Quantifying the Incompatibility of Quantum Measurements Relative to a Basis

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Quantifying the incompatibility of quantum measurements relative to a basis

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Motivated by quantum resource theories, we introduce a notion of incompatibility for quantum measurements relative to a reference basis. The notion arises by considering states diagonal in that basis and investigating whether probability distributions associated with different quantum measurements can be converted into one another by probabilistic post-processing. The induced preorder over quantum measurements is directly related to multivariate majorization and gives rise to families of monotones, i.e., scalar quantifiers that preserve the ordering. For the case of orthogonal measurement we establish a quantitative connection between incompatibility, quantum coherence and entropic uncertainty relations. We generalize the construction to arbitrary POVM measurements and report complete families of monotones.

Introduction.— One of the cornerstones of quantum theory is the concept of incompatibility between observables [1]. A pair of quantum observables is deemed incompatible if the corresponding self-adjoint operators fail to commute. Operationally, incompatibility implies that there exist pure quantum states for which it is impossible to simultaneously predict with certainty the measurement outcomes of two incompatible observables. Finite-dimensional observables that share the same eigenbasis are fully compatible, while any pair of observables associated with bases that are mutually unbiased are maximally incompatible: certain knowledge for the outcome of one assures complete randomness for the possible outcomes of the other.

Incompatibility is famously captured through uncertainty relations, that may involve variances [2–4], entropies [5–11] or other information-theoretic quantities [12–17]. A quantitative description of incompatibility in quantum mechanics was persued recently, from the perspective of state discrimination and quantum steering [18–26]. In this approach, one of the central notions is that of a *parent* measurement, i.e., one that can simulate the original one through probabilistic post-processing.

Quantum resource theories provide a framework to systematically characterize and quantify quantum properties (for example, entanglement). There, such a property is fully described by the conversion relations among states under a class of quantum processes that, suitably chosen, cannot enhance it [27]. The transformation relations among quantum states can be mathematically described by a *preorder*: if a state can be transformed into another under the distinguished class of processes, then it lies “higher” in the ordering [28]. In turn, the preorder induces a family of scalar functions, called *monotones*, that cannot increase under the allowed state transitions and therefore jointly quantify the resourcefulness of states.

In this work, we introduce a notion of incompatibility of quantum measurements relative to a reference basis by means of a preorder. More specifically, considering states that are diagonal in the reference basis, we investigate whether the probability distributions associated to different measurements can be transformed into one another, by means of probabilistic post-processing. The aforementioned question of convertibility generates a preorder over quantum measurements which, in turn, gives rise to families of scalar functions that jointly quantify the introduced notion of incompatibility relative to a basis. We first consider the special case of orthogonal measurements in which the ordering provides a quantitative, as well as conceptual, connection between incompatibility, quantum coherence and entropic uncertainty relations. We then extend to include generalized measurements and we relate the resulting notion to parent measurements.

Preliminaries.— Consider a non-degenerate observable A over a finite dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^d$ with spectral decomposition $A = \sum_{i=1}^d a_i P_i$ (we denote $P_i := |i\rangle\langle i|$). The role of the eigenvalues a_i is to label the possible outcomes and, as long as they are distinct, this role is unimportant from the point of view of the measurement process, since the probability distribution $\boldsymbol{p}_{\mathbb{B}}(\rho)$ with components $[\boldsymbol{p}_{\mathbb{B}}(\rho)]_i := \mathrm{Tr}\,(P_i \rho)$ (representing a measurement of A in state ρ) only depends on the set of projectors $\{P_i\}_i$ [29]. We will henceforth use the term *basis* (always meaning orthonormal) to refer to a set of rank-1 orthogonal projectors $\mathbb{B} = \{P_i\}_{i=1}^d$, with $\sum_i P_i = I$ [30]. A generalized measurement (POVM) is represented by a set of operators $\mathbb{F} = \{F_i\}_i$ such that $F_i \geq 0$ and $\sum_i F_i = I$. We associate with every basis $\mathbb{B}$ the real abelian algebra of observables $\mathcal{A}_{\mathbb{B}}$ generated by $\{P_i\}_i$. The set of bases over the Hilbert space is denoted by $\mathcal{M}(\mathcal{H})$.

Preorder and monotones.— The idea of deriving families of scalar functions that quantify some feature (for instance, the degree of uniformity of a probability distribution) by invoking a preorder has its roots in the mathematical theory of majorization [31]. Such a paradigm has been extensively employed in quantum information in the context of resource theories for quantifying features of quantum systems, such as entanglement [32], coherence

[33] and out-of-equilibrium thermodynamics [34].

In this approach, one distinguishes a class of quantum operations, deemed as "easy", motivated by some practical consideration. For example, in the case of entanglement, the easy operations are local quantum operations between two parties together with classical communication (LOCC). This set of maps induces a preorder "$\geq$" in the set of quantum states, defined by the allowed transitions under easy operations, namely $\rho \geq \sigma$ if and only if there exists an easy operation $\mathcal{E}$ such that $\sigma = \mathcal{E}(\rho)$. The binary relation induced is a preorder since, by definition, the identity quantum channel is always an easy operation and also the composition of easy operations is again an easy operation. Moreover, $\rho \geq \sigma$ should intuitively correspond in our example to a statement like "ρ is more entangled than σ." This quantification is rigorously captured by the notion of *monotones*, i.e., scalar functions f over states, non-increasing under allowed state transitions ($\rho \geq \sigma \implies f(\rho) \geq f(\sigma)$). Families of monotones $\{f_a\}_\alpha$ are said to form a *complete set*, if they satisfy $f_\alpha(\rho) \geq f_\alpha(\sigma)\ \forall\alpha \iff \rho \geq \sigma$.

A preorder over orthonormal bases.— Our goal is to define a notion of incompatibility relative to a basis. Let us begin with the case of orthogonal measurements. Consider a basis $\mathbb{B}_0 = \{P_i^{(0)}\}_i$ and a state $\rho_0 = \sum_i p_i P_i^{(0)} \in \mathcal{A}_{\mathbb{B}_0}$ diagonal over it, described by the probability distribution $\boldsymbol{p}$. Given another basis $\mathbb{B}_1 = \{P_i^{(1)}\}_i$, one can also associate with ρ_0 the probability distribution $\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)$ corresponding to a measurement over $\mathbb{B}_1$. In fact, $\boldsymbol{p}_{\mathbb{B}_1}(\rho_0) = X(\mathbb{B}_1,\mathbb{B}_0)\boldsymbol{p}$, where $X(\mathbb{B}_1,\mathbb{B}_0)$ denotes the bistochastic matrix [35] with elements

$$[X(\mathbb{B}_1,\mathbb{B}_0)]_{ij} \coloneqq \mathrm{Tr}\left(P_i^{(1)}P_j^{(0)}\right) \quad . \tag{1}$$

Moreover, the probability distribution $\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)$ is always "more uniform" than $\boldsymbol{p}$. This is precisely captured by the majorization statement $\boldsymbol{p} \succ \boldsymbol{p}_{\mathbb{B}_1}(\rho_0)$ that is true for any basis $\mathbb{B}_1$ and follows directly from the bistochasticity of X [36].

Let us now introduce another measurement, over a basis $\mathbb{B}_2$, such that there exists some bistochastic matrix M with

$$X(\mathbb{B}_2,\mathbb{B}_0) = MX(\mathbb{B}_1,\mathbb{B}_0) \quad . \tag{2}$$

This relation has a rather strong implication: for all states ρ_0 diagonal in $\mathbb{B}_0$, the distribution $\boldsymbol{p}_{\mathbb{B}_2}(\rho_0)$ can be obtained from $\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)$ through "uniforming" classical post-processing, represented by some bistochastic M which is independent of the state.

Motivated by the above, if Eq. (2) holds, we declare that "an orthogonal measurement over $\mathbb{B}_1$ is more compatible than over $\mathbb{B}_2$, relative to states diagonal in $\mathbb{B}_0$". We introduce the following notation.

Definition 1. We denote $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2,\mathbb{B}_0) = MX(\mathbb{B}_1,\mathbb{B}_0)$.

The definition has the following immediate consequences. (i) The binary relation "$\succ^{\mathbb{B}_0}$" over $\mathcal{M}(\mathcal{H})$ is a preorder, i.e., $\mathbb{B} \succ^{\mathbb{B}_0} \mathbb{B}\ \forall\mathbb{B}$ (reflexivity) and $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$, $\mathbb{B}_2 \succ^{\mathbb{B}_0} \mathbb{B}_3 \implies \mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_3$ (transitivity). (ii) $\mathbb{B}_0 \succ^{\mathbb{B}_0} \mathbb{B}$ for all bases $\mathbb{B}$ ("measurement over $\mathbb{B}_0$ is more compatible than over any other basis") (iii) $\mathbb{B} \succ^{\mathbb{B}_0} \mathbb{B}_{\mathrm{MU}}$ for all bases $\mathbb{B}$, where $\mathbb{B}_{\mathrm{MU}}$ is any basis mutually unbiased to $\mathbb{B}_0$ ("measurement over any basis is more compatible than over a mutually unbiased one").

The preorder "$\succ^{\mathbb{B}_0}$" is not in general a partial order, i.e., $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ and $\mathbb{B}_2 \succ^{\mathbb{B}_0} \mathbb{B}_1$ do not necessarily imply $\mathbb{B}_1 = \mathbb{B}_2$. For example, any $\mathbb{B}_1$ and $\mathbb{B}_2$ that are unbiased relative to $\mathbb{B}_0$ satisfy the aforementioned relations but can be taken to be distinct.

The ordering (2) over matrices has been studied in the context of multivariate majorization, called *matrix majorization* [37]. There, $A \succ C$ for matrices A and C if there exists a bistochastic B such that $C = BA$. We now connect the aforementioned preorder with quantum measurements.

"$\succ^{\mathbb{B}_0}$" from non-selective measurements.— Definition 1 can be operationally understood in terms of classical post-processing of probability distributions. Here we show that the ordering "$\succ^{\mathbb{B}_0}$" also admits a quantum operational interpretation in terms of emulation of a non-selective measurement via additional such measurements.

Any basis $\mathbb{B}$ gives rise to a corresponding *dephasing* or *measurement* quantum map

$$\mathcal{D}_{\mathbb{B}}(X) \coloneqq \sum_i P_i X P_i \quad . \tag{3}$$

The latter can be though of as a non-selective orthogonal measurement of any non-degenerate observable belonging in $\mathcal{A}_{\mathbb{B}}$, while a composition $\mathcal{D}_{\mathbb{B}_n} \ldots \mathcal{D}_{\mathbb{B}_1}$ represents the quantum operation associated with n such successive measurements [38].

We are now ready to state the result. The ordering $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ holds if and only if, for any initial state diagonal in $\mathbb{B}_0$, the output of a non-selective $\mathbb{B}_2$ measurement can be emulated by a non-selective $\mathbb{B}_1$ measurement, followed possibly by an additional sequence of measurements and a unitary rotation. More specifically:

Proposition 1. $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exist a unitary superoperator $\mathcal{U}$ and a (possibly trivial) sequence of measurements $\{\mathcal{D}_{\mathbb{B}'_\alpha}\}_\alpha$ such that

$$\mathcal{D}_{\mathbb{B}_2}\mathcal{D}_{\mathbb{B}_0} = \mathcal{U}\left[\prod_\alpha \mathcal{D}_{\mathbb{B}'_\alpha}\right]\mathcal{D}_{\mathbb{B}_1}\mathcal{D}_{\mathbb{B}_0} \quad . \tag{4}$$

All proofs can be found in [39].

The auxiliary sequence of measurements needed might be, in fact, infinite. Eq. (4) should be understood as "$\left\|\mathcal{D}_{\mathbb{B}_2}\mathcal{D}_{\mathbb{B}_0} - \mathcal{U}\left[\prod_\alpha \mathcal{D}_{\mathbb{B}'_\alpha}\right]\mathcal{D}_{\mathbb{B}_1}\mathcal{D}_{\mathbb{B}_0}\right\|$ can be made arbitrarily small", i.e., the state transformation of the RHS can approximate arbitrarily well the one of the LHS.

We now analyze the $d = 2$ case, by invoking Proposition 1 together the usual Bloch ball representation of quantum states $\rho = \frac{1}{2}(I + \boldsymbol{v} \cdot \boldsymbol{\sigma})$, where different bases are in one to one correspondence with lines passing from the center. In this representation, the action of $\mathcal{D}_{\mathbb{B}_1}$ on a state ρ coincides with projecting $\boldsymbol{v}$ onto the $\mathbb{B}_1$ line while the action of $\mathcal{U}$ is translated into an $SO(3)$ rotation. Clearly, Eq. (4) can be satisfied (in fact, by means of a single $\mathcal{D}_{\mathbb{B}_1'}$) if and only if $\theta_1 \leq \theta_2$; here θ_i is the (acute) angle between the lines corresponding to $\mathbb{B}_0$ and $\mathbb{B}_i$. In particular, for $d = 2$ the ordering "$\succ^{\mathbb{B}_0}$" is a total preorder, but not for $d > 2$.

Measures of relative (in)compatibility.— A preorder gives rise to a distinguished class of scalar functions, i.e., monotones. We adopt the following definition.

Definition 2. A function $f_{\mathbb{B}_0} : \mathcal{M}(\mathcal{H}) \to \mathbb{R}_0^+$ is *measure of compatibility (incompatibility)* relative to $\mathbb{B}_0$ if it convex (concave) with respect to the preorder "$\succ^{\mathbb{B}_0}$", i.e., $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2 \implies f_{\mathbb{B}_0}(\mathbb{B}_1) \geq f_{\mathbb{B}_0}(\mathbb{B}_2)$ $\big(\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2 \implies f_{\mathbb{B}_0}(\mathbb{B}_1) \leq f_{\mathbb{B}_0}(\mathbb{B}_2)\big)$. Moreover, if $f_{\mathbb{B}_0}(\mathbb{B}_1) = f_{\mathbb{B}_1}(\mathbb{B}_0)$, we call it a *symmetric* measure of relative compatibility (incompatibility).

The following Proposition gives a construction for measures of relative compatibility arising from convex functions. It is a direct consequence of a result from [40], derived in the context of matrix majorization.

Proposition 2. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous convex function. Then,

$$f^{\phi}_{\mathbb{B}_0}(\mathbb{B}_1) := \sum_i \phi(X_i^R(\mathbb{B}_1, \mathbb{B}_0)) \tag{5}$$

is a measure of relative compatibility; here, X_i^R stand for the row vectors of the matrix X_{ij}.

An analogous claims hold for the incompatibility case in terms of concave functions.

In fact, the family $\{f^{\phi}_{\mathbb{B}_0}(\mathbb{B}_1)\}_\phi$ for all continuous convex ϕ is known to be a complete family of monotones for matrix majorization [40], i.e., joint monotonicity $f^{\phi}_{\mathbb{B}_0}(\mathbb{B}_1) \geq f^{\phi}_{\mathbb{B}_0}(\mathbb{B}_2)$ for all such functions is enough to imply $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$. In that sense, the existence of a probabilistic uniforming process M such that Eq. (2) holds is fully captured by this family of functions.

Incompatibility and coherence.— Quantum coherence refers to the property of quantum systems to exist in a linear superposition of different physical states. It is a notion defined with respect to some preferred, physically relevant basis, which we will denote as $\mathbb{B}_0$. A state ρ is said to be coherent if there exist non-vanishing off-diagonal elements when ρ is expressed as a matrix in $\mathbb{B}_0$. Recently, coherence was formulated as a resource theory [41]. One of the central measures in the theory is *relative entropy of coherence*, $c^{(\mathrm{rel})}_{\mathbb{B}_0}(\rho) := S(\rho \,\|\, \mathcal{D}_{\mathbb{B}_0}\rho)$ that admits several operational interpretations in terms of conversion rates [42, 43]. Later, we will also invoke the 2-coherence $c^{(2)}_{\mathbb{B}_0} := \sum_{i \neq j} |\rho_{ij}|^2$ [44].

Although quantum coherence refers to states and relative incompatibility to measurements, the two notions are closely connected. In fact, the ordering "$\succ^{\mathbb{B}_0}$" has rather strong implications in terms of quantum coherence, both for state conversion under Incoherent Operations [45] (i.e., easy operation in the resource theory of coherence [33]) and coherence monotones. We define the action of a unitary superoperator over a basis as $\mathcal{U}(\mathbb{B}) := \{\mathcal{U}(P_i)\}_i$.

Proposition 3. Let $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$.

(i) Consider a pair of unitary quantum maps $\mathcal{U}, \mathcal{V}$ such that $\mathcal{U}(\mathbb{B}_1) = \mathbb{B}_0$ and $\mathcal{V}(\mathbb{B}_2) = \mathbb{B}_0$ and a pure state $P_j \in \mathbb{B}_0$. Then, $\mathcal{V}(P_j)$ can be transformed to $\mathcal{U}(P_j)$ via incoherent operations over $\mathbb{B}_0$. Consequently, all coherence measures over such states are non-increasing.

(ii) $c_{\mathbb{B}_1}(\rho_0) \leq c_{\mathbb{B}_2}(\rho_0)$ for all ρ_0 diagonal in $\mathbb{B}_0$, where $c_{\mathbb{B}}$ denotes either the relative entropy of coherence or the 2-coherence over $\mathbb{B}$.

In addition to the interpretation of Proposition 3 in the framework of coherence, one can also infer from (ii) above that a $\mathcal{D}_{\mathbb{B}_1}$ measurement disturbs less ρ_0 compared to a $\mathcal{D}_{\mathbb{B}_2}$ measurement, if $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$, as it is precisely captured by statistical meaning of the relative entropy [46].

In the light of the interpretation of $c^{(\mathrm{rel})}_{\mathbb{B}}$ as distillable coherence [42], (ii) above demonstrates a quantitative trade-off between compatibility and coherence. Moreover, any coherence average $C_{\mathbb{B}_0}(\mathbb{B}) := \int d\mu(\rho_0) c_{\mathbb{B}}(\rho_0)$ is a measure of incompatibility of $\mathbb{B}$ relative to $\mathbb{B}_0$. In fact, these averages over the uniform distribution have been performed, verifying explicitly that $C_{\mathbb{B}_0}(\mathbb{B}) = f^{\phi}_{\mathbb{B}_0}(\mathbb{B}_1)$ is of the form indicated in the (concave analogue of) Proposition 2. Indeed, ϕ coinsides with the subentropy [47] for the case of the relative entropy of coherence [48], while $\phi(p_1, \ldots, p_d) \propto \sum_i (\frac{1}{d} - p_i^2)$ for the 2-coherence [49].

Finally, we note that in [50], the authors considered a geometrically motivated measure of "mutual unbiasedness" between pairs of orthonormal bases. Their measure is proportional to the 2-coherence average above, hence is also a symmetric measure of relative incompatibility.

Incompatibility and uncertainty.— We now consider implication of the preorder "$\succ^{\mathbb{B}_0}$" in terms of uncertainty and fluctuations.

By its definition, the ordering $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ assures that the distribution $\boldsymbol{p}_{\mathbb{B}_2}(\rho_0)$ is "more uniform" than $\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)$, for any state ρ_0 diagonal in $\mathbb{B}_0$. An immediate consequence is that all Schur-concave functions [51], which for instance include α-Rényi entropies for ($\alpha = 1$ corresponds to the usual Shannon entropy), satisfy $S_\alpha(\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)) \leq S_\alpha(\boldsymbol{p}_{\mathbb{B}_2}(\rho_0))$ [52].

Quantum fluctuations over different bases can be quantified via entropic uncertainty relations [10]. There, one tries to impose bounds over entropic quantities, such as $S_\alpha(\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)) + S_\beta(\boldsymbol{p}_{\mathbb{B}_2}(\rho_0)) \geq r_{\mathbb{B}_0}(\mathbb{B}_2, \mathbb{B}_1)$ ($\alpha = \beta = 1$ corresponds to the usual Shannon entropy), as a function of the bases. The most well-known inequality is due to Maassen and Uffink [6] and states that a ($\mathbb{B}_0$ independent) choice for the above bound is $r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_1) \coloneqq -\log(\max_{i,j} X_{ij}(\mathbb{B}_2, \mathbb{B}_1))$ for any $\alpha, \beta \geq 1/2$ with $1/\alpha + 1/\beta = 2$. The bound has recently been improved by Coles *et al.* [8] for the case of Shannon/von Neumann entropy, as $S(\boldsymbol{p}_{\mathbb{B}_1}(\rho_0)) + S(\boldsymbol{p}_{\mathbb{B}_2}(\rho_0)) \geq S(\rho_0) + r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_1)$.

Let us also consider the quantity

$$Q_{\mathbb{B}_0}(\mathbb{B}_1) \coloneqq \sup_{A \in \mathcal{A}_{\mathbb{B}_1}, \|A\|_2 = 1} \max_{i=1,\dots,d} \mathrm{Var}_i(A) \ , \tag{6}$$

where $\mathrm{Var}_i(A) \coloneqq \mathrm{Tr}\left(P_i^{(0)} A^2\right) - \left[\mathrm{Tr}\left(P_i^{(0)} A\right)\right]^2$,

that captures the strength of the fluctuations of a pure state diagonal in $\mathbb{B}_0$ over a $\mathbb{B}_1$ measurement. In [39] we derive the upper bound

$$Q_{\mathbb{B}_0}(\mathbb{B}_1) \leq 1 - \lambda_{\min}\left(X(\mathbb{B}_1, \mathbb{B}_0) X^T(\mathbb{B}_1, \mathbb{B}_0)\right) \coloneqq q(\mathbb{B}_1, \mathbb{B}_0) \tag{7}$$

($\lambda_{\min}(X)$ stands for the minimum eigenvalue of X). The bound is symmetric and satisfies $q(\mathbb{B}_1, \mathbb{B}_0) = 0$ if and only if $\mathbb{B}_1 = \mathbb{B}_0$, hence it vanishes if and only if $Q_{\mathbb{B}_0}(\mathbb{B}_1)$ vanishes.

In words, $r^{(\mathrm{MU})}$ and q provide bounds on uncertainty and fluctuations that arise due to the incompatibility between the bases of measurement (for $r^{(\mathrm{MU})}$) or state preparation and measurement (for q), and can be thought of as playing a role analogous to the commutator term in the usual uncertainty relations for observables. As such, they both turn out to be (symmetric) measures of relative incompatibility, monotonic relative to the ordering "$\succ^{\mathbb{B}_0}$".

Proposition 4. Let $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$. Then, $q(\mathbb{B}_1, \mathbb{B}_0) \leq q(\mathbb{B}_2, \mathbb{B}_0)$ and $r^{(\mathrm{MU})}(\mathbb{B}_1, \mathbb{B}_0) \leq r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_0)$.

Generalized measurements.— The ordering "$\succ^{\mathbb{B}_0}$" can be directly extended to include generalized measurements described by POVMs. Consider a state $\rho_0 = \sum_i p_i P_i^{(0)} \in \mathcal{A}_{\mathbb{B}_0}$ and a measurement $\mathbb{F} = \{F_i\}_i$. The probability distribution of possible outcomes is $\boldsymbol{p}_{\mathbb{F}}(\rho_0) = X(\mathbb{F}, \mathbb{B}_0)\boldsymbol{p}$, where now $[X(\mathbb{F}, \mathbb{B}_0)]_{ij} \coloneqq \mathrm{Tr}(F_i P_j^{(0)})$ is just column stochastic [53]. The analogous ordering over POVMs $\mathbb{F}$ and $\mathbb{G}$ relative to a basis $\mathbb{B}_0$ can be defined as $\mathbb{F} \succ^{\mathbb{B}_0} \mathbb{G}$ if and only if there exists a bistochastic M such that $X(\mathbb{G}, \mathbb{B}_0) = M X(\mathbb{F}, \mathbb{B}_0)$. In fact, the family $\{f^{\phi}_{\mathbb{B}_0}(\mathbb{F}) \coloneqq \sum_{i=1}^d \phi(X_i^R(\mathbb{F}, \mathbb{B}_0))\}_\phi$ for all continuous convex ϕ still forms a complete family of monotones for the ordering "$\succ^{\mathbb{B}_0}$", now considered over POVMs.

However, in contrast with the orthogonal measurement case, now it does not hold that $\boldsymbol{p}_{\mathbb{B}_0}(\rho_0) \succ \boldsymbol{p}_{\mathbb{F}}(\rho_0)$ for all $\mathbb{F}$, namely generalized measurements can "purify" the initial probability distribution [54]. For this reason, we consider as the appropriate meaningful generalization of "incompatibility relative to a basis" to POVMs the less restraining ordering that occurs by relaxing the constraint of bistochasticity on the matrix M, and instead requiring only column stochasticity. In this case, if $\mathbb{F}$ lies "higher" in the ordering than $\mathbb{G}$, then $\boldsymbol{p}_{\mathbb{G}}(\rho_0)$ can be obtained by probabilistic post-processing (not necessarily a uniforming one) from $\boldsymbol{p}_{\mathbb{F}}(\rho_0)$, independently of $\rho_0 \in \mathcal{A}_{\mathbb{B}_0}$.

Definition 3. We denote $\mathbb{F} \succ\!\!\succ^{\mathbb{B}_0} \mathbb{G}$ if and only if there exists a stochastic matrix M such that $X(\mathbb{G}, \mathbb{B}_0) = M X(\mathbb{F}, \mathbb{B}_0)$.

The ordering is a preorder and clearly $\mathbb{F} \succ^{\mathbb{B}_0} \mathbb{G} \implies \mathbb{F} \succ\!\!\succ^{\mathbb{B}_0} \mathbb{G}$. As such, the corresponding monotones for "$\succ\!\!\succ^{\mathbb{B}_0}$" are related to Eq. (5). The following is a direct implication of a result by Alberti *et al.* [55] (see also [56]).

Proposition 5. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a function that is simultaneously convex and homogeneous in all its arguments. Then,

$$g^{\psi}_{\mathbb{B}_0}(\mathbb{F}) \coloneqq \sum_i \psi(X_i^R(\mathbb{F}, \mathbb{B}_0)) \tag{8}$$

is a monotone over "$\succ\!\!\succ^{\mathbb{B}_0}$", i.e., $\mathbb{F} \succ\!\!\succ^{\mathbb{B}_0} \mathbb{G} \implies g^{\psi}_{\mathbb{B}_0}(\mathbb{F}) \geq g^{\psi}_{\mathbb{B}_0}(\mathbb{G})$; here, X_i^R stand for the row vectors of the matrix X_{ij}. Moreover, the family $\{g^{\psi}_{\mathbb{B}_0}(\mathbb{F})\}_\psi$ forms a complete set of monotones for "$\succ\!\!\succ^{\mathbb{B}_0}$".

An analogue of Proposition 1 for the ordering "$\succ\!\!\succ^{\mathbb{B}_0}$" is reported in [39].

Basis-independent incompatibility.— Finally, we connect the orderings describing measurement incompatibility relative to a basis with the notion of a *parent measurement* [26, 57]. In this context, $\mathbb{F}$ is called a parent of $\mathbb{G}$ if there exists a stochastic M such that $G_i = \sum_j M_{ij} F_j \ \forall i$, while a family of measurements are jointly measurable if they admit a common parent.

Proposition 6. $\mathbb{F}$ is a parent of $\mathbb{G}$ if and only if $\mathbb{F} \succ\!\!\succ^{\mathbb{B}_0} \mathbb{G}$ for all $\mathbb{B}_0 \in \mathcal{M}(\mathcal{H})$ and the post-processing matrix M can be chosen to be the same for all $\mathbb{B}_0$.

Conclusions.— Quantum resource theories seem to suggest that an appropriate quantification of quantum properties, even conceptually simple ones such as the "uniformity" of a state [58], cannot be achieved by means of a single scalar quantifier. Instead, only an infinite set of functions is able to capture such properties in their wholeness, as they naturally result out of preorders. In this work, we defined operationally motivated preorders over quantum measurements that capture a notion of incompatibility relative to a basis. Our approach uncovers a quantitative, as well as conceptual, connection between incompatibility, uncertainty and quantum coherence unified under the prism of multivariate majorization.

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[53] The POVMs are allowed have arbitrary number of elements. If this number is different, it is understood that the POVM with the least number of elements is padded with zeros until the cardinality of the sets becomes equal, so that the (rectangular) matrices $X(\mathbb{F}, \mathbb{B}_0)$ and $X(\mathbb{G}, \mathbb{B}_0)$ have equal dimensions.

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