillation is possible for $\bar{n} < \bar{n}_{\mathrm{thresh},H} = 0.366$. (d) $\hat{U}$ maps logical square-lattice GKP states to equivalent logical states in the hexagonal-lattice GKP encoding [31]. A vacuum state on the hexagonal lattice (1-σ error ellipse shown in blue) is mapped to a squeezed state on the square lattice under $\hat{U}^\dagger$. (e) Probability of producing a resource state distillable to $|{+T_L^{\mathrm{hex}}}\rangle$ of at least fidelity F by performing GKP$^{\mathrm{hex}}$ error correction on a thermal state of mean occupation $\bar{n}$. Resource states whose Bloch vectors lie on or within the stabilizer octohedron ($F \leq 0.789$) cannot be distilled, which occurs at $\bar{n}_{\mathrm{bound},T} = 0.468$. For T states, a distillation threshold has been proven for $F > 0.8273$ [43], which occurs for $\bar{n}_{\mathrm{thresh},T} = 0.391$.

Bloch vector for the error-corrected state—Using the logical basis in Eq. (1) we represent the output state $\hat{\rho}(\mathbf{t}) = \frac{1}{2}\sum_\mu r_\mu(\mathbf{t})\hat{\sigma}_L^\mu$ by a 4-component Bloch vector $\mathbf{r}(\mathbf{t})$ with outcome-dependent coefficients $r_\mu(\mathbf{t}) \coloneqq \mathrm{Tr}[\hat{\rho}(\mathbf{t})\hat{\sigma}_L^\mu]$. For the unnormalized state, $\bar{r}_0(\mathbf{t}) = \mathrm{pdf}(\mathbf{t})$, and for the normalized state, $r_0(\mathbf{t}) = 1$. In what follows, we use the notation $\mathbf{r} = (r_0, \vec{r})$, where $\vec{r}$ is the ordinary (3-component) Bloch vector within $\mathbf{r}$.

We employ the Wigner functions for the logical basis states [21], shown in Fig. 1(a), to find the Wigner functions for the GKP-encoded Pauli operators and the GKP logical identity, Eq. (1). Their explicit form is

$$W_{\sigma_L^\mu}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \frac{(-1)^{\mathbf{n}\cdot\bar{\boldsymbol{\ell}}_\mu}}{2}\delta^{(2)}\left[\mathbf{x} - \left(\mathbf{n} + \frac{\boldsymbol{\ell}_\mu}{2}\right)\sqrt{\pi}\right], \quad (4)$$

where $\mathbf{x} = (q,p)^{\mathrm{T}}$, $\boldsymbol{\ell}_0 = (0,0)^{\mathrm{T}}$, $\boldsymbol{\ell}_1 = (1,0)^{\mathrm{T}}$, $\boldsymbol{\ell}_2 = (1,1)^{\mathrm{T}}$, $\boldsymbol{\ell}_3 = (0,1)^{\mathrm{T}}$, and $\bar{\boldsymbol{\ell}}_\mu$ is just $\boldsymbol{\ell}_\mu$ with its entries swapped. The Wigner functions are shown in Fig. 1(b).

Since $\hat{\Pi}_{\mathrm{GKP}}\hat{\sigma}_L^\mu\hat{\Pi}_{\mathrm{GKP}} = \hat{\sigma}_L^\mu$, we skip the projection using $\hat{\Pi}_{\mathrm{GKP}}$ and directly calculate the unnormalized Bloch-vector components from the overlap of the unnormalized error-corrected state $\hat{\bar{\rho}}(\mathbf{t})$ with the logical Paulis. We find the overlaps in the Wigner representation:

$$\begin{aligned}\bar{r}_\mu(\mathbf{t}) &= \mathrm{Tr}[\hat{\bar{\rho}}(\mathbf{t})\hat{\sigma}_L^\mu] = \mathrm{Tr}[\hat{V}(-\mathbf{t})\hat{\rho}_{\mathrm{in}}\hat{V}^\dagger(-\mathbf{t})\hat{\sigma}_L^\mu] \\ &= 2\pi \iint d^2\mathbf{x}\, W_{\mathrm{in}}(\mathbf{x}+\mathbf{t})W_{\sigma_L^\mu}(\mathbf{x}),\end{aligned} \quad (5)$$

where $W_{\mathrm{in}}(\mathbf{x})$ is the Wigner function of the input state $\hat{\rho}_{\mathrm{in}}$. Note that $\bar{r}_0(\mathbf{t}) = \mathrm{Tr}[\hat{\bar{\rho}}(\mathbf{t})] = \mathrm{pdf}(\mathbf{t})$, which is normalized over a unit cell of size $(2\sqrt{\pi}) \times (2\sqrt{\pi})$ (since the full pdf is periodic). The normalized Bloch 4-vector is $\mathbf{r}(\mathbf{t}) \coloneqq \bar{\mathbf{r}}(\mathbf{t})/\bar{r}_0(\mathbf{t})$.

GKP error correction of Gaussian states—In what follows, we apply GKP error correction to a general Gaussian state—i.e., an input state whose Wigner function is $W_{\mathrm{in}}(\mathbf{x}) = G_{\mathbf{x}_0,\boldsymbol{\Sigma}}(\mathbf{x})$, where $G_{\mathbf{x}_0,\boldsymbol{\Sigma}}$ is a normalized Gaussian with mean vector $\mathbf{x}_0$ and covariance matrix $\boldsymbol{\Sigma}$.

Equation (5) can be evaluated analytically when the input state is Gaussian:

$$\bar{r}_\mu(\mathbf{t}) = \frac{1}{4\pi}\left[G_{\mathbf{0},(4\pi\boldsymbol{\Sigma})^{-1}}(\mathbf{v})\right]^{-1}\Theta\left(\mathbf{v} + \frac{\bar{\boldsymbol{\ell}}_\mu}{2}, \boldsymbol{\tau}\right), \quad (6)$$

where $\boldsymbol{\tau} = \frac{i}{2}\boldsymbol{\Sigma}^{-1}$, $\mathbf{v} = \boldsymbol{\tau}\left[\frac{1}{2}\boldsymbol{\ell}_\mu - \frac{1}{\sqrt{\pi}}(\mathbf{x}_0 + \mathbf{t})\right]$, and the Riemann (a.k.a. Siegel) theta function is defined as

$\Theta(\mathbf{z},\boldsymbol{\tau}) := \sum_{\mathbf{m}\in\mathbb{Z}^n} \exp\left[2\pi i\left(\frac{1}{2}\mathbf{m}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{m} + \mathbf{m}^{\mathrm{T}}\mathbf{z}\right)\right]$ for $\boldsymbol{\tau} \in \mathbb{H}_n$. The set $\mathbb{H}_n$ denotes the Siegel upper half space—i.e., the set of all complex, symmetric, $n \times n$ matrices with positive-definite imaginary part (see Ref. [44], for example). The overall coefficient $\frac{1}{4\pi}$ ensures that $\mathrm{pdf}(\mathbf{t})$ is normalized over a single unit cell.

GKP magic states from error correction—GKP error correction of a Gaussian state yields a known, random state encoded in the GKP logical subspace. Unless that state is highly mixed or too close to a logical Pauli eigenstate, it can be used as a (noisy) magic state along with GKP Clifford QC for fault-tolerant universal QC [5]. Reference [37] suggested coupling a vacuum mode to an external qubit to perform GKP error correction and then postselecting an outcome close to $\mathbf{t} \approx \mathbf{0}$ to produce a logical H-type state [5]. In fact, neither postselection nor interaction with a material qubit is required.

With access to a supply of $|0_L\rangle$ states, there is no need for any resources beyond Gaussian QC, since nearly any outcome $\mathbf{t}$ from applying GKP error correction to the vacuum state produces a distillable H-type magic state [5, 41], as shown in Fig. 2(a). This is because there are 12 H-type magic states (all related by Cliffords to $|{+H_L}\rangle$), and any of them will do the job [5]. The relevant quantity is the fidelity F to the closest H-type state [41]. Without loss of generality, assume this is $|{+H_L}\rangle$, whose Bloch 3-vector is $\vec{r}_H = \frac{1}{\sqrt{2}}(1, 0, 1)$. (If not, apply GKP Cliffords until it is.) Then, $F = \langle +H_L|\,\hat{\rho}(\mathbf{t})\,|{+H_L}\rangle = \frac{1}{2}[1 + \vec{r}_H \cdot \vec{r}(\mathbf{t})]$. States of sufficient fidelity can be twirled onto the H_L-axis, depolarized to make them identical, and then distilled [5, 42].

Input-state purity is not required either. Applying GKP error correction to a thermal state also produces a distillable mixed state with nonzero probability as long as its mean occupation number $\bar{n} < 0.366 =: \bar{n}_{\mathrm{thresh},H}$; see Fig. 2(c). (A thermal state is Gaussian with $\mathbf{x}_0 = \mathbf{0}$ and $\boldsymbol{\Sigma} = (\bar{n} + \frac{1}{2})\mathbf{I}$, which we plug into Eq. (6) to produce this plot.) Most high-purity, Gaussian states can be GKP-error corrected into a distillable magic state because most states do not preferentially error correct to a Pauli eigenstate. For the vacuum, $\mathrm{pdf}(\mathbf{t})$ is always between 0.066 and 0.094—*i.e.*, all outcomes, and thus a wide variety of states, are roughly equally likely.

Hexagonal-lattice GKP code—Our results can be extended to the hexagonal-lattice GKP code [31] by simply modifying the Gaussian state to be error corrected as follows. Define $\hat{U}$ as the Gaussian unitary such that $\hat{U}\,|\psi_L^{\mathrm{square}}\rangle = |\psi_L^{\mathrm{hex}}\rangle$, where the logical state is the same although the encoding differs. Let $\hat{\rho}$ be a Gaussian state to be GKP error corrected using the hexagonal lattice, with $\mathbf{x}_0 = \mathbf{0}$ and covariance $\boldsymbol{\Sigma}$. Then, the equivalent state to be GKP error corrected on the square lattice is $\hat{\rho}'_{\mathrm{in}} = \hat{U}^\dagger \hat{\rho}_{\mathrm{in}} \hat{U}$, which is Gaussian with $\mathbf{x}_0 = \mathbf{0}$ and covariance $\boldsymbol{\Sigma}' = \mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-\mathrm{T}}$ [45], where $\mathbf{S} = (2\sqrt{3})^{-\frac{1}{2}}\begin{pmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{pmatrix}$. This mapping is shown for $\hat{\rho}_{\mathrm{in}} = |\mathrm{vac}\rangle\langle\mathrm{vac}|$ in Fig. 2(d).

Using this mapping, we can get results for hexagonal-lattice GKP error correction by reusing Eq. (6) with the modified state. Vacuum and thermal states are biased towards the xz-plane of the Bloch sphere in the square-lattice encoding but unbiased with respect to all three Pauli axes in the hexagonal-lattice encoding. Thus, in Fig. 2(e), we plot the fidelity of hexagonal-lattice GKP error correction of a thermal state with T-type magic states [5] such as $|{+T_L^{\mathrm{hex}}}\rangle$, which has Bloch 3-vector $\vec{r}_T = \frac{1}{\sqrt{3}}(1, 1, 1)$.

Error correction as heterodyne detection—We note that an alternate description of what we are proposing is to perform heterodyne detection (measurement in the coherent-state basis) on half of a GKP-encoded Bell pair. This is similar to what GKP proposed [21], but with photon counting replaced by heterodyne detection, which is Gaussian. To see this, note that a Bell state can be written (ignoring normalisation) as $\sum_{\mu\nu} \eta_{\mu\nu} \hat{\sigma}_L^\mu \otimes \hat{\sigma}_L^\nu$, where $\boldsymbol{\eta} = \mathrm{diag}(1, -1, -1, -1)$. A coherent-state measurement on the first mode with outcome α produces $\sum_{\mu\nu} \eta_{\mu\nu} \operatorname{Tr}(|\alpha\rangle\langle\alpha|\,\hat{\sigma}_L^\mu)\hat{\sigma}_L^\nu$ on the second mode, which agrees with Eq. (5) using $\hat{\rho}_{\mathrm{in}}$ as vacuum and $\mathbf{t} = -\sqrt{2}(\operatorname{Re}\alpha, \operatorname{Im}\alpha)^{\mathrm{T}}$ but with opposite sign of the resulting $\vec{r}$. Intuitively, this is just Knill-type error correction [46], which involves teleporting the state to be corrected through an encoded Bell pair and reinterpreting vacuum teleportation as heterodyne detection.

Noise and imperfections—We employ ideal GKP Clifford QC in our analysis because the fidelity requirements for fault-tolerant Clifford QC (our actual assumption) are orders of magnitude stricter than those for magic-state distillation [5], so small additional noise will not qualitatively change our main result [38]. Using finite-precision GKP states for error correction causes uncertainty in the measurement outcome $\mathbf{t}$ [39], which can be modeled as additive Gaussian noise on the input state—i.e., by replacing $\bar{n} \mapsto \bar{n} + \Delta\bar{n}$, where $\Delta\bar{n}$ equals the $\hat{q}$- or $\hat{p}$-variance of an individual GKP spike. $\Delta\bar{n}$ is between 0.05 and 0.016 for 10- to 15-dB GKP states ($\Delta\bar{n} = \frac{1}{2}10^{-(\#\mathrm{dB})/10}$) [24], and fault tolerance is unlikely to be possible with lower-quality states than these [24, 47, 48]. Since $\bar{n}_{\mathrm{thresh}} > 0.36$, this additional noise is qualitatively unimportant to our main result. Furthermore, the resulting magic states will be the same quality as the GKP ancillas [39], which are sufficient for fault tolerance by assumption. Having established our result's robustness to imperfections, we leave detailed elaboration to further work.

Discussion—We have deployed GKP error correction in a nonstandard way to extract the magic from easy-to-prepare Gaussian states that extend into the "wilderness space" outside a bosonic code's logical subspace. The wilderness space may be rich in other resources, too—e.g., providing the means to produce other logical states or perform logical operations more easily than would be possible by restricting to the logical subspace. This feature is likely to extend beyond GKP to other bosonic codes such as rotation-symmetric codes [23] (including cat [49, 50] and binomial [51] codes), bosonic subsystem codes [52], and multi-mode codes [22, 31, 53].

Our result is an example of the fact that two efficiently simulable subtheories (GKP Clifford QC and Gaussian QC here) together can contain all the ingredients for universality. For qubits this is straightforward [54, 55]: combine Clifford QC from different Pauli frames since stabilizer states of one are magic states for the other. In CV systems, dual-rail photonic qubits [56] also exhibit this feature: Clifford QC (requiring several non-Gaussian elements) and Gaussian single-qubit gates together give universality.

The GKP encoding stands out among bosonic codes as the only known code for which Clifford QC is implemented entirely with Gaussian operations given a supply of encoded Pauli eigenstates. We show that once high-quality GKP Clifford QC is achieved—a challenging task already in progress [14, 29, 30]—then fault-tolerant universality is just a trivial Gaussian state away. This means there is no longer any need to pursue creating cubic phase states for the GKP encoding. Focus on making high-quality GKP Pauli eigenstates, and the rest is all Gaussian.

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