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We generalize the Landauer formula to describe the dissipative electron transport through a superconducting point contact. The finite-temperature, linear-in-bias, dissipative DC conductance is expressed in terms of the phase- and energy-dependent scattering matrix of the Bogoliubov quasiparticles in the quantum point contact. The derived formula is also applicable to hybrid superconducting-normal structures and normal contacts, where it agrees with the known limits of Andreev reflection and normal-state conductance, respectively.

The celebrated Landauer formula[1] relates the conductance of a mesoscopic sample to the transmission coefficient for electrons passing through it, and is valid for arbitrary transmission strength. The derivation is usually approached via a scattering formalism, or the Kubo formula[2] applied to an ensemble of non-interacting fermions. The former method relies on charge conservation; the latter requires performing the calculation at a finite frequency ω, followed by taking the limit $\omega \to 0$ at small but fixed bias $\mathcal{V}$ in order to obtain the DC conductance.

In the case of a superconducting junction, both of these approaches are problematic. The asymptotic scattering states are free-propagating Bogoliubov quasiparticles with no well-defined charge, which precludes a direct application of scattering theory. In the linear-response theory, the instantaneous current across the junction depends on the phase difference φ; and the phase perturbation, $2e\mathcal{V}/\hbar\omega$, diverges in the limit $\omega \to 0$. This divergence is an indication of the AC Josephson effect[3], which predicts a non-dissipative current oscillating in time with frequency $2e\mathcal{V}/\hbar$. The non-perturbative in $\mathcal{V}$, dissipationless alternating current component, however, generally coexists with a linear-in-$\mathcal{V}$ dissipative one. Indeed, for the case of weak tunnelling, the current at finite bias $\mathcal{V}$ and any temperature T was found[4] to the lowest order in transmission coefficient. A linear-in-$\mathcal{V}$ expansion of the current-voltage characteristic[4] of a tunnel junction between two superconductors[5] yields a finite value of the linear conductance[6] at $T \neq 0$. This dissipative conductance $G(T)$ is caused by Bogoliubov quasiparticles tunnelling across the junction.

The perturbative-in-tunneling results are adequate for conventional large-area Josephson junctions, but are not applicable to point contacts having one or a few channels with high transmission coefficient. Such junctions are presently actively studied in a variety of platforms, including proximitized nanowires[7] and cold fermions[8,9]. The purpose of this work is to free the evaluation of $G(T)$ from the assumption of weak tunneling. Our main result, Eq (12), expresses $G(T)$ in terms of the quasiparticle scattering matrix. This generalization of the Landauer formula is valid for a junction between leads made of superconductors or normal conductors, in any combination. Additionally, the derived relation provides a lucid interpretation of the dissipative, so-called[10] "$\cos\varphi$" component[3] of the AC Josephson current.

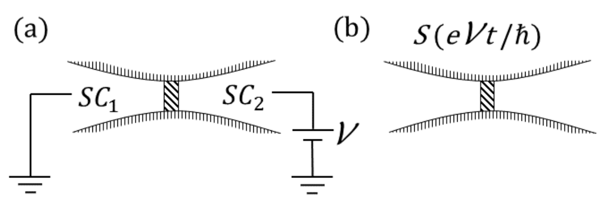

FIG. 1. (a) A point contact between two superconductors SC_1 and SC_2 under applied bias $\mathcal{V}$. (b) To evaluate the dissipative current due to the quasiparticles at finite temperature T, we absorb the bias voltage in the time dependence of the quasiparticle scattering matrix $S(\Omega\, t)$, where $\Omega = e\mathcal{V}/\hbar$. The main general expression for dissipative conductance G is given in Eq. (12) and application to a specific model of a superconducting point contact (SPC) in Eq. (15).

Aiming at evaluation of $G(T)$ for a system with broken gauge invariance, it is useful to reformulate the problem so that the chemical potentials of the leads are not affected by the bias. This is achieved by introducing a time-dependent phase $e\mathcal{V}t/\hbar$ in the definition of the creation operators for electrons to which bias is applied, $\psi^\dagger \to \psi^\dagger \exp(ie\mathcal{V}t/\hbar)$ and thus endowing the scattering matrix describing the contact with a periodic dependence on time, see Fig. 1. The time dependence allows for energy absorption by electrons passing through the junction, *i.e.*, introduces channels of inelastic scattering. The energy transfer is quantized in units of $\hbar\Omega = e\mathcal{V}$, small in the limit $\mathcal{V} \to 0$. Our strategy consists of two steps. First, we relate the scattering matrix for such "soft" inelastic processes to the conventionally-defined elastic scattering matrix of the system in the absence of time dependence. Next, we evaluate the absorbed power $\mathcal{P}$ in terms of scattering matrix and find $G(T)$ from the relation $\mathcal{P} = G\mathcal{V}^2$ for Ohmic losses. This method avoids problems associated with the charge non-conservation and presence of large non-dissipative currents. The result, Eq. (12), is applicable to superconducting and hybrid normal metal–superconductor structures. For such structures, Eq. (12) has the same status as that of the standard Landauer formula for the normal-state contacts; in the absence of superconductivity, Eq. (12) readily reduces to the conventional form of the Landauer formula.

Inelastic quasiparticle scattering in channel N is associated with absorption of N quanta ($N = 0, \pm 1, \pm 2, \dots$) and is characterized by scattering matrix S_N. In order to relate S_N to the elastic scattering matrix, we consider a generic scattering problem with a Hamiltonian

$$H = H_0 + W(t), \quad (1)$$
$$W(t) = V e^{-i\phi(t)} + V^\dagger e^{i\phi(t)} + V_0,$$

where H_0 describes the two leads, and $W(t)$ represents the coupling between them (V and $V^\dagger$ terms) and backscattering off the junction (term V_0)[11]. In the case of the time-independent phase, $\phi(t) = \phi$, scattering is elastic and described by an instantaneous scattering matrix $S(\phi)$. At a finite bias, the phase $\phi(t) = \Omega\, t$ winds with frequency Ω, allowing for inelastic transitions with energy transfer $N\hbar\Omega$.

To relate S_N to $S(\phi)$, we compare their respective representations by infinite-order series in W. For that, we inspect the time evolution of the wave-function $|\psi(t)\rangle = U(t)|m\rangle$ with the initial state $|m\rangle$ at $t = -\infty$; here $|m\rangle$ is an eigenstate of H_0 with energy ε_m. The evolution operator is given by the usual time-ordered exponential $U(t) = \mathcal{T} \exp\left[\frac{1}{i\hbar}\int_{-\infty}^{t} dt_1 W_I(t_1)\right]$, and the subscript I stands for the interaction representation. The k-th order expansion term of the evolution operator[12] reads

$$U_k(t) = \frac{1}{(i\hbar)^k}\int_{-\infty}^{t} dt_k W_I(t_k) \cdots \int_{-\infty}^{t_2} dt_1 W_I(t_1).$$

At this point, it is convenient to introduce a variable s taking values $0, \pm 1$ and rewrite $W(t) = \sum_s V^s e^{is\phi(t)}$, where $V^{-1} \equiv V$, $V^{+1} \equiv V^\dagger$, and $V^0 \equiv V_0$. That allows one to further specify the form of the expansion term. For $\phi(t) = \Omega\, t$, we may write $U_k(t)$ as a sum of harmonics,

$$U_k(t) = \sum_N \frac{e^{iN\Omega t}}{(i\hbar)^k} \sum_{s_1,\dots s_k} \delta_{\sigma_k, N} \int_{-\infty}^{t} dt_k e^{i\sigma_k \Omega(t_k - t)} V_I^{s_k}(t_k) \cdots \int_{-\infty}^{t_2} dt_1 e^{i\sigma_1\Omega(t_1 - t_2)} V_I^{s_1}(t_1), \quad (2)$$

with $\sigma_k = s_k + \ldots + s_1$. A similar result for the static problem, $\phi(t) = \phi$, is obtained from Eq. (2) by replacing the factor $e^{iN\Omega t} \to e^{iN\phi}$ and setting $\Omega = 0$ in all the integrands.

This form of $U_k(t)$ allows a direct comparison of the perturbative expansion of the wavefunctions for linearly winding phase $\phi(t) = \Omega\, t$, and for fixed phase $\phi(t) = \phi$, which we denote $|\tilde{\psi}(t)\rangle$ and $|\psi(t)\rangle$, respectively. Projecting the two wave functions onto the energy eigenstate $|n\rangle$ of H_0 with energy ε_n, we find

$$\langle n|\tilde{\psi}(t)\rangle \equiv \langle n| \left[U(t)|_{\phi(t)=\Omega t}\right] |m\rangle \quad (3)$$
$$= \delta_{nm} + \sum_N \frac{1}{i\hbar}\int_{-\infty}^{t} dt'\, e^{i(\varepsilon_{n,m} + \hbar\Omega N)t'/\hbar - 0|t'|}\, \tilde{\mathcal{T}}_{nm}(N, \Omega)$$

and

$$\langle n|\psi(t)\rangle \equiv \langle n| \left[U(t)|_{\phi(t)=\phi}\right] |m\rangle \quad (4)$$
$$= \delta_{nm} + \frac{1}{i\hbar}\int_{-\infty}^{t} dt'\, e^{i\varepsilon_{n,m}t'/\hbar - 0|t'|}\, \mathcal{T}_{nm}(\phi)\, .$$

The $\mathcal{T}$-matrices introduced above are given by the following series:

$$\tilde{\mathcal{T}}_{nm}(N, \Omega) = \sum_{k=1}^{\infty} \sum_{\substack{m_{k-1},\dots,m_1 \\ s_k + \ldots + s_1 = N}} \quad (5)$$
$$\times \frac{V^{s_k}_{nm_{k-1}} \ldots V^{s_1}_{m_1 m}}{(\varepsilon_{m,m_{k-1}} - \hbar\Omega\, \sigma_{k-1} + i0)\ldots(\varepsilon_{m,m_1} - \hbar\Omega\, \sigma_1 + i0)}$$

and

$$\mathcal{T}_{nm}(\phi) = \quad (6)$$
$$\sum_N e^{i\phi N} \sum_{k=1}^{\infty} \sum_{\substack{m_{k-1},\dots,m_1 \\ s_k + \ldots + s_1 = N}} \frac{V^{s_k}_{nm_{k-1}} \ldots V^{s_1}_{m_1 m}}{(\varepsilon_{m,m_{k-1}} + i0)\ldots(\varepsilon_{m,m_1} + i0)}.$$

Here, we introduced the notation $\varepsilon_{m,n} = \varepsilon_m - \varepsilon_n$ and wrote the matrix elements as $V^s_{mn} = \langle m|V^s|n\rangle$. A finite Ω brings about inelastic transitions with an arbitrary integer number N of energy quanta $\hbar\Omega$ being released ($N > 0$) or absorbed ($N < 0$). The corresponding transition amplitudes are given by $\tilde{\mathcal{T}}_{nm}(N, \Omega)$. In the case of fixed-phase, $\phi(t) = \phi$, the scattering is elastic.

By comparing the *inelastic* (5) and *elastic* (6) $\mathcal{T}$-matrices, we note that in the limit $\Omega \to 0$

$$\tilde{\mathcal{T}}_{nm}(N, 0) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, \mathcal{T}_{nm}(\phi)\, e^{-i\phi N}. \quad (7)$$

The utility of this expression is that the scattering matrix of a time-independent problem may be easier to evaluate. The use of Eq. (7) is justified as long as the effect of $\hbar\Omega$ in the energy denominators of Eq. (5) is negligible. An applicability criterion specific to a superconducting junction is discussed in the end of the paper. We note in passing that Eq. (7) agrees with the "frozen scattering matrix" principle set forward in Refs. [13,14]. Next, we evaluate dissipative conductance using Eq. (7).

The dissipated power may be written using scattering theory, where the absorbed power, averaged over states in equilibrium, is[15]

$$\mathcal{P} = \frac{2\pi}{\hbar} \sum_N N\hbar\Omega \sum_{n,m} \left|\tilde{\mathcal{T}}_{nm}(N, \Omega)\right|^2 \quad (8)$$
$$\times \left[f(\varepsilon_n) - f(\varepsilon_m)\right]\, \delta(\varepsilon_n - \varepsilon_m + \hbar\Omega N).$$

Each term in the sum over N here has a simple meaning: it is a product of the energy $N\hbar\Omega$ absorbed in a transition, multiplied by the transition rate (here $f(\varepsilon_{n,m})$ are fermionic occupation factors). In the framework of scattering theory, it is customary to work in the continuous energy representation instead of the discrete indices n

and m. Therefore, we replace $n \to (\varepsilon'\alpha)$, $m \to (\varepsilon\beta)$ and introduce the density of states $\rho_\alpha(\varepsilon')$ and $\rho_\beta(\varepsilon)$ to re-write Eq. (8) in the form

$$\mathcal{P} = \frac{2\pi\hbar\Omega}{\hbar}\sum_N N \sum_{\alpha,\beta}\iint d\varepsilon' d\varepsilon\, \rho_\alpha(\varepsilon')\rho_\beta(\varepsilon) \times \left|\tilde{\mathcal{T}}_{\varepsilon'\alpha\,,\,\varepsilon\beta}(N,\Omega)\right|^2 [f(\varepsilon') - f(\varepsilon)]\; \delta(\varepsilon' - \varepsilon + \hbar\Omega N). \tag{9}$$

Here, α and β are the residual discrete indices; they may label channels, leads, particle-hole branches, etc. We integrate Eq. (9) over ε' and expand to the lowest (second) order in Ω

$$\mathcal{P} = \frac{2\pi(\hbar\Omega)^2}{\hbar}\sum_N N^2 \sum_{\alpha\beta}\int d\varepsilon\, \rho_\alpha(\varepsilon)\rho_\beta(\varepsilon) \times \left|\tilde{\mathcal{T}}_{\varepsilon\alpha\,,\,\varepsilon\beta}(N,0)\right|^2 [-\partial_\varepsilon f(\varepsilon)]. \tag{10}$$

Crucially, the *inelastic* $\mathcal{T}$-matrix $\tilde{\mathcal{T}}(N,\Omega=0)$ is evaluated at $\Omega = 0$ in Eq. (10). So we may express it via the *elastic* $\mathcal{T}$-matrix according to Eq. (7),

$$\mathcal{P} = \frac{2\pi(\hbar\Omega)^2}{\hbar}\int d\varepsilon\,[-\partial_\varepsilon f(\varepsilon)] \iint_0^{2\pi} \frac{d\phi' d\phi}{(2\pi)^2} \times \sum_N e^{iN(\phi'-\phi)} N^2 \sum_{\alpha\beta} \rho_\alpha(\varepsilon)\rho_\beta(\varepsilon)\, \mathcal{T}^*_{\varepsilon\alpha\,,\,\varepsilon\beta}(\phi')\mathcal{T}_{\varepsilon\alpha\,,\,\varepsilon\beta}(\phi). \tag{11}$$

Next we use the relation[12],[16] between the $\mathcal{T}$-matrix and the on-shell *elastic* scattering matrix and replace the derivatives $-2\pi i\sqrt{\rho_\alpha(\varepsilon)\rho_\beta(\varepsilon)}\,\partial_\phi \mathcal{T}_{\varepsilon\alpha\,,\,\varepsilon\beta}(\phi) \to \partial_\phi S_{\alpha\beta}(\phi,\varepsilon)$, which allows one to express the summation over α and β as a trace. Further simplification comes from noticing that $\sum_N e^{iN(\phi'-\phi)}N^2 = 2\pi\partial_\phi\partial_{\phi'}\delta(\phi-\phi')$ in Eq. (11). Finally, recalling that $\Omega = \mathcal{V}e/\hbar$ and $G = \mathcal{P}/\mathcal{V}^2$, we obtain the dissipative conductance, which is the main result of this work:

$$G = \frac{e^2}{h}\int d\varepsilon\,[-\partial_\varepsilon f(\varepsilon)] \int_0^{2\pi} \frac{d\phi}{2\pi}\, \mathrm{Tr}\left\{\partial_\phi S^\dagger(\phi,\varepsilon)\,\partial_\phi S(\phi,\varepsilon)\right\}. \tag{12}$$

Consistently with Eq. (1), the gauge in Eq. (12) is fixed by associating the phase factor $e^{i\phi}$ with the transmission amplitude of the normal-state scattering matrix. For a superconducting junction, the order parameter phase difference across the junction is $\varphi = 2\phi$.

It is instructive to relate the DC conductance G to the dissipative part of the low-frequency admittance $Y(\omega \to 0, \phi, T)$ of the same junction.[17] In evaluating $\mathrm{Re}\, Y(\omega \to 0, \phi, T)$, the perturbation $\delta\phi(t) = eU\cos(\omega t)/\hbar\omega$ of the phase $\phi(t) = \phi + \delta\phi(t)$ across the junction is a small parameter, as the limit $U \to 0$ is taken first. Applying the same technique as above, we find that only single-quantum transitions occur to linear order in U, with amplitudes $\propto \partial_\phi S$. Evaluation of the absorption power yields[18]

$$\mathrm{Re}\, Y(\omega \to 0, \phi, T) = \frac{e^2}{h}\int d\varepsilon\,[-\partial_\varepsilon f(\varepsilon)]\,\mathrm{Tr}\left\{\partial_\phi S^\dagger(\phi,\varepsilon)\,\partial_\phi S(\phi,\varepsilon)\right\}. \tag{13}$$

Comparing Eq. (12) with (13) and recalling that the phase winds with time as $e\mathcal{V}t/\hbar$, we conclude that G may be viewed as a time-averaged value

$$G = \overline{\mathrm{Re}\, Y(\omega \to 0, e\mathcal{V}t/\hbar, T)} \tag{14}$$

of the instantaneous conductance given by the dissipative part of the admittance. It generalizes the known relation in normal junctions[2] between the DC Landauer conductance and the $\omega \to 0$ limit of the Kubo formula.

Equation (12) is non-perturbative in tunneling, which is one of its advantages over the known[4,5] results. We illustrate the utility of Eq. (12) by finding the conductance between two superconductors connected by a short channel of arbitrary transmission coefficient, see Fig. 1. Finite temperature induces a thermal population of quasiparticles in each of the two leads. To start with, we focus on the case of equal gaps $\Delta_1 = \Delta_2 = \Delta$. We follow Ref. [19] and evaluate the corresponding S-matrix. In the Bogoliubov-de Gennes representation, the quasiparticle excitations have positive energy $\varepsilon > \Delta$, and the S-matrix is 4-by-4 due to the 2 leads and 2 particle-hole branches, see [20] for details. We apply Eq. (12) and evaluate the conductance at arbitrary transmission coefficient τ of the junction,

$$\frac{G_{SPC}}{G_n} = \int_\Delta^\infty d\varepsilon\,[-\partial_\varepsilon f(\varepsilon)]\,\frac{2\,\varepsilon^2}{\sqrt{(\varepsilon^2-\Delta^2)(\varepsilon^2-\Delta^2(1-\tau))}}. \tag{15}$$

Here $G_n = 2e^2\tau/h$ is the normal-state conductance. An alternative way to derive Eq. (15) is to use Eq. (14) and the result[21] for $\mathrm{Re}\, Y(\Omega, \phi, T)$.

It is instructive to consider first the low-temperature asymptote, $\Delta/T \gg 1$,

$$\frac{G_{SPC}(\Delta/T,\tau)}{G_n(\tau)} \approx \sqrt{\frac{2\Delta}{\Delta+\varepsilon_A(\tau)}}\,\frac{\Delta}{T}\, e^{-\frac{\Delta+\varepsilon_A(\tau)}{2T}} K_0\left[\frac{\Delta-\varepsilon_A(\tau)}{2T}\right], \tag{16}$$

where $K_0(x)$ is the modified Bessel function. Note that the superconducting contact supports Andreev levels with energies $\varepsilon_A(\tau,\phi) = \Delta\sqrt{1-\tau\sin^2\phi}$ carrying the Josephson current, which is not the subject of this work. However the indirect effect of the Andreev levels is observed in Eqs. (15) and (16), where we denote $\varepsilon_A(\tau) \equiv \varepsilon_A(\tau,\pi/2) = \Delta\sqrt{1-\tau}$. The Andreev levels lead to a strong modification of the density of states of the delocalized quasiparticles and thus influence their transport. The low-temperature conductance (16) displays a

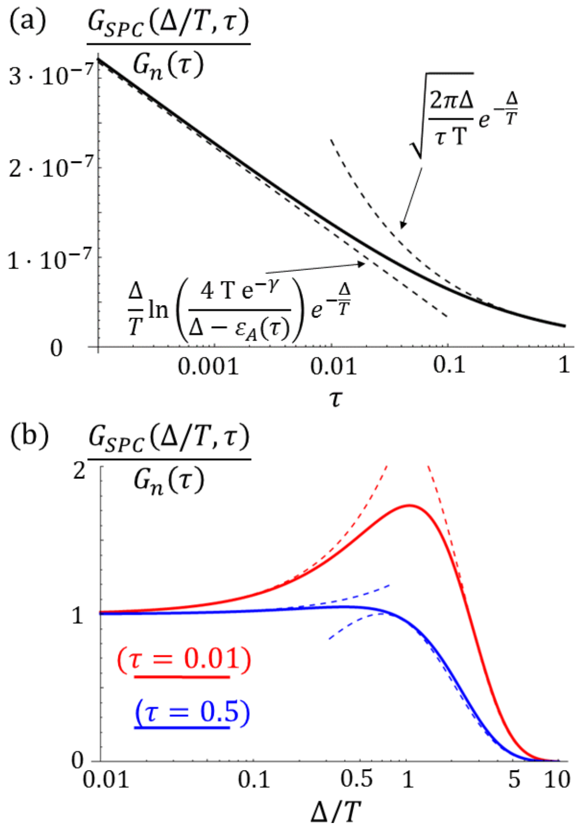

FIG. 2. (a) Conductance G_{SPC} of a superconducting point contact as a function of transmission coefficient τ evaluated from Eq. (15) at a low temperature, $T/\Delta = 0.05$. The two asymptotes of Eq. (16), shown in dashed lines, are valid, respectively, at transmission $\tau \ll T/\Delta$ and $\tau \gg T/\Delta$. (b) G_{SPC} as a function of Δ/T (solid lines) at two fixed values of τ, along with the asymptotes (16) and (17), shown by dashed lines.

crossover between two asymptotes defined by a dimensionless ratio $\frac{\Delta-\varepsilon_A}{T} \propto \frac{\tau\Delta}{T}$. Above the crossover temperature ($T \gg \tau\Delta$), the conductance may be approximated as $G_{SPC} = G_n \frac{\Delta}{T} \ln[4\, e^{-\gamma}\, T/(\Delta - \varepsilon_A)]e^{-\frac{\Delta}{T}}$ (here γ is the Euler-Mascheroni constant). We note that the perturbative-in-τ result[4,5] which diverges as $2e\mathcal{V} \to 0$, is cut off by the scale $\Delta - \varepsilon_A$. Below the crossover temperature ($T \ll \tau\Delta$), one finds $G_{SPC} = G_n\sqrt{\frac{2\pi\Delta}{\tau T}}e^{-\frac{\Delta}{T}}$. Both asymptotes are illustrated in Fig. 2(a).

The high-temperature $T \gg \Delta$ (i.e. small-gap) asymptote is

$$\frac{G_{SPC}(\Delta/T,\tau)}{G_n(\tau)} \approx 1 + \frac{\Delta}{T}\, k(\tau)\,. \tag{17}$$

Note that the coefficient $k(\tau) \geq 0$ (see Ref. [20] for the full expression). It is logarithmically large, $k(\tau) \sim -\frac{1}{4}\ln\tau$, for $\tau \to 0$, and $k(\tau = 1) = 0$. Therefore, at any $\tau < 1$ the conductance G_{SPC} initially grows with the opening of the superconducting gap Δ. We plot the dependence of G_{SPC} on Δ/T in Fig. 2(b) and observe that the conductance reaches maximum at $\Delta \sim T$. Note that the thermoelectric transport coefficients of SPC exhibit similar behavior[22].

The dissipative conductance Eq. (15) involves an unusual type of multiple Andreev reflection processes. In such events, quasiparticles are not created but rather gain energy exceeding $e\mathcal{V}$ at $N > 1$. In the context of Eqs. (8)–(10), N represents the number of energy quanta $\hbar\Omega$ absorbed or emitted during the quasiparticle tunnelling. Because of the relation $\hbar\Omega = e\mathcal{V}$, integer N also has the meaning of the number of electrons passing through the junction in a scattering event. The corresponding probabilities are given by the appropriately thermally-averaged[20] values of $|\tilde{\mathcal{T}}(N,\Omega)|^2$, see Eq. (10). At $T \ll \Delta{\cdot}\tau$, the averaged $|\tilde{\mathcal{T}}(N,\Omega)|^2$ depend weakly on N for $N < N^* = \sqrt{\Delta\tau/T}$ and decay as $\tilde{T}(N,\Omega) \sim 1/N^4$ for $N > N^*$. This indicates that processes with a transfer of a large number of electrons gain significance at low temperatures.

If both leads are superconducting, the series for the absorbed power (10) contains infinitely many terms in N, and the trace formula (12) is an agile way to calculate G. If at least one of the leads is non-superconducting, the sum over N in Eq. (10) truncates. As an example, we consider an NS junction, i.e. set $\Delta_1 = 0$, $\Delta_2 = \Delta$. It is easy to see[20] that the highest harmonics of the elastic S-matrix are $e^{\pm 2i\phi}$, truncating the series at $|N| = 2$. Evaluating the sum or using the trace formula (12), and accounting for the unitarity of the S-matrix, we recover the known[23] expression,

$$G_{NS} = \frac{2e^2}{h}\int_0^\infty d\varepsilon\,[-\partial_\varepsilon f(\varepsilon)]\,\big[(1-|r^{ee}|^2+|r^{he}|^2) + (1-|r^{hh}|^2+|r^{eh}|^2)\big]\ , \tag{18}$$

where $r^{ee}(\varepsilon)$, $r^{hh}(\varepsilon)$, and $r^{he}(\varepsilon)$, $r^{eh}(\varepsilon)$ are, respectively, the particle, hole, and two Andreev reflection amplitudes[24]. The S-matrix of a normal junction ($\Delta_1 = \Delta_2 = 0$) contains only $e^{\pm i\phi}$ harmonics, along with a ϕ-independent part. As a result, $r^{he}(\varepsilon) = r^{eh}(\varepsilon) = 0$ and Eq. (18) reduces to the standard Landauer formula in the particle-hole representation.

In the derivation of Eq. (12), we relied upon the relation between elastic and "soft" inelastic scattering matrices, cf. Eq. (7). This is justified as long as $\hbar\Omega$ is negligible compared to the typical energy differences $\varepsilon_m - \varepsilon_{m'}$ involved in the summation over virtual states. In the context of a tunnel junction between two superconductors with gaps $\Delta_1 \neq \Delta_2$, one may estimate the significance of the next-order in $\Omega = e\mathcal{V}/\hbar$ terms by expanding in $\mathcal{V}$ the known[4] expression, $I(\mathcal{V}) = I_1(\mathcal{V}) + I_3(\mathcal{V}) + \mathcal{O}(\mathcal{V}^5)$, where $I_n(\mathcal{V}) \propto \mathcal{V}^n$. We evaluate the ratio of the consecutive terms in the expansion of current[20] and find $\frac{I_3}{I_1} \propto \frac{(e\mathcal{V})^2}{T^2}$ and $\frac{I_3}{I_1} \propto \frac{(e\mathcal{V})^2}{(\Delta_1-\Delta_2)^2}$ in the cases $|\Delta_1 - \Delta_2| \gg T$ and $|\Delta_1 - \Delta_2| \ll T$, respectively. In other words, the next-order corrections in $e\mathcal{V}$ may be dropped as long as $\hbar\Omega = e\mathcal{V}$ is the smallest energy scale in the problem. At finite transmission τ and equal gaps, for which Eq. (15) is derived, this applicability criterion amounts to $e\mathcal{V} \ll \min[\,T, \Delta - \varepsilon_A(\tau)]$.

It is worth emphasizing that the derived dissipative conductance G_{SPC}, Eq. (15), is entirely due to the itinerant Bogoliubov quasiparticles passing through the junction. The associated Andreev levels do not contribute to the dissipation in the absence of relaxation. The latter creates an additional channel of dissipation via the Debye mechanism[25]. To quantify this, we introduce a phenomenological relaxation rate γ for an occupied Andreev level[26] and estimate the ratio $\frac{I_A}{I_{qp}}$ of the dissipative current $I_A(\mathcal{V})$ due to the Andreev levels[20] and the current $I_{qp}(\mathcal{V}) = G_{SPC}\mathcal{V}$ due to the quasiparticles. In the limit $\frac{\tau\Delta}{T} \ll 1$, we estimate $\frac{I_A}{I_{qp}} \propto \frac{\tau}{\ln(T/\tau\Delta)} \frac{\Delta\,\hbar\gamma}{(\hbar\gamma)^2+(2e\mathcal{V})^2}$, indicating that the quasiparticle current I_{qp} dominates even in the linear-in-$\mathcal{V}$ regime ($e\mathcal{V} \ll \hbar\gamma$) provided the relaxation rate $\hbar\gamma \gtrsim \tau\Delta$. In the limit of low temperatures $T/\Delta \ll 1$ and intermediate τ, we find that the ratio of currents scales as $\frac{I_A}{I_{qp}} \propto \frac{T}{\hbar\gamma}\exp[\frac{\Delta}{T}(1-\sqrt{1-\tau})]$ and $\frac{I_A}{I_{qp}} \propto \frac{T\,\hbar\gamma}{(e\mathcal{V})^2}\exp[\frac{\Delta}{T}(1-\sqrt{1-\tau})]$ in the opposite regimes of small ($e\mathcal{V} \ll \hbar\gamma$) and large ($e\mathcal{V} \gg \hbar\gamma$) bias, respectively. In the latter regime, the large exponential factor may be mitigated by a small γ. Note that in the absence of the relaxation due to phonons as, e.g., in the cold atom experiments[9], the relaxation is itself determined by the quasiparticle population and is, therefore, exponentially suppressed at low temperatures, $\gamma \propto \exp(-\Delta/T)$.

In summary, we have expressed the dissipative linear conductance G of a superconducting quantum point contact in terms of the scattering matrix for Bogoliubov quasiparticles, see Eq. (12). At a finite temperature, G is finite; Eq. (12) adequately accounts for the thermally-excited quasiparticles passing through the junction. It generalizes the Landauer formula and is valid for junctions with normal or superconducting leads. In addition, we uncovered the relation (14) between the DC conductance and the phase-averaged real part of the AC admittance of a junction.

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[16] We follow Ref. [12], where the relation is derived $\mathcal{S}_{nm} = \delta_{nm} - 2\pi i\,\delta(\varepsilon_n - \varepsilon_m)\,\mathcal{T}_{nm}$ between matrix elements of scattering $\mathcal{S}$ and transition $\mathcal{T}$ operators. In the energy ε representation, this relation translates into the following identification of the scattering matrix $S_{\alpha\beta}(\varepsilon) = \delta_{\alpha\beta} - 2\pi i\,\sqrt{\rho_\alpha(\varepsilon)\rho_\beta(\varepsilon)}\;\mathcal{T}_{\alpha\varepsilon,\beta\varepsilon}$.

[17] The admittance is defined as a linear AC response to an applied bias, $I(\omega) = Y(\omega, \phi, T)U(\omega)$.

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