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Coherence distillation is one of the central problems in the resource theory of coherence. In this Letter, we complete the deterministic distillation of quantum coherence for a finite number of coherent states under strictly incoherent operations. Specifically, we find the necessary and sufficient condition for the transformation from a mixed coherent state into a pure state via strictly incoherent operations, which recovers a connection between the resource theory of coherence and the algebraic theory of majorization lattice. With the help of this condition, we present the deterministic coherence distillation scheme and derive the maximum number of maximally coherent states obtained via this scheme.

Introduction.– Quantum coherence is a valuable resource in performing quantum information processing tasks [1]. It can implement various information processing tasks that cannot be accomplished classically, such as quantum computing [2, 3], quantum cryptography [4], quantum metrology [5, 6], and quantum biology [7]. Recently, the resource theory of coherence has attracted a growing interest due to the development of quantum information science [8–18].

All quantum resource theories have two fundamental ingredients: free states and free operations [19, 20]. For the resource theory of coherence, the free states are the quantum states that are diagonal in a prefixed reference basis. However, there is no general consensus on the set of free operations. Based on different physical and mathematical considerations, a number of free operations were proposed [8, 9, 11–14]. Here, we focus our discussion on the strictly incoherent operations. This type of free operation was first given in Ref. [11] and was shown that it can neither create nor use coherence and has a physical interpretation in terms of interferometry in Ref. [12]. Thus, the strictly incoherent operations are a physically well-motivated set of free operations for coherence and a strong candidate for free operations.

One of the central problems in the resource theory of coherence is the coherence distillation [9, 11, 19, 21–30], which is the process that extracts pure coherent states from general states via free operations. This problem was approached in two different settings: the asymptotic regime [11, 19, 21, 25–28] and the one-shot regime [23, 24, 29, 30]. Although many interesting results have been obtained, however, there are still some open fundamental questions remaining to be solved. One of which is the deterministic coherence distillation, whose aim is to find the condition of conversion from a general mixed state to the maximally coherent state with certainty [18, 29, 31]. Investigations on this topic have been started in Ref. [29], where the deterministic coherence distillation of pure coherent states under several classes of incoherent operations was introduced. However, the deterministic coherence distillation of general mixed states has been left as an open question.

In this Letter, we address the above question by completing the framework for deterministic coherence distillation under strictly incoherent operations. We first recall some notions of the resource theory of coherence and the notions of majorization lattice which are related to our topic. Then, we present the necessary and sufficient condition for the transformation from a general state into a pure state via strictly incoherent operations, which recovers a connection between the resource theory of coherence and the algebraic theory of majorization lattice. With the help of this condition, we present the deterministic coherence distillation scheme. Then, we derive the maximum number of maximally coherent states that can be obtained in this deterministic coherence distillation scheme.

Resource theory of coherence.–Let $\mathcal{H}$ represent the Hilbert space of a d-dimensional quantum system. A particular basis of $\mathcal{H}$ is denoted as $\{|i\rangle,\ i = 0, 1, \cdots, d-1\}$, which is chosen according to the physical problem under discussion. Specifically, a state is said to be incoherent if it is diagonal in the basis. We represent the set of incoherent states as $\mathcal{I}$. Any state that cannot be written as a diagonal matrix is defined as a coherent state. Note that the term coherent state here is different from the canonical coherent state or the spin coherent state [1]. For a pure state $|\varphi\rangle$, we will denote $|\varphi\rangle\langle\varphi|$ as φ, i.e., $\varphi := |\varphi\rangle\langle\varphi|$ and we will denote $|\varphi_m^d\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle$ as a d-dimensional maximally coherent state.

A strictly incoherent operation is a completely positive trace-preserving map, expressed as $\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger$, where the Kraus operators K_n satisfy not only $\sum_n K_n^\dagger K_n = I$ but also $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$ and $K_n^\dagger \mathcal{I} K_n \subset \mathcal{I}$ for K_n, i.e., each K_n as well as $K_n^\dagger$ maps an incoherent state to an incoherent state. With this definition, it is elementary to show that a projector is an incoherent operator if and only if it has the form $\mathbb{P}_{\mathrm{I}} = \sum_{i\in \mathrm{I}} |i\rangle\langle i|$ with $\mathrm{I} \subset \{0, 1, ..., d-1\}$. In what follows, we will denote $\mathbb{P}_{\mathrm{I}}$ as strictly incoherent projective operators. The the dephasing map, which we will denote as $\Delta(\cdot)$, is defined as $\Delta\rho = \sum_{i=0}^{d-1} |i\rangle\langle i|\rho|i\rangle\langle i|$.

Majorization and majorization lattice.– Majorization [32] is a mathematical tool widely used in quantum information theory [33–35]. For the n-dimensional probability distributions $\mathcal{P}^n$, we say that a probability distribution $\mathbf{p} = (p_1, p_2, ..., p_n)$ is majorized by $\mathbf{q} = (q_1, q_2, ..., q_n)$, in symbols

$\mathbf{p} \prec \mathbf{q}$, if there are $\sum_{i=1}^{l} p_i^{\downarrow} \leq \sum_{i=1}^{l} q_i^{\downarrow}$, for all $1 \leq l \leq n$, where $\downarrow$ indicates that the elements are to be taken in descending order. The majorization lattice [36–38] is a quadruple $(\mathcal{P}^n, \prec, \vee, \wedge)$. Here $\prec$ is the relation introduced above. For every pair of $\mathbf{p}, \mathbf{q} \in \mathcal{P}^n$, $\mathbf{p} \wedge \mathbf{q}$ is the unique greatest lower bound of $\mathbf{p}, \mathbf{q}$ up to a permutation transformation which is defined as a probability distribution, for every $\mathbf{s} \in \mathcal{P}^n$ with $\mathbf{s} \prec \mathbf{p}$, $\mathbf{s} \prec \mathbf{q}$, then there is $\mathbf{s} \prec \mathbf{p} \wedge \mathbf{q}$; and $\mathbf{p} \vee \mathbf{q}$ is the unique least upper bound of $\mathbf{p}, \mathbf{q}$ which is defined as a probability distribution for every $\mathbf{t} \in \mathcal{P}^n$ with $\mathbf{p} \prec \mathbf{t}$ and $\mathbf{q} \prec \mathbf{t}$, then there is $\mathbf{p} \vee \mathbf{q} \prec \mathbf{t}$. Similarly, we write $\bigwedge \mathcal{S}$ as the unique greatest lower bound of $\mathcal{S}$ and $\bigvee \mathcal{S}$ as the unique least upper bound of $\mathcal{S}$, where $\mathcal{S}$ is a subset of $\mathcal{P}^n$. Hereafter, we will apply majorization to density operators and write $\rho_1 \prec \rho_2$ if and only if the corresponding majorization relation holds for the eigenvalues of ρ_1 and ρ_2. And $\bigvee \mathcal{S} \prec \rho$ means that the least upper bound (up to a unitary transformation) of $\mathcal{S}$ is majorized by ρ.

Determined state transformation.–In the following, we will give the necessary and sufficient condition for a state ρ can be transformed into a pure coherent state $|\varphi\rangle$ via strictly incoherent operations.

Theorem 1. We can transform a mixed state ρ into a pure coherent state φ via strictly incoherent operations if and only if there exists an orthogonal and complete set of incoherent projectors $\{\mathbb{P}_\alpha\}$ such that, for all α, there are

$$\frac{\mathbb{P}_\alpha \rho \mathbb{P}_\alpha}{\mathrm{Tr}(\mathbb{P}_\alpha \rho \mathbb{P}_\alpha)} = \psi_\alpha \ \text{ and } \ \Delta\psi_\alpha \prec \Delta\varphi, \tag{1}$$

where ψ_α are all pure coherent states. In other words, there exists $\{\mathbb{P}_\alpha\}$ such that

$$\bigvee \mathcal{S} \prec \Delta\varphi, \tag{2}$$

where $\mathcal{S}$ is the set of $\{\Delta\psi_\alpha\}$.

Proof. First, we show that ρ can be transformed into φ via a strictly incoherent operation if and only if $P\rho P^t$ (superscript t means transpose) can be transformed into φ via a strictly incoherent operation with P being a permutation matrix.

For any two strictly incoherent operations Λ_1 with Kraus operators $\{K_n^1\}$ and Λ_2 with Kraus operators $\{K_m^2\}$, the operation $\Lambda = \Lambda_1 \circ \Lambda_2$ is also a strictly incoherent operation with Kraus operators $\{K_l = K_n^1 K_m^2\}$, since we can easily verify it by examining $K_l \mathcal{I} K_l^\dagger \subseteq \mathcal{I}$ and $K_l^\dagger \mathcal{I} K_l \subseteq \mathcal{I}$. It is straightforward to verify that, for any permutation matrix, both P and its inverse are strictly incoherent operations. With these knowledge, it is easy to show that ρ can be transformed into φ via a strictly incoherent operation if and only if $P\rho P^t$ can be transformed into φ via a strictly incoherent operation. Hence, without loss of generality, we let

$$\rho = \bigoplus_\mu p_\mu \rho_\mu, \tag{3}$$

corresponding to the Hilbert space $\mathcal{H} = \bigoplus_\mu \mathcal{H}_\mu$ with each ρ_μ being irreducible. Here, an irreducible matrix ρ_μ means that it cannot be transformed into a block diagonal matrix by using a permutation matrix.

Second, we show the *if* part of the theorem, i.e., if the state ρ satisfies the condition in the theorem above, then we can transform a mixed state ρ into a pure state φ via a strictly incoherent operation.

Let ρ be a state satisfying the condition in the theorem above. Then, according to the result in Ref. [13, 33, 34] which says that a pure coherent state $|\psi\rangle$ can be transformed into another pure coherent state $|\varphi\rangle$ via strictly incoherent operations if and only if there is $\Delta\psi \prec \Delta\varphi$, we can always find strictly incoherent operations $\Lambda_\alpha(\cdot)$, which act on the support of $\mathbb{P}_\alpha$, with $\Lambda_\alpha(\cdot) = \sum_n K_\alpha^n (\cdot) K_\alpha^{n\dagger}$, such that

$$\Lambda_\alpha(\psi_\alpha) = \varphi,$$

for all α. With this result, we transform ρ into $|\varphi\rangle$ by using the operation

$$\Lambda(\cdot) = \bigoplus_\alpha \Lambda_\alpha(\cdot),$$

where the corresponding Kraus operators are

$$K_{\alpha,n} = K_\alpha^n \oplus \mathbf{0}.$$

Here, $\mathbf{0}$ represents a square matrix with all its elements being zero. It is straightforward to show that $\Lambda(\cdot)$ is a strictly incoherent operation.

Third, we show the *only if* part of the theorem, i.e., if φ can be obtained from a state ρ via a strictly incoherent operation, then the state ρ should satisfy the condition in the theorem above.

Let us assume that we can obtain a pure coherent state φ from a mixed state ρ by using a strictly incoherent operation $\Lambda(\cdot)$. Then, there is

$$\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger = \varphi. \tag{4}$$

Substituting Eq. (3) into (4), we can obtain that

$$\Lambda(\rho) = \sum_{n,\mu} p_\mu K_n \rho_\mu K_n^\dagger = \varphi. \tag{5}$$

Since pure states are extreme points of the set of states, there must be

$$K_n \rho_\mu K_n^\dagger = q_{n,\mu} \varphi,$$

for all n and μ, where $q_{n,\mu} = \mathrm{Tr}(K_n \rho_\mu K_n^\dagger)$.

According to the definition of the strictly incoherent operations, there is at most one nonzero element in each column (row) of a strictly incoherent Kraus operator. Thus, any K_n can always be decomposed into

$$K_n = P_\pi K_n^D \mathbb{P}_n, \tag{6}$$

where the operator P_π is a permutation matrix, $K_n^D = \mathrm{diag}(a_1, ..., a_n, 0, 0, ...)$ is a diagonal matrix with a_i being nonzero complex numbers, and $\mathbb{P}_n$ is a projective operator corresponding to K_n^D, i.e., $\mathbb{P}_n = \mathrm{diag}(1, ..., 1, 0, 0, ...)$. Let

$\{p_{\mu,i}, |\psi_{\mu,i}\rangle\}$ be an arbitrary ensemble decomposition of ρ_μ. Then, there is

$$K_n\rho_\mu K_n^\dagger = \sum_{\mu,i} p_{\mu,i} P_\pi K_n^D \mathbb{P}_n \psi_{\mu,i} \mathbb{P}_n K_n^{D^\dagger} P_\pi^\dagger. \tag{7}$$

From Eqs. (5) and (7), we obtain that

$$\Lambda(\rho) = \sum_{n,\mu,i} p_\mu p_{\mu,i} P_\pi K_n^D \mathbb{P}_n \psi_{\mu,i} \mathbb{P}_n K_n^{D^\dagger} P_\pi^\dagger = \varphi.$$

Again, by using the fact that pure states are extreme points of the set of states, we immediately obtain that

$$\frac{P_\pi K_n^D \mathbb{P}_n \psi_{\mu,i} \mathbb{P}_n K_n^{D^\dagger} P_\pi^\dagger}{\mathrm{Tr}(P_\pi K_n^D \mathbb{P}_n \psi_{\mu,i} \mathbb{P}_n K_n^{D^\dagger} P_\pi^\dagger)} = \varphi \ \text{ or } \ \mathbf{0}, \tag{8}$$

for all μ, i, and n. Clearly, $|\psi_{\mu,i}\rangle$ are states of the subspace $\mathcal{H}_\mu$. Thus, we only need to consider the projective operator $\mathbb{P}_n$ in Eq. (6) corresponding to the subspace $\mathcal{H}_\mu$ and we denoted it as $\mathbb{P}_{n,\mu}$. Since Λ is a trace preserving map, we can get that $\sum_n K_n^\dagger K_n = I$ and, furthermore, $\sum_n \mathbb{P}_{n,\mu} K_n^\dagger K_n \mathbb{P}_{n,\mu} = I_\mu$ with I_μ being the identity matrix of the subspace $\mathcal{H}_\mu$. Here, since every ρ_μ is irreducible, $P_\pi K_n^D \mathbb{P}_n |\psi_{\mu,i}\rangle$ cannot be a zero vector at the same time.

From Eq. (8) and $\sum_n \mathbb{P}_{n,\mu} K_n^\dagger K_n \mathbb{P}_{n,\mu} = I_\mu$, we get that

$$\mathbb{P}_{n,\mu}\psi_{\mu,i}\mathbb{P}_{n,\mu} = \mathbb{P}_{n,\mu}\psi_{\mu,j}\mathbb{P}_{n,\mu} \ \text{ or } \ \mathbf{0}, \tag{9}$$

for all i and j. Both these two cases mean that $\frac{\mathbb{P}_{n,\mu}\rho_\mu\mathbb{P}_{n,\mu}}{\mathrm{Tr}(\mathbb{P}_{n,\mu}\rho_\mu\mathbb{P}_{n,\mu})}$ is a pure coherent state and we denoted it as $\psi_{n,\mu}$ for the sake of simplicity. By using the condition that $\Lambda(\rho) = \varphi$ and the condition in Eq. (9), we immediately derive that

$$\Lambda(\psi_{n,\mu}) = \varphi,$$

for every n and μ. Since the state $\psi_{n,\mu}$ can be transformed into φ via a strictly incoherent operation if and only if $\Delta\psi_{n,\mu} \prec \Delta\varphi$, we immediately obtain the conclusion in our theorem. This completes the proof of the *only if* part. □

From Theorem 1, we infer the following corollary:

Corollary. We can transform ρ into a pure coherent state ψ via strictly incoherent operations if and only if ψ_α are all coherent states for some $\{\mathbb{P}_\alpha\}$.

Proof. The *only if* part follows directly from Theorem 1. To prove the *if* part, without loss of generality, let us assume that $|\psi_\alpha\rangle = \sum_{i=1}^{d_\alpha} c_i^\alpha |i\rangle$ with the number of $c_i^\alpha > 0$ being $d_\alpha \geq 2$, and $c_1^\alpha \geq \cdots \geq c_{d_\alpha}^\alpha$. From the definition of the majorization lattice, we can immediately obtain that $\bigvee \mathcal{S} \prec \bigvee \mathcal{S}'$, where $\mathcal{S}' = \{\Delta\psi'_\alpha\}$ with $|\psi'_\alpha\rangle = c_1^\alpha|1\rangle + \sum_{i=2}^{d_\alpha} c_i^\alpha|i\rangle$. Noting that the set $\mathcal{S}'$ is an ordered set [32] and $c_1^\alpha < 1$, we then obtain that $\bigvee \mathcal{S}'$ equals to one of $\Delta\psi'_\alpha$ and this corresponds to a coherent state $|\psi\rangle$ where $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$ with $0 < c_1 < 1$. □

Deterministic coherence distillation.–Next, let us move to the deterministic coherence distillation of a finite number of coherent states.

Suppose that we have n coherent states

$$\rho_1, \rho_2, ..., \rho_n,$$

where $\rho_1, \rho_2, ..., \rho_n$ are not necessarily identical and n is a finite number. The deterministic coherence distillation process is the process that extracts pure coherent states from them with certainty. Here, we concentrate our discussion on the task that extracts as more 2-dimensional maximally coherent state $|\varphi_m^2\rangle$ as possible from $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$ via strictly incoherent operations.

Based on the result above, we take the distillation procedure as the following three steps (See Fig.1).

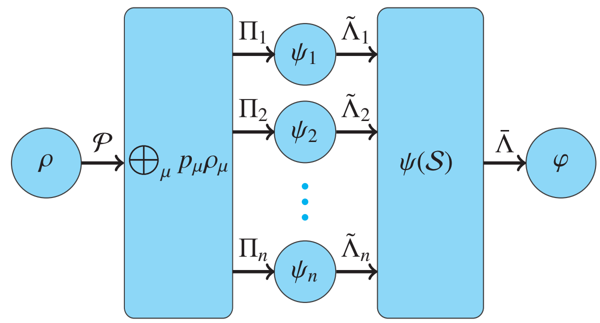

FIG. 1. (Color online). Schematic picture of the deteiministic coherence transformation via strictly incoherent operations. Here, $\Pi_\alpha = \mathbb{P}_\alpha \cdot \mathbb{P}_\alpha$ for incoherent projective operator $\mathbb{P}_\alpha$, $\psi(\mathcal{S})$ is the pure coherent state determined by $\bigvee \mathcal{S}$, $\tilde{\Lambda}_\alpha$ are the strictly incoherent operations such that $\tilde{\Lambda}_\alpha(\psi_\alpha) = \psi$, $\bar{\Lambda}$ is the strictly incoherent operation such that $\bar{\Lambda}(\psi) = \varphi$, and all the others are the same as in the main text.

First, for the given $\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$, we should transform ρ into a block diagonal matrix.

To this end, one should calculate out the permutation matrix P that can transform ρ into a block diagonal matrix, i.e., the permutation matrix P such that

$$\mathcal{P}(\rho) = P\rho P^t = \bigoplus_{\mu=1}^{L} p_\mu\rho_\mu \bigoplus \mathbf{0}, \tag{10}$$

where each $\rho_\mu = \sum_{i,j} \rho_{ij}^\mu |i\rangle\langle j|$ $(\mu = 1, 2, \cdots, n)$ is an irreducible density operator defined on the d_μ-dimensional subspace $\mathcal{H}_\mu$, $p_\mu > 0$ satisfies $\sum_{\mu=1}^L p_\mu = 1$, and $\mathbf{0}$ represents a square matrix of dimension $d_0 = d - \sum_{\mu=1}^L d_\mu$ with all its elements being zero.

Second, we should calculate out an incoherent projective operators set $\{\mathbb{P}_\alpha\}$ in Theorem 1.

To this end, let us first introduce the following three matrices, which are useful to obtain the corresponding $\{\mathbb{P}_\alpha\}$. For $\rho = \sum_{ij} \rho_{ij}|i\rangle\langle j|$, we can define two matrices $|\rho|$ and $(\Delta\rho)^{-\frac{1}{2}}$, where $|\rho|$ reads $|\rho| = \sum_{ij} |\rho_{ij}||i\rangle\langle j|$ and $(\Delta\rho)^{-\frac{1}{2}}$ is a diagonal matrix with elements

$$(\Delta\rho)_{ii}^{-\frac{1}{2}} = \begin{cases} \rho_{ii}^{-\frac{1}{2}}, & \text{if } \rho_{ii} \neq 0; \\ 0, & \text{if } \rho_{ii} = 0. \end{cases}$$

Next, we recall the following matrix with the help of $|\rho|$ and $(\Delta\rho)^{-\frac{1}{2}}$

$$\mathcal{A} = (\Delta\rho)^{-\frac{1}{2}}|\rho|(\Delta\rho)^{-\frac{1}{2}}. \tag{11}$$

A useful property of $\mathcal{A}$ is that all the elements of $\mathcal{A}$ are 1 if and only if ρ is a pure coherent state [24]. By substituting the expression in Eq. (10) into Eq. (11), we obtain that

$$\mathcal{A} = (\Delta\rho)^{-\frac{1}{2}}|\rho|(\Delta\rho)^{-\frac{1}{2}} = \bigoplus_{\mu=1}^{L}\mathcal{A}_\mu \bigoplus \mathbf{0},$$

where $\mathcal{A}_\mu = (\Delta\rho_\mu)^{-\frac{1}{2}}|\rho_\mu|(\Delta\rho_\mu)^{-\frac{1}{2}}$ are also irreducible nonnegative matrices. Next, we should find out all the maximally dimensional principal submatrices $\mathcal{A}_\mu^n$ of $\mathcal{A}_\mu$ with all its elements being 1, where the maximal dimension means that the dimension of $\mathcal{A}_\mu^n$ cannot be enlarged. Let the corresponding Hilbert subspaces of principal submatrices $\mathcal{A}_\mu^n$ be $\mathcal{H}_\mu^n$ spanned by $\{|i_\mu^1\rangle, |i_\mu^2\rangle, \cdots, |i_\mu^{d_n}\rangle\} \subset \{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$. Then, the corresponding incoherent projective operators are

$$\mathbb{P}_\alpha = |i_\mu^1\rangle\langle i_\mu^1| + |i_\mu^2\rangle\langle i_\mu^2| + \cdots + |i_\mu^{d_n}\rangle\langle i_\mu^{d_n}|.$$

Performing $\{\mathbb{P}_\alpha\}$ on the state ρ, we obtain $\{\psi_\alpha\}$, i.e.,

$$\frac{\mathbb{P}_\alpha\rho\mathbb{P}_\alpha}{\mathrm{Tr}(\mathbb{P}_\alpha\rho\mathbb{P}_\alpha)} = \psi_\alpha.$$

By the way, we note that the set of $\{\mathbb{P}_\alpha\}$ in Theorem 1 is not necessarily unique, and we denote the set of $\{\Delta\psi_\alpha\}$ corresponding to the maximally dimensional principal submatrices $\mathcal{A}_\mu^n$ as $\mathcal{S}_m$.

Third, we should calculate out the least upper bound of the set $\mathcal{S}_m = \{\Delta\psi_\alpha\}$, i.e., $\bigvee \mathcal{S}_m$.

Without loss of generality, suppose that $|\psi_\alpha\rangle = \sum_{i=1}^{d_n} c_\alpha^i|i\rangle$ and the corresponding probability distributions of $|\psi_\alpha\rangle$ are $\mathbf{p}_\alpha^\downarrow = (|c_\alpha^1|^2, |c_\alpha^2|^2, ..., |c_\alpha^{d_n}|^2, 0, 0, ...0)$. Let us show how to calculate out the least upper bound of $\mathcal{S}_m$, i.e., $\bigvee \mathcal{S}_m$. To this end, we first define a probability distribution $\mathbf{a} = (a_1, a_2, ..., a_d)$, where

$$a_i = \max\{\sum_{j=1}^{i}|c_1^j|^2, \sum_{j=1}^{i}|c_2^j|^2, ..., \sum_{j=1}^{i}|c_L^j|^2\} - \sum_{j=1}^{i-1}a_j.$$

We note that the elements of $\mathbf{a} = (a_1, a_2, ..., a_d)$ might not be in nonincreasing order, i.e., it is not true in general that $a_j \geq a_{j+1}$. Apart from $\mathbf{a}$, we also need the following lemma, which was proved in Ref. [36].

Lemma. Let $\mathbf{a} = (a_1, a_2, ...a_d)$ be a given probability distribution, and let j be the smallest integer in $\{2, ..., n\}$ such that $a_j > a_{j-1}$. Moreover, let i be the greatest integer in $\{1, 2, ..., j-1\}$ such that $a_{i-1} \geq \frac{\sum_{r=i}^{j} a_r}{j-i+1} = a$. Let the probability distribution $\mathbf{q} = (q_1, q_2, ..., q_d)$ be defined as

$$q_r = \begin{cases} a, & \text{for } r = i, i+1, ..., j; \\ a_r, & \text{otherwise.} \end{cases}$$

Then for the probability distribution $\mathbf{q}$, we have that $q_{r-1} \geq q_r$, for all $r = 2, ..., j$, and $\sum_{s=1}^k q_s \geq \sum_{s=1}^k a_s$, $k = 1, ..., d$. Moreover, for all $\mathbf{t} = (t_1, t_2, ..., t_d)$ such that $\sum_{s=1}^k t_s \geq \sum_{s=1}^k a_s$, $k = 1, ..., n$, we also have $\sum_{s=1}^k t_s \geq \sum_{s=1}^k q_s$, $k = 1, ..., n$.

By using the definition of $\mathbf{a}$ and the iterate application of the above Lemma, we can obtain the least upper bound of $\mathcal{S}_m = \{\Delta\psi_\alpha\}$, i.e., $\bigvee \mathcal{S}_m$ and we denoted it as $\Delta\psi$.

Without loss of generality, let the maximum number of φ_m^2 we can distill from $\rho_1\otimes\rho_2\otimes\cdots\otimes\rho_n$ be N. The generalization to $d > 2$ is straightforward. From Theorem 1, this distillation can be accomplished if the following majorization relation holds:

$$\Delta\psi \prec \mathrm{diag}(2^{-N}, ..., 2^{-N}, 0..., 0). \tag{12}$$

The above relation can be fulfilled if and only if

$$||\psi||_\infty \leq 2^{-N}, \tag{13}$$

where $\|\cdot\|_\infty$ is the max norm on the matrix space. This can be examined directly since if the first inequality of majorization relation in Eq. (12) holds, then the other inequalities for Eq. (12) are automatically satisfied.

Thus, the inequality in Eq. (13) gives the maximum number of 2-dimensional maximally coherent state that can be distilled from $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$ and the maximum number is

$$N_{\max} = \lfloor \log_2 ||\psi||_\infty^{-1} \rfloor,$$

where $\lfloor x \rfloor$ represents the largest integer equal to or less than x.

We can then summarize the above results as Theorem 2.

Theorem 2. The maximum number of 2-dimensional maximally coherent state that can distill from a set of states, such as $\rho_1, \rho_2, ..., \rho_n$, is

$$N_{\max} = \lfloor \log_2 ||\psi||_\infty^{-1} \rfloor. \tag{14}$$

In particular, if the states we chose are all pure coherent states $\{|\varphi_\gamma\rangle\}$ with $\gamma = 1, ..., n$, then the maximum number of 2-dimensional maximally coherent state that we can be distilled is $N_{\max} = \lfloor \log_2 \otimes_{\gamma=1}^n ||\varphi_\gamma||_\infty^{-1} \rfloor$, which corresponds to the result in [29]. This is reminiscent of the case of entanglement [35, 39, 40], where the deterministic entanglement distillation of pure entangled states was studied.

We should note that there is a class of states that cannot be distilled into any pure coherent state via strictly incoherent operations. If we can transform $\rho = \sum_{ij}\rho_{ij}|i\rangle\langle j|$ with the number of $\rho_{ii} \neq 0$ being m into a pure coherent state $|\varphi\rangle = \sum_i c_i|i\rangle$ with the number of $c_i \neq 0$ being n via a strictly incoherent operation, then the rank of ρ is at most $\frac{m}{n}$. To see this, suppose that we can distill a pure coherent state φ from ρ, according to Theorem 1, there must be an orthogonal and complete set of incoherent projectors $\{\mathbb{P}_\alpha\}$ fulfilling the condition in Eq. (1). Let the corresponding decomposition of the Hilbert space of $\{\mathbb{P}_\alpha\}$ be $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$, where the dimension of $\mathcal{H}_\alpha$ is d_α, the projections $\{\mathbb{P}_\alpha\}$ of ρ onto each $\mathcal{H}_\alpha$ are $\{\psi_\alpha\}$, respectively, and $\rho = \sum_{i=1}^l \lambda_i|\lambda_i\rangle\langle\lambda_i|$ is a spectral decomposition for ρ. Then, there are

$$\frac{\mathbb{P}_\alpha|\lambda_i\rangle\langle\lambda_i|\mathbb{P}_\alpha}{\mathrm{Tr}(\mathbb{P}_\alpha|\lambda_i\rangle\langle\lambda_i|\mathbb{P}_\alpha)} = \psi_\alpha,$$

for all $i = 1, ..., l$, with $|\psi_\alpha\rangle = \sum_i c^i_\alpha |i\rangle$. This means that the number, D_ρ, of the linear independent vectors of the set $\{|\lambda_i\rangle\}$ must satisfy $D_\rho = l - \sum_\alpha (d_\alpha - 1) \leq m - \sum_\alpha d_\alpha + \sum_\alpha 1 = \sum_\alpha 1$. From Theorem 1 and the definition of $\Delta\psi_\alpha \prec \Delta\varphi$, we can obtain that the number of $c^i_\alpha \neq 0$ is at least as many as that of $c_i \neq 0$. Thus, there is $D_\rho = \sum_\alpha 1 \leq \frac{m}{n}$.

In passing, we would like to point that the phenomenon of bound coherence under strictly incoherent operations was uncovered in Refs. [23, 27, 28] recently, i.e., there are coherent states from which no coherence can be distilled via strictly incoherent operations in the asymptotic regime. The necessary and sufficient condition for a state being bound state was presented in Refs. [27, 28]. Their result shows that a state is a bound state if and only if it cannot contain any rank-one submatrix. Comparing this result with the *Corollary*, we obtain that, for any mixed state ρ, if we can transform it into a pure coherent state $|\varphi\rangle$, then it cannot be a bound state. However, in general, the converse is not true. Thus, the set of states that can be transformed into a pure coherent state $|\varphi\rangle$ is a strictly smaller set of the set of distillable states.

Conclusions.–We have completed the operational task of deterministic coherence distillation for a finite number of coherent states under strictly incoherent operations. Specifically, we have presented the necessary and sufficient condition for the transformation from a mixed coherent state into a pure coherent state via strictly incoherent operations, which recovers a connection between the resource theory of coherence and the algebraic theory of majorization lattice. With the help of this condition, we have presented the deterministic coherence distillation scheme and we have derived the maximum number of maximally coherent states that can be obtained via this scheme.

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